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An Approximate solution of Nonlinear Differential Equations with Variable Coefficients by using Modified-Laplace Variational Iteration Method

Razaq.A. ODERINU [1], Waliu.A. TIJANI [2], Kafilat.A. SALAUDEEN [3]

[1]*Ladoke Akintola University of Technology, Ogbomoso, Department of Pure and Applied Mathematics, Nigeria. E-mail: raoderinu@lautech.edu.ng*
[2]*Ladoke Akintola University of Technology, Ogbomoso, Department of Pure and Applied Mathematics, Nigeria. E-mail: adiguntijani111@gmail.com*
[3]*Emmanuel Alayande University of Education, Oyo, Department of Mathematics, Nigeria. E-mail: salaudeenka@gmail.com*

Abstract

The modified Laplace method, an integral transform, was enhanced to effectively solve nonlinear differential equations with variable coefficients. Some fundamental theorems have been established about the transforms of function with variable coefficients and nth order derivatives. Theorems that had already been established were employed in conjunction with variational iteration method to formulate a correction functional capable of simplifying nonlinear functions. The correctness of the suggested method was evaluated by considering three exemplary situations. The findings were then compared to the referenced solutions, showing a favourable comparison.
Keywords: Modified Laplace transform method, variable coefficient, variational iteration method.
Corresponding author:Waliu. A. TIJANI (email: adiguntijani111@gmail.com)

1. INTRODUCTION

Problems of nonlinearity are of fundamental importance in different fields of science, engineering and technology. Nonlinear models of real-world phenomena are complex to solve either approximately or analytically. Many years back, several numerical techniques such as Differential Transform Method (DTM), Adomian Decomposition Methods (ADM), Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) have been used to solve various linear and nonlinear Ordinary Differential Equations (ODE), Partial Differential Equations (PDE) and integro-differential equations. Many studies have shown that these methods are reliable and efficient for a wide range of scientific applications. VIM has been successfully applied to many initial and boundary value problems, the steps involved in applying VIM to solve differential equations are; obtaining the correction functional; identifying the Lagrange multiplier; determining a good initial approximation, (Yisa *et al.*, 2018).

Much attention has been paid to the search for better and more efficient approximate or exact, analytical or numerical methods for solving nonlinear problems. Integral transformations are one of the most commonly used mathematical techniques to determine the solutions of linear advance problems of space, science, technology and engineering .The importance of these integral transforms is that they provide powerful operational methods for solving initial value problems as well as boundary value problems for linear differential and integral equations, this transforms also play a crucial role in control engineering. There are many standard integral transform methods for solving linear differential equations, for example, the Laplace transform method and its various modifications [10].

Laplace transform is employed in many field of science, Engineering and Technology. In 1782 during the study of theory of probability Laplace transform was introduced to solve linear ordinary, or partial differential equations with suitable initial and boundary conditions. Modified Laplace transform for a piece-wise continuous function of exponential order which reduces to Laplace transform for $e = a$ where $a \neq 1$, $a > 0$ and $t > 0$, was introduced by [6].

Recently many researchers have combined Laplace transforms with other numerical methods because of its inability to handle nonlinear problems. Among the methods are: VIM and HPM by [7] and [15] respectively. This limitation of Laplace transform also affects other integral transforms such as Elzaki transform, Sadik transform, Sumudu transform, Kamal transform and Mohand transform . Modified Laplace transform by [6] is one of these integral transforms, thus, it has to be combined with another method to effectively solve nonlinear differential equations and this call for it combination with VIM.

In this paper a combined Modified Laplace transform with Variational Iteration Method (MLVIM) is proposed to solve nonlinear differential equations with variable coefficient. This method gives an approximate solution in the form of a rapidly convergent series. Using combined modified Laplace variational iteration method, the solutions to three problems at third iteration were compared with exact solution. The obtained results show the reliability, accuracy and efficiency of the MLVIM.

