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The Sum of Three Cubes Problem and the Distribution of the Primes

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Abstract

Variants of the sum of three cubes problem ($a^3 + b^3 + c^3 = K$) are introduced and a relationship with the primes is investigated. Two modulo classes involving the factorization of $b^3 + c^3$ are identified. Cubic curves associated with the problem are generated for every K value.

1. INTRODUCTION

Mordell [1] popularized the sum of cubes problem.
The variant is

$$a^3 + (b^3 + c^3)/(b + c) = K \tag{1}$$

where a, b, and c are relatively prime in pairs. Primes of the form $6k + 1$ can be expressed as $(b^3 + c^3)/(b + c)$ for some b and c.
A plot of $(b^3 + c^3)/(b + c)$ versus a for solutions when K=2, 3, 4, 5, 7, 10, 11, 12, 14, 15, 18, 23, 24, 25, 27, 28, 30, 32, 34, 37, 38, 41, 42, 45, 48, and 49 (each one less than usually defined) is

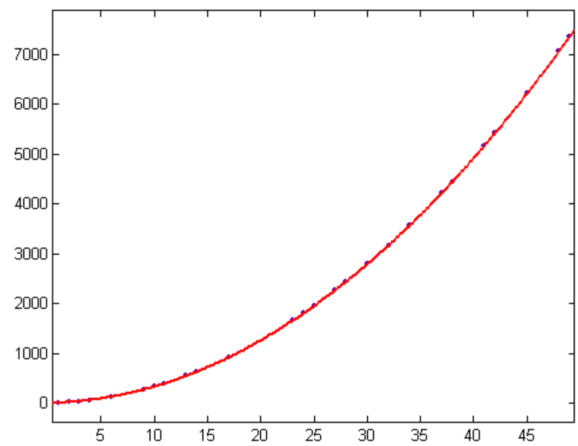

Figure 1:

The a values are one less than the K values. For a quadratic least-squares fit of the curve, $p_1 = 3$, $p_2 = 3$ and $p_3 = 1$ (the fit is perfect). The values of $(b^3 + c^3)/(b + c)$ (primes of the form $6k + 1$) are 7, 19, 37, 61, 127, 271, 331, 397, 547, 631, 919, 1657, 1801, 1951, 2269, 2437, 2791, 3169, 3571, 4219, 4447, 5167, 5419, 6211, 7057, and 7351.

Apparently this is a condition for there to be solutions of $x^3 + y^3 + z^3 = K$.

Let p be an odd prime. Every prime factor of $(a^p + b^p)/(a + b)$ other than p is of the form $pk + 1$. p (and no higher power of p) divides $(a^p + b^p)/(a + b)$ if and only if p divides $a + b$. If $a^p + b^p = c^p$ where p divides $a + b$, then p^{pk-1} divides c. For $p = 3$, a possibility is that $k = 1$ and p^2 divides c.

Primes of the form $6k - 1$ can be expressed as $(b^3 - c^3)/(b - c)$ for some b and c where 3 divides $b - c$. A plot of $(b^3 - c^3)/(b - c)/3$ versus a for solutions when K=3, 6, 9, 18, 24, and 30 (each one less than usually defined) is

