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The Derivatives of Riemann's Epsilon Function and an Integral Involving a Theta Function

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Abstract

The derivatives of the Riemann epsilon function and the integral involving the theta function are determined. Related functions involving the ratio of derivatives are "three-dimensional" for complex values other than those of the Riemann zeta function zeros. These functions have zeros for any complex value having a real part of 1/2.

Keywords: Riemann zeta function, Riemann's epsilon function, Riemann hypothesis

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [2] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.)

2. RIEMANN'S EPSILON FUNCTION AND THE FUNCTIONAL EQUATION OF THE ZETA FUNCTION

Equation (1) in section 1.8 of Edwards' book is

$$\epsilon(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s) \tag{2}$$

The equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3}$$

can be substituted into the equation above.

A variant of $\epsilon(s)$ is

$$\epsilon_1(s) = \frac{\Pi(\frac{s}{2})(n+1)^s(s-1)\zeta(s)^2}{2\pi} \tag{4}$$

A plot of $\epsilon_1(s)$ along with the related function $\frac{\Pi(\frac{s}{2})\Pi(s)(s-1)\zeta(s)^2}{2\pi}$ for the first zeta function zero and $n \leq 200$ is

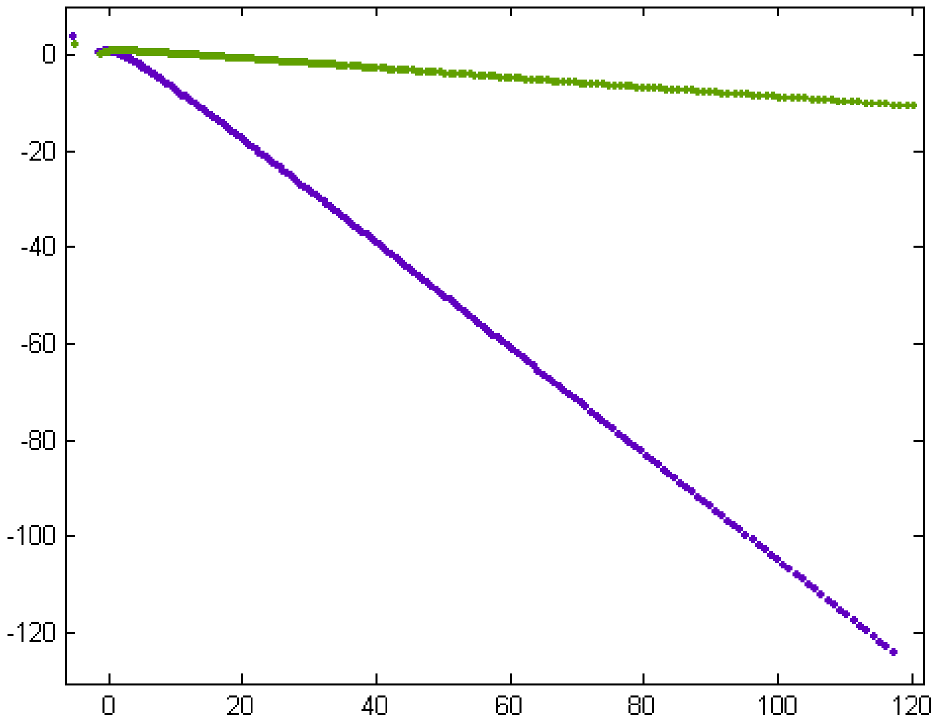

Figure 1:

$\epsilon_1(s)$ was selected because it has the more negative slope.

Riemann's functional equation (equation (5) in section 1.6 of Edwards' book) is

$$\Pi(\frac{s}{2}-1)\pi^{-s/2}\zeta(s) = \Pi(\frac{1-s}{2}-1)\pi^{-(1-s)/2}\zeta(1-s) \tag{5}$$

A plot of the left and right sides of the equation for the first non-trivial zeta function zero ($s = (0.5, 14.134725191735)$) and $n \leq 100000$ is

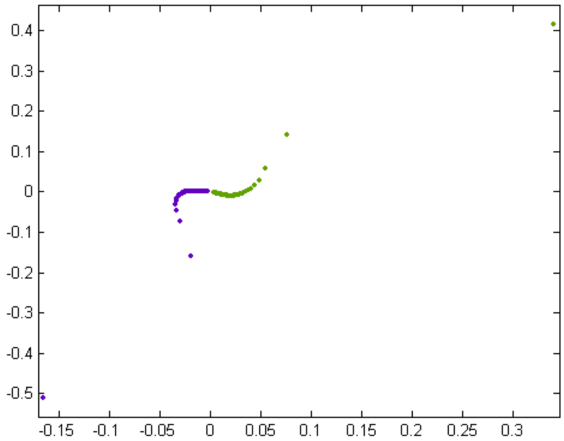

Figure 2:

Such curves are typical for zeta function zeros.