2. BASIC IDEA OF VIM

Consider the differential equation in an operator form as:

$$Ry(t) + Uy(t) + Ny(t) = g(t). \tag{1}$$

Where R is a linear operator with the highest derivative, U is remaining linear operator with derivative less than R, N is a nonlinear operator and g(t) is nonhomogeneous term. According to [10], VIM is constructed as:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(k)[Ry_n(k) + U\bar{y}_n(k) + N\bar{y}_n(k) - f(k)]dk. \tag{2}$$

Where y_0 is an initial approximation and $\bar{y}_n$ is a restricted variation. While the integral part in equation (2) is called correction functional and subscript n denotes nth approximation. With Lagrange multiplier defined by Abbasbandy (2008) as

$$\lambda = \frac{(-1)^m(s-t)^{m-1}}{(m-1)!}. \tag{3}$$

where m is the highest order derivatives.

Definition 3.1 Laplace transform of $f((t))$ for a piecewise continuous functions of exponential order is defined as:

$$Lf(t) = \int_0^\infty e^{-st} f(t)dt \quad R(s) > 0, [10]. \tag{4}$$

Definition 3.2 Modified Laplace transform of $f((t))$ for a piecewise continuous functions of exponential order is defined as:

$$L_a f(t) = \int_0^\infty a^{-st} f(t)dt \quad R(s) > 0, a\epsilon(0,\infty)/\{1\} \tag{5}$$

Where eqn. (5) reduces to simple Laplace transform for $a = e$, [6].

Some fundamental properties of modified Laplace transform:

(1) if $f(t)$=1, then, $L_a(1)$=$\frac{1}{slog_e a}$, $(s > 0)$.

(2) if $f(t) = t$, then, $L_a(t)$=$\frac{1}{s^2(log_e a)^2}$, $(s > 0)$.

(3) if $f(t)$=t^n, then, $L_a(t^n)$=$\frac{n!}{s^{n+1}(log_e a)^{n+1}}$, $(s > 0, \quad n = 0, 1, 2, ...)$.

(4) if $f(t)$=e^{bt}, then, $L_a(e^{bt})$=$\frac{1}{slog_e a - b}$, $slog_e a > |b|$.

(5) if $f(t)$=$sinbt$, then, $L_a(sinbt)$=$\frac{b}{s^2(log_e a)^2 + b^2}$, $(slog_e a > 0)$.

(6) if $f(t)$=$cosbt$, then, $L_a(cosbt)$=$\frac{s}{s^2(log_e a)^2 + b^2}$, $(slog_e a > 0)$

Theorem 1

If $f(t)$ is a piecewise continuous function and of exponential order, where

$$L_a(f(t)) = \int_0^\infty a^{-st} f(t)dt, \tag{6}$$

then

$$L_a\{t^n f(t)\} = (-1)^n \frac{f^n(s,a)}{(log_e a)^n} \quad where\, n = 1, 2, 3, ... \tag{7}$$

Proof,

The proof of theorem 1 is achieved by the principle of mathematical induction. that is when n = 1, in eqn. (7), gives

$$L_a(tf(t)) = \frac{-f(s,a)}{log_e a}. \tag{8}$$

Differentiating eqn. (6) with respect to s, gives

$$\frac{d}{ds}(f(t)) = \frac{d}{ds}\int_0^\infty a^{-st} f(t)dt \tag{9}$$

Simplifying eqn. (8), gives

$$\frac{d}{ds}f(t) = \int_0^\infty \frac{d}{ds} e^{log_e a - st} f(t)dt$$

$$= \int_0^\infty (-tlog_e a) e^{-stlog_e a} f(t)dt$$

$$= \int_0^\infty (-tlog_e a) e^{log_e a - st} f(t)dt$$

$$= -log_e a \int_0^\infty a^{-st} t(f(t)dt$$

$$f^{'}(s,a) = (-loga Ł_a t f(t))$$

$$L_a(tf(t)) = -\frac{f^{'}(s,a)}{log_e a}. \tag{10}$$

Since equation (8) agrees with eqn. (10) then $n = 1$, is true.
Assuming $n = k$ in eqn. (7) then

$$L_a(t^k f(t)) = \frac{(-1)^k f^k(s,a)}{(log_e a)^k}. \tag{11}$$

is true .