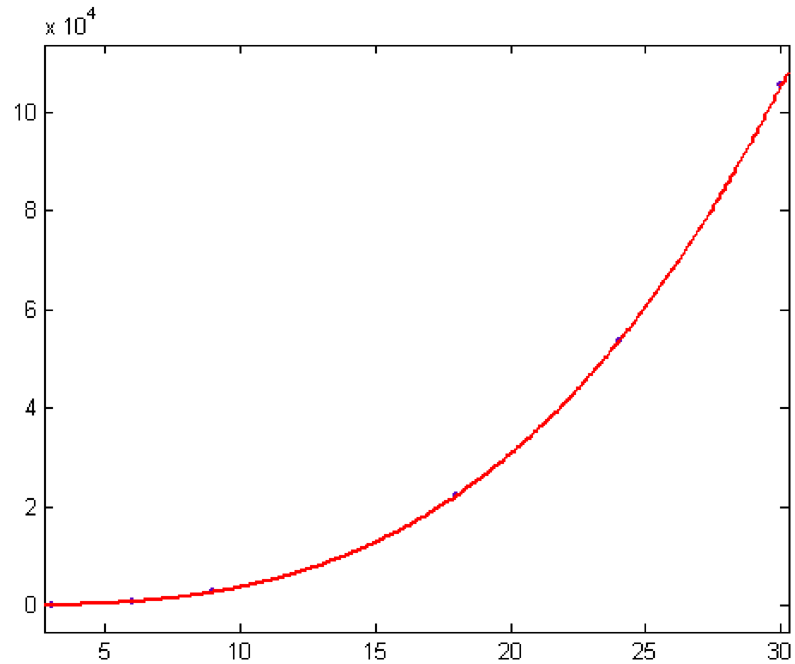

Figure 2:

The a values are one less than the K values. For a cubic least-squares fit of the curve, $p_1 = 4$, $p_2 = -3$, $p_3 = 3$, and $p_4 = -1$ (the fit is perfect). The $(b^3 - c^3)/(b - c)/3$ values are 89, 773, 2699, 22409, 53639, and 105389.

A plot of $(b^3 - c^3)/(b-c)/3$ versus a for $K = 2$ is

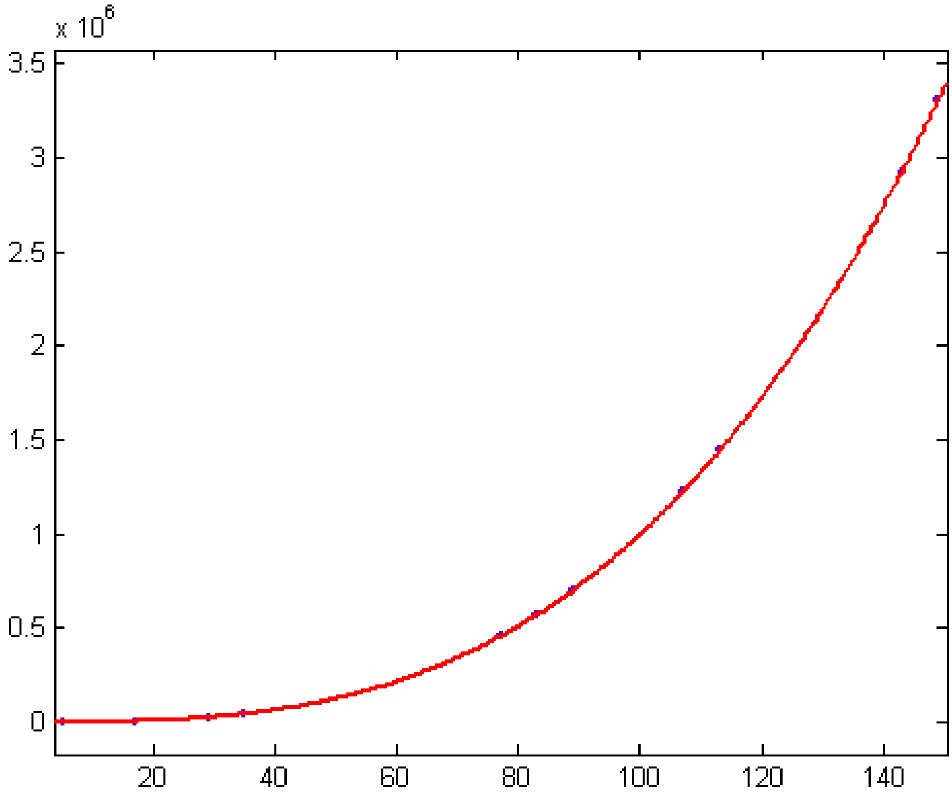

Figure 3:

The a values are 5, 17, 29, 35, 77, 83, 89, 107, 113, 143, and 149. For a cubic least-squares fit of the curve, $p_1 = 1$, $p_2 = 0$, $p_3 = 0$, and $p_4 = 24$. The $(b^3 - c^3)/(b-c)/3$ values (primes of the form $6k-1$) are 149, 4937, 24413, 42899, 456557, 571811, 704993, 1225067, 1442921, 2924231, and 3307973. In general, the p_4 parameters are $2^K \cdot 3$.

2. OTHER VARIANTS

There is a variant for $p = 2$ where the curve is linear. Primes of the form $4k+1$ can be expressed as $(b^2 - c^2)/(b-c)$ for some b and c. A plot of $(b^2 - c^2)/(b-c)$ versus a for solutions when K=3, 7, 9, 15, 19, 21, 27, 31, 37, 45, and 49 (for $K < 50$) is

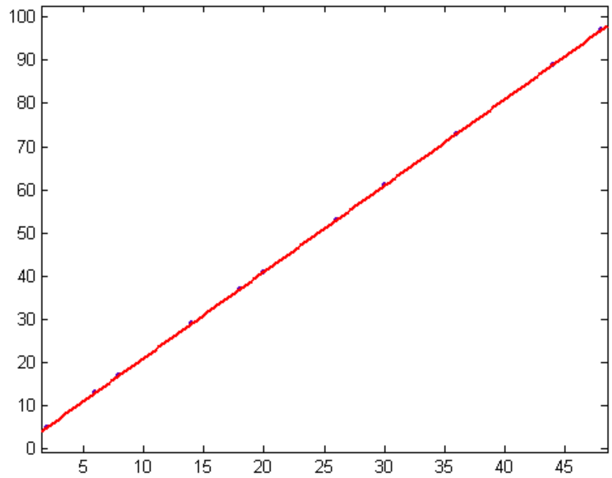

Figure 4:

The a values are one less than the K values. For a linear least-squares fit of the curve, $p_1 = 2$ and p_2 =1 (the fit is perfect). The $(b^2 - c^2)/(b - c)$ values are 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, and 97.

The primes of the form $4k - 1$ have a similar curve with the same linear least-squares fit. The K values are 2, 4, 6, 10, 12, 16, 22, 24, 30, 34, 36, 40, and 42. The a values are one less. The primes of the form $4k - 1$ are 3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 73, 89, and 97. Since the curves have the same linear least-squares fit, the a values can be combined and the $(b^2 - c^2)/(b - c)$ values can be combined. A plot of the curve is

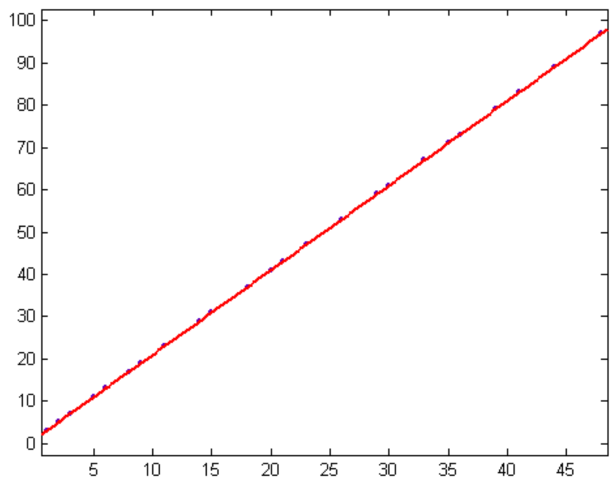

Figure 5:

The linear least-squares fit is perfect. These account for all the primes except 2.