The function $\Pi((s/2)-1)\pi^{-s/2}\zeta(s)$, which occurs in the symmetrical form of the functional equation, has poles at $s=0$ and $s=1$. Riemann multiplies it by $s(s-1)/2$ and defines

$$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s). \tag{6}$$

Similarly, multiplying $\epsilon_2(s) = \Pi(-\frac{s}{2})\pi^{-s/2}\zeta(-\frac{s}{2})$ by $-s(-s-1)/2$ and adding π (to both the real and imaginary components) gives $\xi_1(s)$. A plot of $\xi(s)$ and $\xi_1(s)$ for the first zeta function zero and $n \le 100$ is

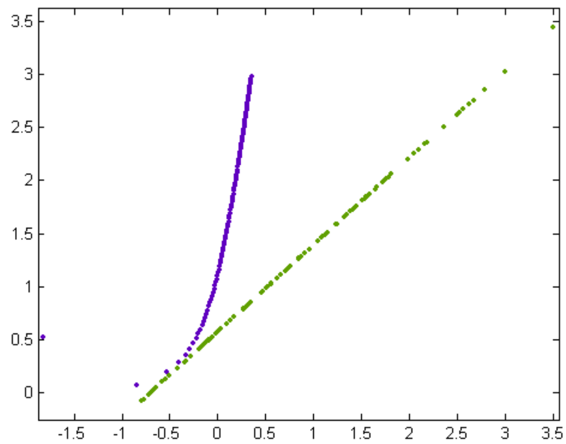

Figure 3:

$\xi_1(s)$ can be interpreted as being the derivative of $\xi(s)$.

It appears that $\frac{\xi_1(s)}{\epsilon_1(s)}$ for s values with imaginary components other than those of the zeros will eventually converge to zero (or close to it).

A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 13.5)$ and $100 \leq n \leq 1000000$ is

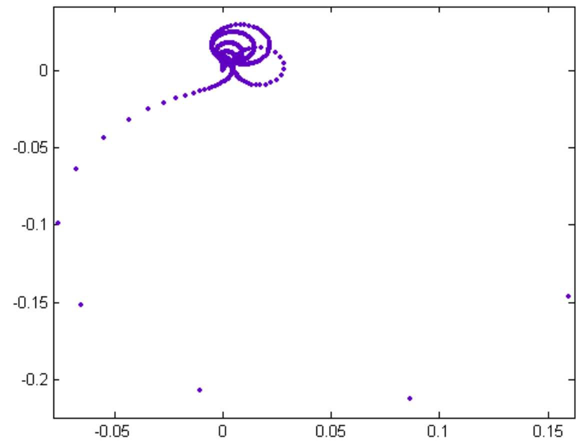

Figure 4:

The curve gets erratic for $n < 100$. A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 13.5)$ and $500 \leq n \leq 1000000$ is

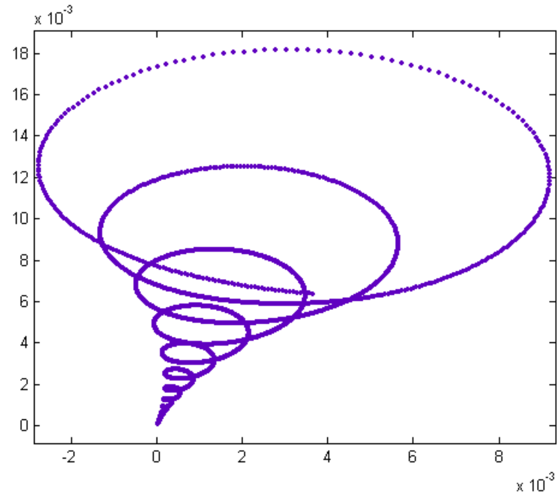

Figure 5:

A plot of the logarithms of the n values of the inflection points for $n \leq 100000$ is

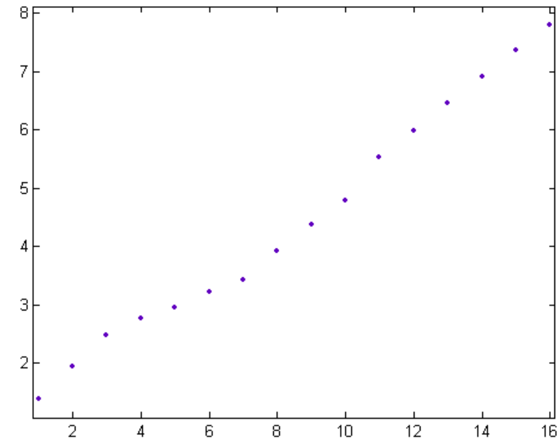

Figure 6:

For a linear least-squares fit of the curve (disregarding the first ten n values), $p_1 = 0.4548$ with a 95% confidence interval of (0.4485, 0,4611), $p_2 = 0.5289$ with a 95% confidence interval of (0.4434, 0.6144), SSE=0.0003587, R-squared=0.9999, and RMSE=0.00949. As expected, the slope is greater than that of the first Riemann zeta function zero (0.4458).
A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 51.3720769775)$ (the average of the tenth and eleventh Riemann zeta function zeros) and $n \leq 1000000$ is

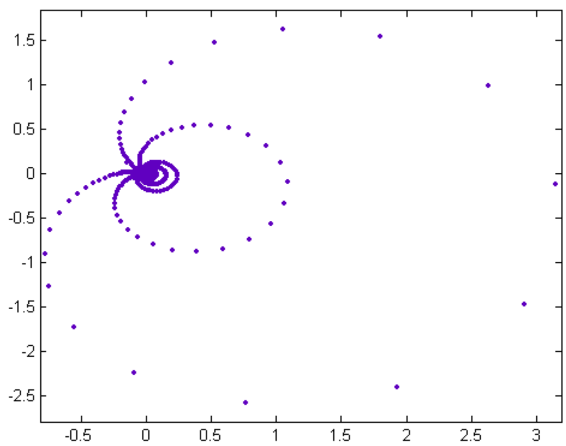

Figure 7:

A plot of the logarithms of the n values of the inflection points for $n \leq 1000000$ is

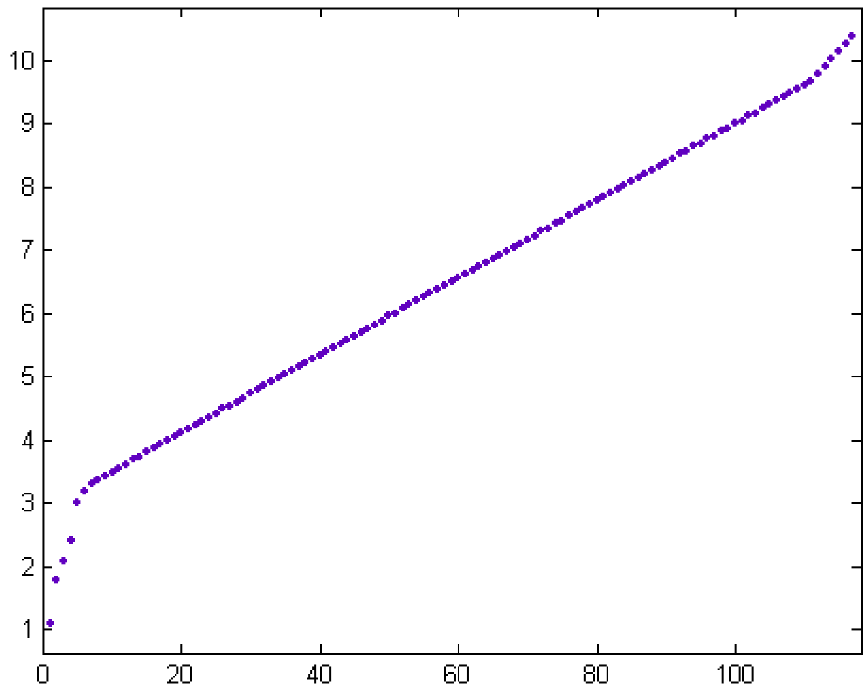

Figure 8:

For a linear least-squares fit of the curve (disregarding the first six n values and the last seven n values), $p_1 = 0.06113$ with a 95% confidence interval of (0.06109, 0.06117), $p_2 = 2.89$ with a 95% confidence interval of (2.887, 2.893), SSE=0.003931, R-squared=1, and RMSE=0.006208. The slope and intercept for the tenth zeta function zero is (0.1261, 1.943), so the above slope is about half of this. For a linear least-squares fit of the curve (disregarding the first 110 n values), $p_1 = 0.1205$ with a 95% confidence interval of (0.1202, 0.1209), $p_2 = -3.72$ with a 95% confidence interval of $(-3.756, -3.682)$, SSE=$2.383 \cdot 10^{-6}$, R-squared=1, and RMSE=0.0006904. The slope and intercept for the eleventh zeta function zero is (0.1186, 2.492), so the above slope is about equal to the average of the slopes for the tenth and eleventh zeta function zeros.

A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 144.556414146)$ (the average of the forty-ninth and fiftyth Riemann zeta function zeros) and $n \leq 1000000$ is

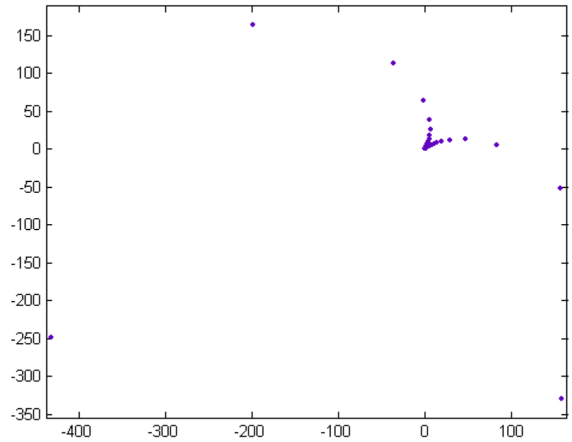

Figure 9:

A plot of the logarithms of the n values of the inflection points for $n \leq 1000000$ is

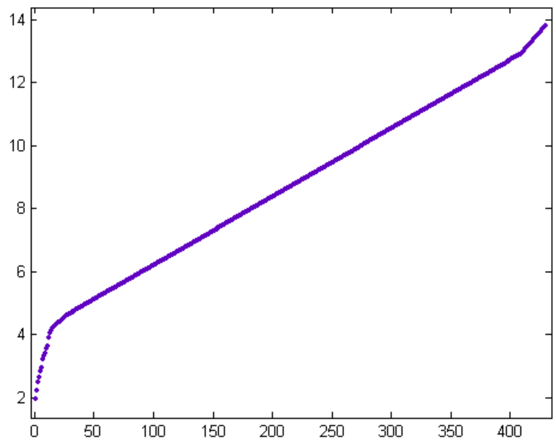

Figure 10:

For a linear least-squares fit of the curve (disregarding the first twenty-four n values and the last nineteenth n values), $p_1 = 0.02175$ with a 95% confidence interval of (0.02175, 0.02176), $p_2 = 4.008$ with a 95% confidence interval of (4.007, 4.008), SSE=0.000883, R-squared=1, and RMSE=0.001516. The slope and intercept for the forty-ninth zeta function zero is (0.0439, 3.469), so the above slope is about half of this. For a linear least-squares fit of the curve (disregarding the first 410 n values), $p_1 = 0.04354$ with a 95% confidence interval of (0.04354, 0.04355), $p_2 = -4.928$ with a 95% confidence interval of $(-4.929, -4.926)$, SSE=$2.02 \cdot 10^{-8}$, R-squared=1, and RMSE=$3.35 \cdot 10^{-5}$. The slope and intercept for the fiftyth zeta function zero is (0.04304, 3.399), so the above slope is about equal to the average of the slopes for the forty-ninth and fiftyth zeta function zeros.