To show that $n = k + 1$ is true whenever $n = k$ is true. That is when n = k+1 in eqn. (7), gives

$$L_a\{t^{k+1}f(t)\} = \frac{(-1)^{k+1}f^{(k+1)}(s,a)}{(log_e a)^{k+1}.} \tag{12}$$

Considering eqn. (11) that is

$$L_a(t^k f(t)) = \int_0^\infty a^{-st}t^k(f(t)dt, \tag{13}$$

Differentiating both sides of eqn.(11) with respect to s gives

$$\frac{d}{ds}L_a(t^k f(t)) = \frac{d}{ds}\frac{(-1)^k f^k(s,a)}{(log_e a)^k} \tag{14}$$

Simplifying eqn. (14), gives

$$\frac{d}{ds}\int_0^\infty a^{-st}t^k(f(t)dt = \frac{(-1)^k f^{(k+1)}(s,a)}{(log_e a)^k}$$

$$\int_0^\infty (-tlog_e a)e^{-stlog_e a}t^k f(t)dt = \frac{(-1)^k f^{(k+1)}(s,a)}{log_e a^k}$$

$$\int_0^\infty e^{log_e a - st}t^{k+1}f(t)dt = \frac{(-1)^k f^{(k+1)}(s,a)}{(log_e a)^k}$$

$$-log_e a \int_0^\infty a^{-st} t^{k+1} (f(t))dt = \frac{(-1)^k f^{(k+1)}(s,a)}{(log_e a)^k}$$

$$\int_0^\infty a^{-st} t^{k+1} (f(t)dt = \frac{(-1)^k f^{(k+1)}(s,a)}{log_e a^k} \frac{(-1)}{(log_e a)}$$

$$L_a\{t^{k+1}(f(t))\} = \frac{(-1)^{k+1} f^{k+1}(s,a)}{(log_e a)^{k+1}}.$$

Proof complete.

Theorem 2

If $f(t)$ is a piecewise continuous function and of exponential order, where modified Laplace transform of function f(t) is equation (6)
then

$$L_a(\frac{f(t)}{t}) = log_e a \int_{(s,a)}^\infty f(u)du. \tag{15}$$

Proof:
Let

$$G(t) = \frac{f(t)}{t} \tag{16}$$

and

$$L_a(G(t)) = L_a(\frac{f(t)}{t}) = g(s,a) \tag{17}$$

Simplifying eqn. (16), gives

$$f(t) = t(G(t)) \tag{18}$$

Taking modified Laplace transform of both sides of eqn. (18), gives

$$L_a f(t) = L_a(tG(t)) \tag{19}$$

Using theorem 1, eqn. (19) becomes

$$f(s,a) = \frac{-1}{log_e a} \frac{d}{ds} g(s,a) \tag{20}$$

since $\frac{d}{ds}g(s,a) = g^{'}(s,a)$ isolating $-g^{'}(s,a)$ in eqn. (20), gives

$$log_e a f(s,a) = -g^{'}(s,a) \tag{21}$$

Integrating both sides of eqn. (21), gives

$$-[g(s,a)]_{(s,a)}^{\infty} = log_e a \int_{(s,a)}^{\infty} f(u)du \tag{22}$$

Simplifying eqn. (22), gives

$$g(s,a) = log_e a \int_{(s,a)}^{\infty} f(u)du \tag{23}$$

Substituting eqn. (17) into eqn. (22), gives eqn. (15), proof complete.

Theorem 3

If $f(t)$ is a piecewise continuous function and of exponential order, where modified Laplace transform of function f(t) is equation (6), then

$$L_a(f^n(t)) = s^n(log_e a)^n L_a(f(t)) - \sum_{k=1}^{n} s^{n-k}(log_e a)^{n-k} f^{k-1}(0) \tag{24}$$

where f^n is the n^{th} order derivative.

Proof:

Applying the principle of mathematical induction.

When n = 1 in eqn. (24), gives

$$L_a(f^{'}(t)) = s(log_e a)L_a(f(t)) - f(0). \tag{25}$$

To achieve eqn. (25),

$$L_a(f^{'}(t)) = \int_0^{\infty} a^{-st}(f^{'}(t))dt \tag{26}$$

Simplifying eqn. (26) using the method of integration by parts, gives

$$= -f(0) + slog_e a \int_0^{\infty} f(t)a^{-st}dt$$

hence

$$L_a(f^{'}(t)) = slog_e a \quad L_a(f(t)) - f(0), \tag{27}$$

$n = 1$ is true.