There is a variant for $p = 5$ where the curve is cubic. Primes of the form $10k + 1$ can be expressed as $(b^5 + c^5)/(b + c)$ for some b and c. A plot of $(b^5 + c^5)/(b + c)$ versus a for solutions when K=2, 3, 6, 11, 12, 17, 20, 25, 28, and 31 is

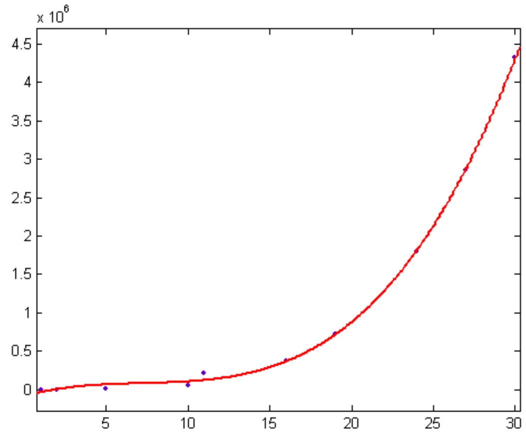

Figure 6:

The a values are one less than the K values. Unlike $p = 3$, the cubic least-squares fit is not perfect. The values of $(b^5 + c^5)/(b + c)$ are 31, 211, 4651, 61051, 210241, 371281, 723901, 1803001, 2861461, and 4329151.

There is a variant for $p = 7$ where the curve is quartic. Primes of the form $14k + 1$ can be expressed as $(b^7 + c^7)/(b + c)$ for some b and c. A plot of $(b^7 + c^7)/(b + c)$ versus a for solutions when K=2, 4, 7, 8, and 9 is

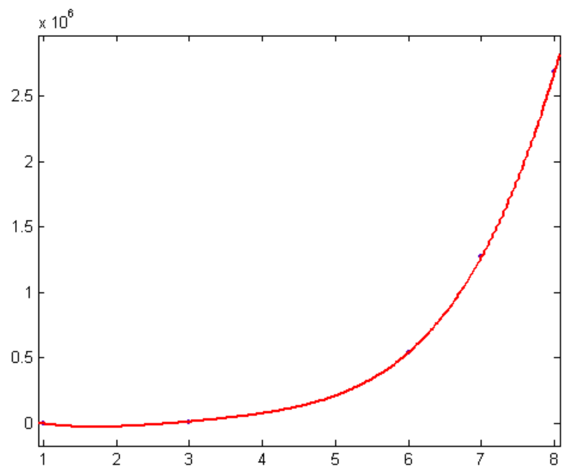

Figure 7:

The a values are one less than the K values. The values of $(b^7+c^7)/(b+c)$ are 127, 14197, 543607, 1273609, and 2685817.

There are similar curves for every p value.

3. A MORE GENERAL VARIANT

The case where the primes are of the form $6k+1$ will be considered first. a and b are not required to be relatively prime. The variant is

$$a^3 + d(b^3+c^3)/(b+c) = K. \quad (2)$$

The case where $d = b + c$ corresponds to the original sum of three cubes problem. An empirically derived result is

$$(a+d) \equiv K-1 (\mod 6) \quad (3)$$

For the original sum of three cubes problem, this gives $a+b+c \equiv K-1 (\mod 6)$.

Usually, $\gcd(a,b) = 2^i 3^j, i \geq 0, j \geq 0$. A plot where $d = K-1$ for K less than 30 is

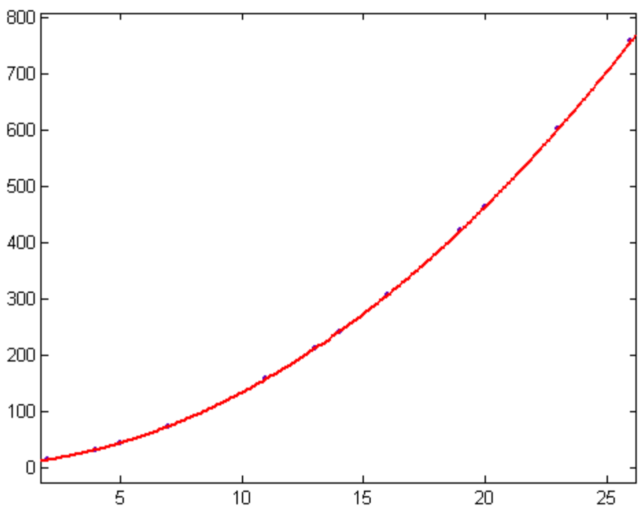

Figure 8:

The a values are 2, 4, 5, 7, 11, 13, 14, 16, 19, 20, 23, and 26. The prime values are 13, 31, 43, 73, 157, 211, 241, 307, 421, 463, 601, and 757. For a quadratic least-squares fit of the curve, $p_1 = 1$, $p_2 = 3$, and $p_3 = 3$ (the fit is perfect).

For a quadratic least-squares fit of the curve where $d = K - 1$, $a = 9$, 18, 36, 54, and 81 and the prime values equal 7, 19, 61, 127, and 271, $p_1 = \frac{1}{3^3}$, $p_2 = \frac{1}{3}$, and $p_3 = 1$.

For a quadratic least-squares fit of the curve where $d = K - 2$, $a = 8$, 16, 56, 88, and 104 and the prime values equal 7, 13, 73, 157, and 211, $p_1 = \frac{1}{2^6}$, $p_2 = \frac{3}{2^3}$, and $p_3 = 3$.

For a quadratic least-squares fit of the curve where $d = K - 2$, a=72, 144, 216, and 288 and the prime values equal 7, 19, 37, and 61, $p_1 = \frac{1}{2^6 . 3^3}$, $p_2 = \frac{1}{2^3 . 3}$ and $p_3 = 1$.

For a quadratic least-squares fit of the curve where $d = K - 3$, a=5, 8, 14, and 20 and the prime values equal 97, 163, 349, and 607, $p_1 = 1$, $p_2 = 3^2$, and $p_3 = 3^3$

For a quadratic least-squares fit of the curve where $d = K - 4$, $a = 8$, 40, 56, and 88 and the prime values equal 19, 67, 103, and 199, $p_1 = \frac{1}{2^6}$, $p_2 = \frac{3}{2^2}$ and $p_3 = 2^2 \cdot 3$.

For a quadratic least-squares fit of the curve when $d = K - 5$, $a = 9$, 18, 27, 54, and 72 and the prime values equal 43, 67, 97, 223, and 337, $p_1 = \frac{1}{3^3}$, $p_2 = \frac{5}{3}$, and $p_3 = 5^2$. Note that the a values are divisible by 9.

For a quadratic least-squares fit of the curve when $d = K - 5$, $a = 125$, 250, and 500 and the prime values equal 7, 13, and 31, $p_1 = \frac{1}{5^6}$, $p_2 = \frac{3}{5^3}$, and $p_3 = 3$. The b values are 5, 5, and 5.

For a quadratic least-squares fit of the curve where $d = K - 6$, $a = 8$, 32, and 56 and the prime values equal 37, 79, and 139, $p_1 = \frac{1}{2^6}$, $p_2 = \frac{3^2}{2^3}$, and $p_3 = 3^3$.

For a quadratic least-squares fit of the curve where $d = K - 7$, $a = 9$, 18, and 27 and the prime values equal 73, 103, and 139, $p_1 = \frac{1}{3^3}$, $p_2 = \frac{7}{3}$, and $p_3 = 7^2$.

For a quadratic least-squares fit of the curve where $d = K - 10$, $K - 30$, $K - 130$, and $K - 350$, $a = 8$, 27, 125, and 343, and the prime values equal 61, 271, 1951, and 7351, $p_1 = 0.02541$, $p_2 = 12.91$, and $p_3 = -66.98$ (the fit is not perfect). The a values are the cubes of the primes and the values subtracted from K are the primes cubed plus the primes. The b values are 8, 27, 125, and 343.