A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 49.773832477672)$ (the tenth zeta function zero) and $n \leq 1000000$ is

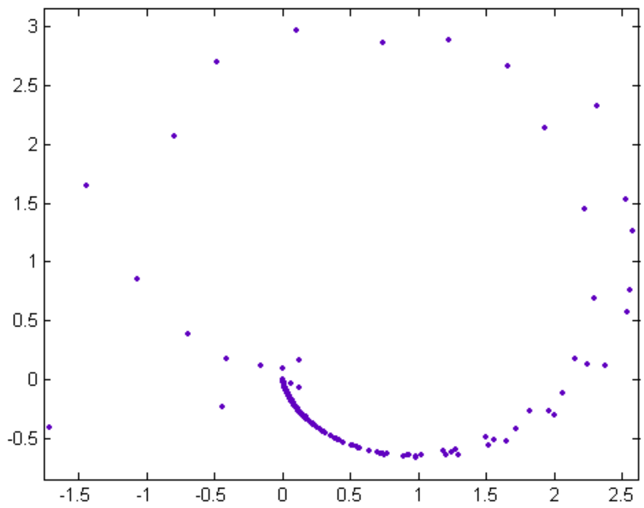

Figure 11:

This is a "two-dimensional" plot.

3. THE INTEGRAL OF A THETA FUNCTION

An equation given in Kargin's [3] article relating $\zeta(s)$ to the series $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$ is

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-n^2 \pi x} dx = \int_0^{\infty} x^{s/2-1} \theta(x) dx \tag{7}$$

A plot of $\frac{\Pi(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}}$ and $\Pi(\frac{s}{2})\zeta(s)$ for the first zeta function zero and $n \leq 100$ is

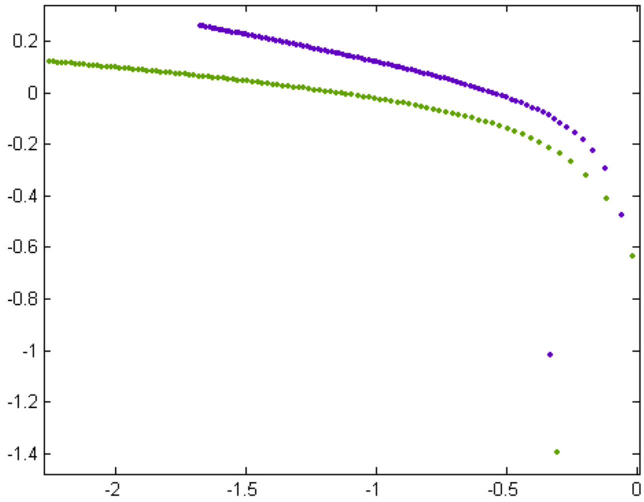

Figure 12:

The bottom curve is $\Pi(\frac{s}{2})\zeta(s)$. Its function is to straighten and rotate curved lines. A plot of the product of these two functions (denoted by $\nu(s)$) and $\xi_2(s) = \epsilon_2(s)$ is

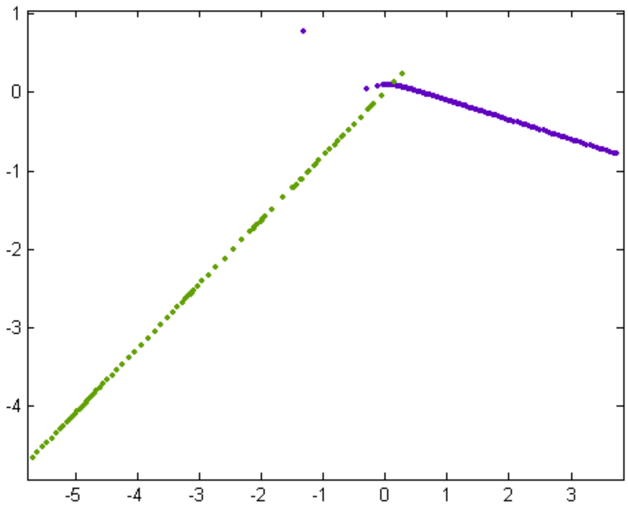

Figure 13:

A plot of $\nu(s)$ and $\xi_2(s)$ for the third zeta function zero ($s = (0.5, 25.01085758015)$) and $n \leq 100$ is