Assuming $n = m$ in eqn. (24), is true.
then

$$L_a(f^m(t)) = s^m(log_e a)^m L_a(f(t)) - \sum_{k=1}^{m} s^{m-k}(log_e a)^{m-k} f^{(k-1)}(0), \tag{28}$$

is assumed to be true.
it has to be shown that n=m+1 in eqn.(24),gives

$$L(f^{(m+1)})(t) = s^{m+1}(log_e a)^{m+1} L_a(f(t)) - \sum_{k=1}^{m+1} s^{m+1-k}(log_e a)^{m+1-k} f^{(k-1)}(0) \tag{29}$$

whenever $n = m$ is true
let

$$L_a(f^{(m+1)}(t)) = \int_0^\infty a^{-st}(f^{(m+1)}(t))dt \tag{30}$$

simplifying equation (30) with the method of integration by part,
then

$$L_a(f^{(m+1)}(t)) = -f^m(0) + slog_e a \int_0^\infty a^{-st} f^m(t)dt \tag{31}$$

since

$$L_a(f^m(t))dt = \int_0^\infty a^{-st} f^m(t)dt \tag{32}$$

Substituting eqn. (32) into eqn. (31), gives

$$L_a(f^{(m+1)}(t)) = -f^m(0) + slog_e a L_a f^m(t) \tag{33}$$

Substituting eqn. (28) into eqn. (33), gives

$$L_a(f^{(m+1)}(t) = slog_e a[s^m(log_e a)^m L_a(f(t)) - \sum_{k=1}^{m} s^{m-k}(log_e a)^{m-k} f^{(k-1)}0] - f^m(0) \tag{34}$$

Simplifying eqn. (34), gives

$$L_a(f^{(m+1)}(t)) = s^{(m+1)}(log_e a)^{(m+1)} L_a(f(t)) - \sum_{k=1}^{m+1} s^{(m+1-k)}(log_e a)^{(m+1-k)} f^{(k-1)}(0),$$

3. THE SCHEME OF THE PROPOSED METHOD (MLVIM).

Taking modified Laplace transform of eqn.(1), gives

$$L_a(Ry(t) + Uy(t) + Ny(t) - f(t)) = 0 \tag{35}$$

Applying modified Laplace derivative property to eqn. (35), gives

$$s^m(log_e a)^m y(slog_e a) - \sum_{k=1}^{m} s^{m-k}(log_e a)^{m-k} y^{(k-1)}(0) =$$

$$-L_a(Uy(t) + Ny(t) - f(t)) \tag{36}$$

isolating $y(log_e a)$ in eqn. (36), gives
$y(slog_e a) = \frac{y(0)}{slog_e a} + ... + \frac{y^{m-1}(0)}{s^m(log_e a)^m}$

$$+(\frac{-1}{s^m(log_e a)^m} L_a(Uy(t) + Ny(t) - f(t)) \tag{37}$$

Taking inverse modified Laplaace transform of eqn.(37), gives
$y(t) = y(0) + ... + \frac{y^{m-1}(0)t^{m-1}}{(m-1)!}$

$$+(\frac{-1}{s^m(log_e a)^m} L_a(Uy(t) + Ny(t) - f(t)) \tag{38}$$

with initial approximation as

$$y_0(t) = y(0) + ... + \frac{y^{m-1}(0)t^{m-1}}{(m-1)!} \tag{39}$$

In order to obtain the Lagrange multiplier associate with this scheme, modified Laplace transform is applied to eqn.(2), to gives

$$y_{n+1}(slog_e a) = y_n(slog_e a) + L_a \int_0^t \lambda(k)[Ry_n(k) + U\bar{y}_n(k) + N\bar{y}_n(k) - f(k)]dk \tag{40}$$