4. CUBIC CURVES

Corresponding cubic curves for primes of the form $6k-1$ less than 1000000 are as follows.

A curve for $d = 2$ is

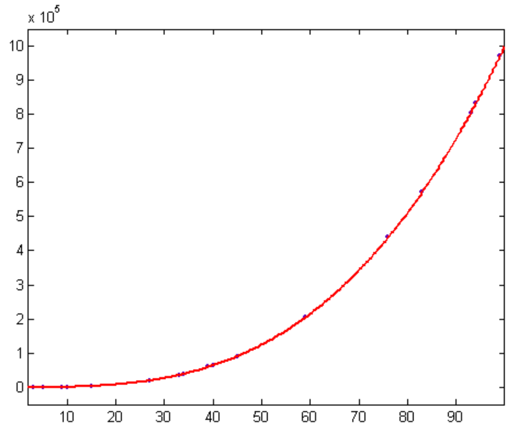

Figure 9:

The a value is 2 and the b values are 3, 5, 9, 10, 15, 27, 33, 34, 39, 40, 45, 59, 76, 83, 93, 94, and 99. The prime values are 41, 137, 743, 1013, 3389, 19697, 35931, 39317, 59333, 64013, 91139, 205391, 438989, 571799, 804371, 830597, and 970313. For a cubic least-squares fit of the curve, $p_1 = 1.0$, $p_2 = 0.0064$, $p_3 = 0.3141$, and $p_4 = 15.09$.

Another curve for $d = 2$ is

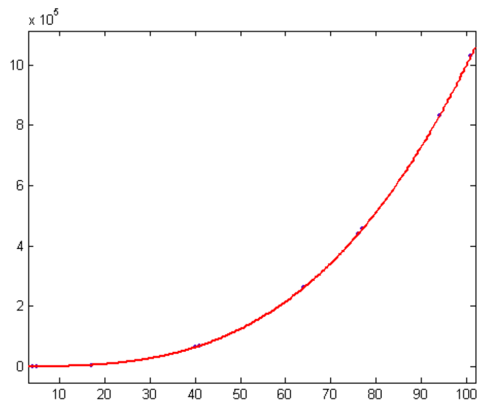

Figure 10:

The a value is 4 and the b values are 4, 5, 17, 40, 41, 64, 76, 77, 94, and 101. The prime values are 71, 131, 4919, 64007, 68927, 262151, 438983, 456539, 830591, and 1030307. For a cubic least-squares fit of the curve, $p_1 = 1.0$, $p_2 = 0.001107$, $p_3 = -0.03722$, and $p_4 = 6.601$.

A curve for $d = 3$ is

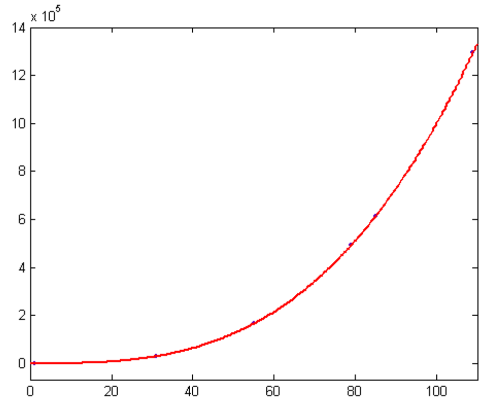

Figure 11:

The a value is 3 and the b values are 1, 31, 55, 79, 85, and 109. The prime values are 29, 29819, 106403, 493067, 614153, and 1295057. For a cubic least-squares fit of the curve, $p_1 = 1$, $p_2 = 0$, $p_3 = 0$, and $p_4 = 28$.