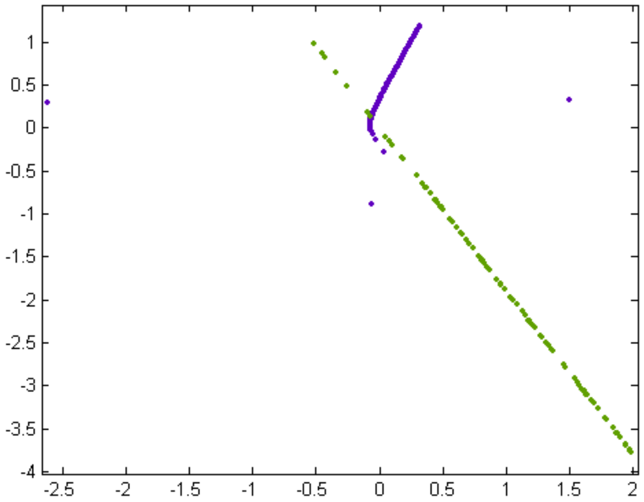

Figure 14:

$\xi_2(s)$ appears to be close to the derivative of $\nu(s)$.

A plot of $\frac{\xi_2(s)}{\epsilon_1(s)}$ for the first zeta function zero and $n \leq 10000$ is

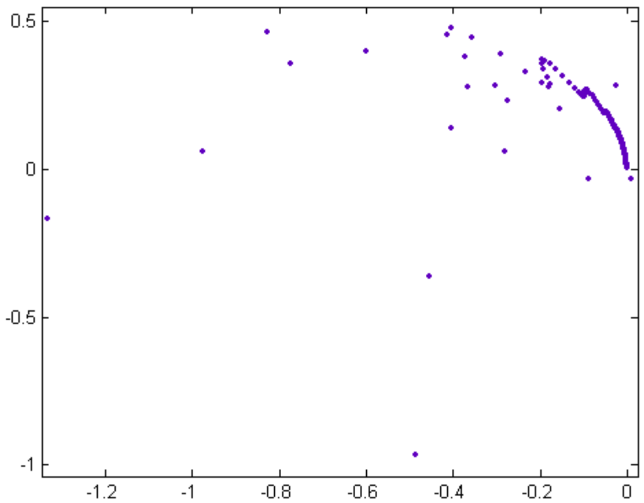

Figure 15:

A plot of $\frac{\xi_2(s)}{\epsilon_1(s)}$ for the third zeta function zero and $n \leq 10000$ is

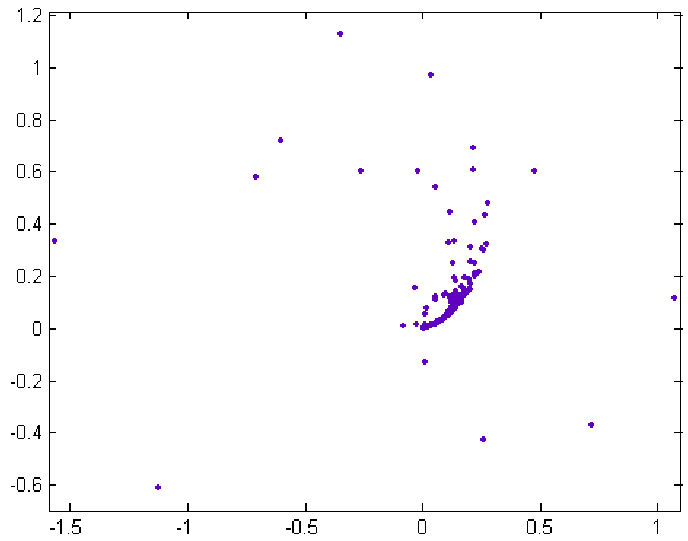

Figure 16:

These are two-dimensional plots.

A plot of $\frac{\xi_2(s)}{\epsilon_1(s)}$ for $s = (0.5, 13.5)$ and $100 \leq n \leq 100000$ is