Regarding the items $L_a(U\bar{y}_n(k) + N\bar{y}_n(k)$ as restricted variation in eqn. (40), and differentiating eqn.(32) with respect to $Y_n(log_e a)$, gives

$$\frac{dy_{n+1}(slog_e a)}{dy_n(slog_e a)} = 1 + [\lambda(k)]_0^t \tag{41}$$

Isolating $\lambda(t)$ by setting $\frac{dy_{n+1}(slog_e a)}{dy_n(slog_e a)} = 0$ in eqn. (41), gives

$$\lambda(t) = \frac{-1}{s^m(loga)^m} \tag{42}$$

substituting eqn.(42) into eqn. (40), gives

$$Y_{n+1}(sloga) = Y_n(sloga) + \frac{-1}{s^m(loga)^m} L_a(Ry_n(t) + Uy_n(t) + Ny_n(t) - f(t)) \tag{43}$$

The successive approximations are obtained by taking the inverse modified Laplace transform of eqn. (43), gives correction functional for combined
modified Laplace transform variational iteration method (CMLVIM) as

$$y_{n+1}(t) = y_n(t) + L_a^{-1}[\frac{-1}{s^m(loga)^m} L_a(Ry_n(k) + Uy_n(k) + Ny_n(k) - f(k))] \tag{44}$$

4. NUMERICAL ILLUSTRATIONS

The above procedure is applied to obtain solutions of certain nonlinear ordinary differential equations (ODE) with variable coefficient.
Illustration 4.1: Consider the differential equation

$$y' + (2t-1)y^2 = 0, y(0) = 1. \tag{45}$$

equation (45) has exact solution, $y(t) = \frac{1}{1-t+t^2}$ [14]
Taking modified Laplace transform (L_a) of eqn. (45) and make use of initial conditions, gives

$$y(slog_e a) = \frac{1}{s(log_e a)} + \frac{1}{s(log_e a)} L_a(-(2t-1)y^2) \tag{46}$$

Taking inverse modified Laplace transform (L_a^{-1}) of eqn. (46), gives

$$y(t) = 1 + L_a^{-1}(\frac{-1}{slog_e a} L_a(-(2t-1)y^2)) \tag{47}$$

Applying the combined modified Laplace variational scheme of eqn. (44) to eqn. (45), gives correction functional as

$$y_{n+1}(t) = y_n(t) + L_a^{-1}(\frac{-1}{slog_e a} L_a(y_n' + (2t-1)y_n^2)). \tag{48}$$

Starting with initial approximation $y_0(t) = 1$ and by using the iteration formula for eqn. (48), gives

$$y_1(t) = 1 + t - t^2$$

$$y_2(t) = 1 + t + t^5 - \frac{t^3}{3} - \frac{t^6}{3}$$

$$y_3(t) = 1 + t + \frac{13t^6}{9} + \frac{2t^5}{3} - t^3 - \frac{t^{14}}{63} + \frac{t^{13}}{9} - \frac{2t^{12}}{9}$$

$$-\frac{t^{11}}{9} + \frac{7t^{10}}{9} - \frac{2t^9}{9} - \frac{10t^8}{9} + \frac{t^7}{63} - \frac{4t^4}{3} + ... \quad (49)$$

Table 1: shows comparison between approximate solution, exact solution and absolute errors for problem 1

t	MLVIM solution	Exact solution	AEMLVIM	AEADM [14]
0.1	1.098874769	1.098901099	0.000026330	0.000729000
0.2	1.190169766	1.190476190	0.000306424	0.004096000
0.3	1.264803492	1.265822785	0.001019293	0.009261000
0.4	1.313923223	1.315789474	0.001866251	0.013824000
0.5	1.331082783	1.333333333	0.002250550	0.015625000
0.6	1.313923223	1.315789474	0.001866251	0.013824000
0.7	1.264803493	1.265822785	0.001019292	0.009261000
0.8	1.190169765	1.190476190	0.000306425	0.004096000
0.9	1.098874768	1.098901099	0.000026331	0.000729000
1.0	1.000000001	1.000000000	0.000000001	0.000000000

Illustration 4.2: Applying CMLVIM to solve nonlinear ODE with variable coefficient described by

$$y^{'} = y^2 - 2ty + t^2 + 1, y(0) = \frac{1}{2}. \quad (50)$$

Exact solution is $t + \frac{1}{2-t}$

Taking modified Laplace transform (L_a) of eqn. (50)and make use of initial conditions, gives

$$y(slog_e a) = \frac{1}{2s(loga)} + \frac{-1}{s(loga)} L_a{-y^2 + 2ty - t^2 - 1} \quad (51)$$