Another curve for $d = 3$ is

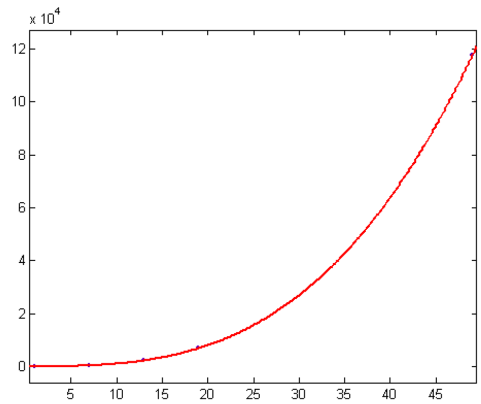

Figure 12:

The a value is 9 and the b values are 1, 7, 19, and 49. The prime values are 11, 353, 2207, 6869, and 117659. For a cubic least-squares fit of the curve $p_1 = 1$, $p_2 = 0$, $p_3 = 0$, and $p_4 = 10$.

A curve for $d = 4$ is

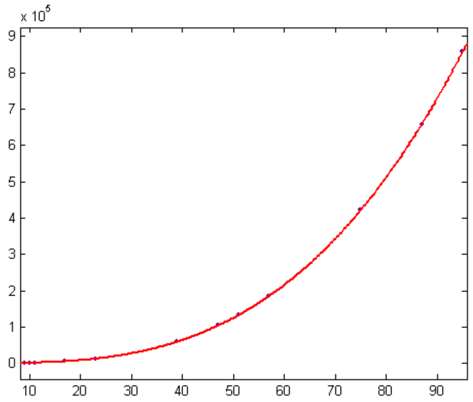

Figure 13:

The a value is 2 and the b values are 9, 10, 11, 17, 23, 39, 47, 51, 57, 75, 87, and 95. The primes are 827, 1097, 1427, 5009, 12263, 59417, 103919, 132749, 185291, 421973, 658601, and 857471. For a cubic least-squares fit of the curve, $p_1 = 1$, $p_2 = 0.006253$, $p_3 = -0.2272$, and $p_4 = 98.64$.

A curve for $d = 9$ is

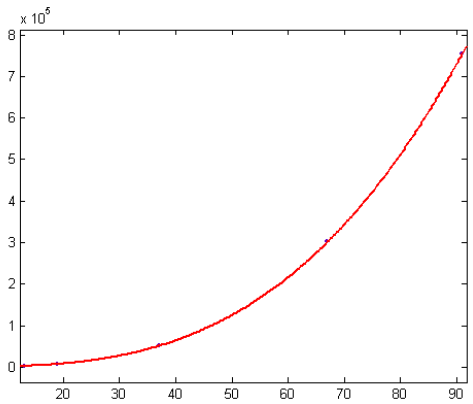

Figure 14:

The a value is 3 and the b values are 13, 19, 37, 67, and 91. The primes are 2927, 7589, 51383, 301493, and 754301. For a cubic least-squares fit of the curve, $p_1 = 1$, $p_2 = 0$,

$p_3 = 0$, and $p_4 = 730$.

A curve for $d = 27$ is

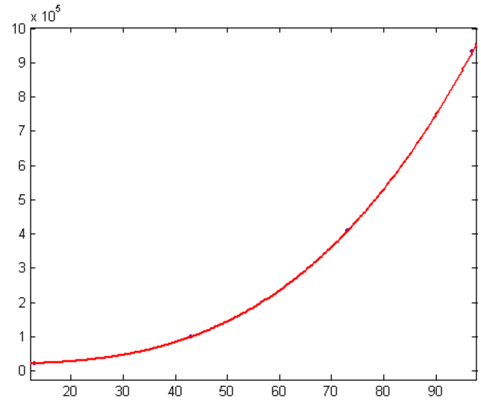

Figure 15:

The a value is 3 and the b values are 13, 43, 73, and 97. The primes are 21881, 99191, 408701, and 932357. For a cubic least-squares fit of the curve, $p_1 = 1$, $p_2 = 0$, $p_3 = 0$, and $p_4 = 19680$.