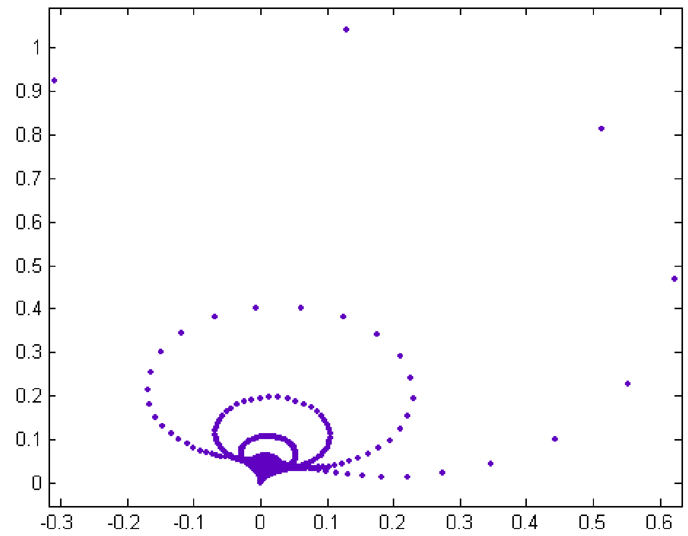

Figure 17:

A plot of the logarithms of the n values of the inflection points (for $n \leq 100000$) is

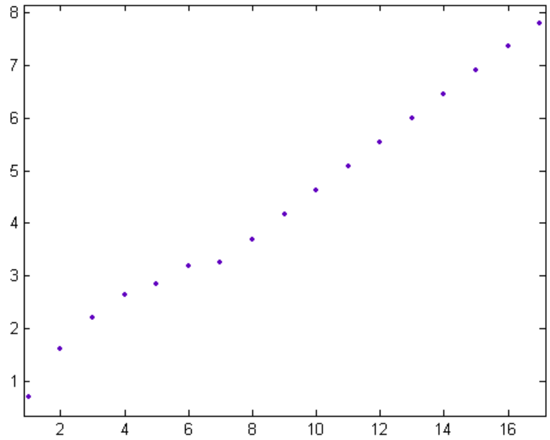

Figure 18:

For a linear least-squares fit of the curve (disregarding the first six n values), $p_1 = 0.4569$ with a 95% confidence interval of (0.4548, 0.4591), $p_2 = 0.05014$ with a 95% confidence interval of (0.02377, 0.7652), SSE=0.0008739, R-squared=1, and RMSE=0.009854. As expected, the slope is greater than that of the first Riemann zeta function zero.

A plot of $\frac{\xi_2(s)}{\epsilon_1(s)}$ for $s = (0.5, 51.3720769775)$ and $n \leq 1000000$ is

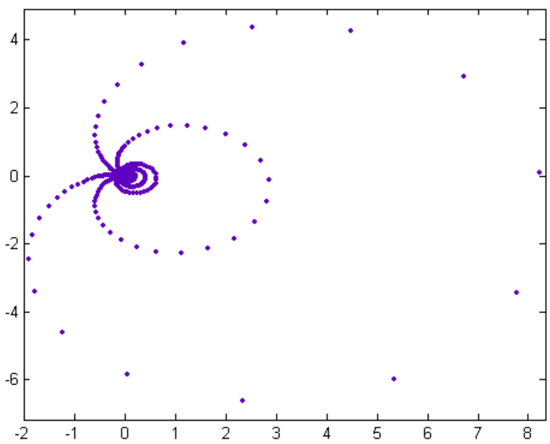

Figure 19:

A plot of the logarithms of the n values of the inflections points is

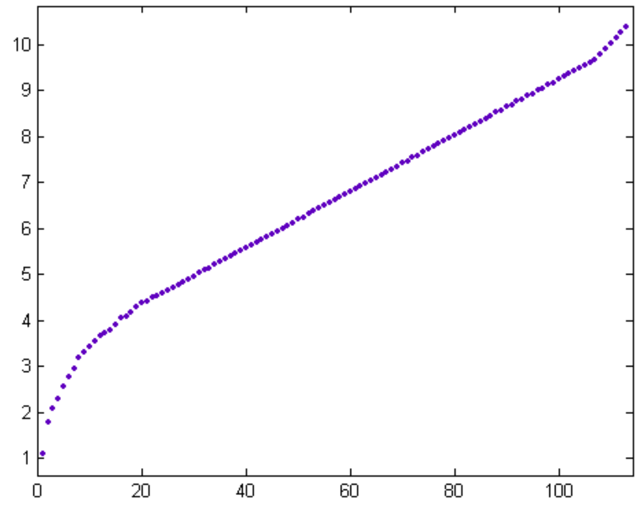

Figure 20:

The first twenty n values of the inflections points increase quadratically, not exponentially. A plot of these values is