Taking inverse modified Laplace transform (L_a^{-1}) of eqn. (51), gives

$$y(t) = \frac{1}{2} + L_a^{-1}(\frac{-1}{s(loga)}L_a{-}y^2 + 2ty - t^2 - 1) \tag{52}$$

Applying the combined modified Laplace variational scheme of eqn. (44) to eqn. (50), gives correction functional as

$$y_{n+1}(t) = y_n(t) + L_a^{-1}(\frac{-1}{s(loga)}L_a(y_n^{'} - y_n^2 + 2ty_n - t^2 - 1)) \tag{53}$$

Starting with initial approximation $y_0(t) = \frac{1}{2}$ and by using the iteration formula for eqn. (52), gives

$$y_1(t) = \frac{1}{2} + \frac{5t}{4} - \frac{t^2}{2} + \frac{t^3}{3}$$

$$y_2(t) = \frac{1}{2} + \frac{5t}{4} + \frac{t^2}{8} - \frac{7t^3}{48} + \frac{t^7}{63} - \frac{t^6}{18} + \frac{t^5}{12} + \frac{t^4}{48}$$

$$y_3(t) = \frac{1}{2} + \frac{5t}{4} + \frac{t^2}{8} + \frac{t^3}{16} + \frac{29t^7}{16128} + \frac{11t^6}{1152} - \frac{7t^5}{960} - \frac{t^4}{48}$$

$$+\frac{t^{15}}{59533} - \frac{t^{14}}{7938} + \frac{t^{13}}{2268} - \frac{13t^{12}}{18144} + \frac{143t^{10}}{66480} - \frac{481t^9}{145152} + \frac{23t^8}{64512} + ... \tag{54}$$

Table 2: shows comparison between approximate solution, exact solution and absolute error of MLVIM for problem 2

t	MLVIM solution	Exact solution	AEMLVIM
0.1	0.626310354	0.626315790	0.000005436
0.2	0.755464970	0.755555556	0.000090589
0.3	0.887758357	0.888235294	0.000476937
0.4	1.023433661	1.025000000	0.001566339
0.5	1.162692899	1.166666667	0.003973768
0.6	1.305714651	1.314285714	0.008571063
0.7	1.452678576	1.469230769	0.016552193
0.8	1.603796517	1.633333333	0.029536816
0.9	1.759351078	1.809090909	0.049739831
1.0	1.919744147	2.000000000	0.080255853

Illustration 4.3: Applying MLVIM to solve third order nonlinear nonhomogeneous ODE with variable coefficient.

$$y^{'''} + y + y^2(t-1) = -t^6 + 4t^5 - 6t^4 + 5t^3 - 6, y(0) = 0, y^{'}(0) = 0, y^{"}(0) = 2 \quad (55)$$

According to Ruslan (2020),the exact solution of eqn.(55) is $t^3 + t^2$.
Taking modified Laplace transform (L_a) of eqn. (55) and make use of initial conditions, gives

$$y(slog_e a) = \frac{2}{s^3(loga)^3} + \frac{1}{s^3(loga)^3} L_a(-y - (y^2(t-1) - t^6 + 4t^5 - 6t^4 + 5t^3 - 6) \quad (56)$$

Taking inverse modified Laplace transform (L_a^{-1}) of eqn. (56), gives

$$y(t) = t^2 + L_a^{-1}(\frac{1}{s^3(loga)^3} L_a(-y - (y^2(t-1) - t^6 + 4t^5 - 6t^4 + 5t^3 - 6)) \quad (57)$$

Applying the combined modified Laplace variational scheme of eqn. (44) to eqn. (55), gives correction functional as

$$y_{n+1}(t) = y_n(t) + L_a^{-1}(\frac{1}{s^3(loga)^3} L_a(-y_n - y_n^2(t-1) - t^6 + 4t^5 - 6t^4 + 5t^3 - 6)) \quad (58)$$