Similar cubic curves occur for other K values. There appear to be perfect cubic least-squares fits when d is a power of 3. For $d = 81$ and primes of the form $6k - 1$ less than 4500000, the p_4 parameter is 531400. A plot of the logarithms of the p_4 values (28, 730, 19680, and 531400) is

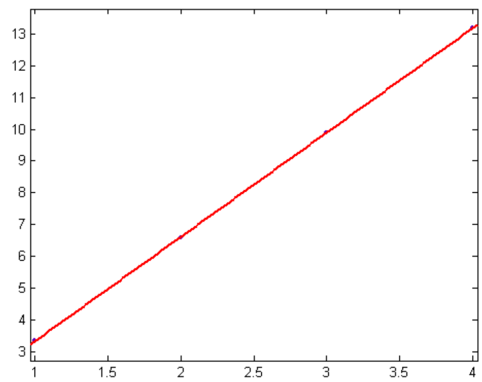

Figure 16:

For a linear least-squares fit of the curve, $p_1 = 3.29$ with a 95% confidence interval of (3.254, 3.327), $p_2 = 0.02786$ with a 95% confidence interval of $(-0.0726, 0.1282)$, SSE=0.0007254, R-squared=1, and RMSE=0.01904.

5. METHODS

The attached C programs generate the cubic curves.

```
#include <math.h>
#include <stdio.h>
#include "util1.h" // prime look-up table (of the form 6k-1)
unsigned int d=2;
unsigned int del=1; // usually set to 1
unsigned int tsize=57142; // size of prime look-up table
void main() {
unsigned int k,l,m,n,a,c,sum,count;
double x,delta,mindel;
FILE *Outfp;
Outfp = fopen("find1n.dat","w");
count=0;
mindel=9999999.0;
for (l=1; l<=2000; l++) {
    for (m=1; m<=2000; m++) {
        c=m*m*m;
        for (n=1; n<=tsize; n++) {
            a=table[n-1];
            sum=(a-c)/3;
            sum=sum*(del+l);
            x=pow((double)sum,1.0/3.0);
            delta=(double)d-x;
            if (delta<0.0)
                delta=-delta;
            if (delta<mindel) {
                mindel=delta;
                if (mindel<0.000001) {
                    count=count+1;
                    printf("l=%d m=%d a=%d \n",l+del,m,a);
                    fprintf(Outfp," %d %d %d \n",l+del,m,a);
                    mindel=999999.0;
                    }
                }
```

```
            }
        }
    }
printf("count=%d\n",count);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
#include "util.h" // prime look-up table (of form 6k+1)
unsigned int d=3;
unsigned int del=1; // usually set to 1
//unsigned int tsize=157901; // size of prime look-up table
//unsigned int tsize=74413; // primes less than 2000000
unsigned int tsize=39232; // primes less than 1000000
void main() {
unsigned int l,m,n,a,c,sum,count;
double x,delta,mindel;
FILE *Outfp;
Outfp = fopen("find1.dat","w");
count=0;
mindel=9999999.0;
for (l=1; l<=1000; l++) {
    for (m=1; m<=1000; m++) {
        c=m*m*m;
        for (n=1; n<=tsize; n++) {
            a=table[n-1];
            sum=c+a*(del+l);
            x=pow((double)sum,1.0/3.0);
            delta=(double)d-x;
            if (delta<0.0)
                delta=-delta;
            if (delta<mindel) {
                mindel=delta;
                if (mindel<0.000001) {
                    count=count+1;
                    printf("l=%d m=%d a=%d \n",l+del,m,a);
                    fprintf(Outfp," %d, %d, %d, \n",l+del,m,a);
```

```
                    mindel=999999.0;
                    }
                }
            }
        }
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] Mordell L.J. (1942) ”On sums of three cubes”,*Journal of the London Mathematical Society*, Second Series **17** (3), 139-144