Starting with initial approximation $y_0(t) = t^2$ and by using the iteration formula for eqn. (58), gives

$$y_1(t) = t^2 + t^3 + \frac{t^5}{60} + \frac{t^6}{24} + \frac{t^7}{30} - \frac{5t^8}{336} + \frac{t^9}{504}$$

$$y_2(t) = t^2 + t^3 + \frac{t^5}{60} + \frac{t^6}{30} + \frac{t^7}{30} - \frac{179t^8}{20160} + \frac{t^9}{12096} - \frac{7t^{10}}{5400} - \frac{t^{11}}{10080} + \frac{89t^{12}}{3326400} + \frac{1357t^{13}}{43243200} - \frac{811t^{14}}{27518400} + \frac{3427t^{15}}{27518400} - \frac{113t^{16}}{37632000} + \frac{1721t^{17}}{1233792000} - \frac{1081t^{18}}{1480550400} + \frac{853t^{19}}{3281886720} - \frac{233t^{20}}{3861043200} + \frac{t^{21}}{126690480} - \frac{t^{22}}{2347107840}$$

$$y_3(t) = t^2 + t^3 + \frac{t^5}{60} + \frac{t^6}{30} + \frac{t^7}{30} - \frac{179t^8}{20160} + \frac{t^9}{15120} - \frac{7t^{10}}{5400} - \frac{1523t^{11}}{19958400} + \frac{2021t^{12}}{79833600} + \frac{557t^{13}}{19958400} - \frac{257t^{14}}{8255520} + \frac{6847t^{15}}{1009008000} - \frac{43872716t^{16}}{290594304000} + \frac{1242403t^{17}}{741015475200} - \frac{588233t^{18}}{1387404016000} + \frac{201282041t^{19}}{1689515283456000} - \frac{601920721t^{20}}{14481559572480000} + \frac{6591949823t^{21}}{354798209526670000} - \frac{62977663673t^{22}}{1561112121913344000} + ... \quad (59)$$

Table 3: Hows comparison between approximate solution, exact solution and absolute error of CMLVIM for problem 3

t	MLVIM solution	Exact solution	AEMLVIM
0.1	0.011000137	0.011000000	0.000000137
0.2	0.048003604	0.048000000	0.000003604
0.3	0.117022898	0.117000000	0.000022898
0.4	0.224082772	0.224000000	0.000082772
0.5	0.375224309	0.375000000	0.000224309
0.6	0.576516016	0.576000000	0.000516016
0.7	0.834072467	0.833000000	0.001072467
0.8	1.154071576	1.152000000	0.002071576
0.9	1.542754833	1.539000000	0.003754833
1.0	2.006377669	2.000000000	0.006377669

MLVIM:Modified Laplace Variational Iteration Method.
AEMLVIM: Absolute Error Modified Laplace Variational Iteration Method.
AEADM: Absolute Error Adomian Decomposition Method.

5. DISCUSSION

Modified Laplace transform of nth order derivative $(f^n(t))$ and variable coefficient function $t^n f(t)$ has been established in theorem 3 and theorems 1 and 2 and this result was used alongside the scheme of VIM to form a new modified scheme capable of solving nonlinear differential equations with variable coefficients. The derived scheme was applied to solve three different problems and results are shown in tables 1 , 2 and 3. It was observed from table 1 that the maximum and minimum errors obtained from the proposed method were 2.25×10^{-3} and 2.63×10^{-5} respectively. These errors were achieved by finding the absolute difference between the computed values and exact values at each points. However, the maximum and minimum errors reported by [14] where Adomian Decomposition Method (ADM) was used are 1.56×10^{-2} and 7.29×10^{-4} respectively. This showed that the proposed scheme gave a smaller error compared to the results of ADM; Table 2 and 3 also justify the effectiveness of the methods as very small errors were obtained when compared with the exact solution.

6. CONCLUSION

A new scheme for solving nonlinear differential equations with variable coefficient has been formulated by coupling modified Laplace transform with VIM. Theorems associated with this scheme were also established. The modified Laplace transform

Lagrange multiplier for this method was also identified and effectively introduced into the derived scheme. Computational results showed that the method is effective and reliable. The method was used without linearization of functions, discretization of solution and unrealistic assumptions.

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