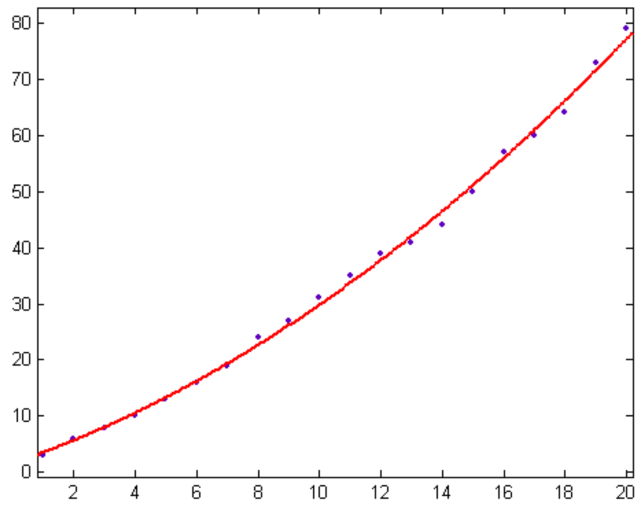

Figure 21:

For a quadratic least-squares fit of the curve, $p_1 = 0.09518$ with a 95% confidence interval of (0.07451, 0.1158), $p_2 = 1.883$ with a 95% confidence interval of (1.436, 2.33), $p_3 = 1.518$ with a 95% confidence interval of $(-0.5188, 3.556)$, SSE=28.64, R-squared=0.99972, and RMSE=1.298.

For a linear least-squares fit of the logarithms of the n values of the inflection points (excluding the first twenty and last seven n values), $p_1 = 0.06117$ with a 95% confidence interval of (0.06112, 0.06122), $p_2 = 3.128$ with a 95% confidence interval of (3.125, 3.132), SSE=0.003041, R-squared=1, and RMSE=0.006107. The slope is about half of that for the tenth zeta function zero.

For a linear least-squares fit of the logarithms of the last seven n values of the inflection points, $p_1 = 0.1205$ with a 95% confidence interval of (0.1201, 0.1209), $p_2 = -3.236$ with a 95% confidence interval of $(-3.278, -3.194)$, SSE=$3.058 \cdot 10^{-6}$, R-squared=1, and RMSE=0.000784. The slope is between that of the tenth and eleventh zeta function zeros.

A plot of $\frac{\xi_2(s)}{\epsilon_1(s)}$ for $s = (0.5, 144.556414146)$ and $n \leq 1000000$ is

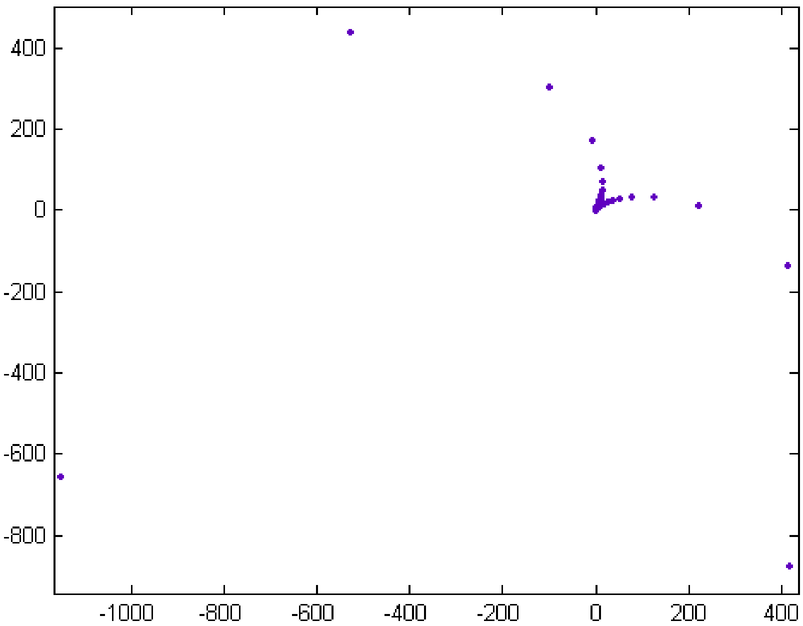

Figure 22:

A plot of the logarithms of the n values of the inflection points is

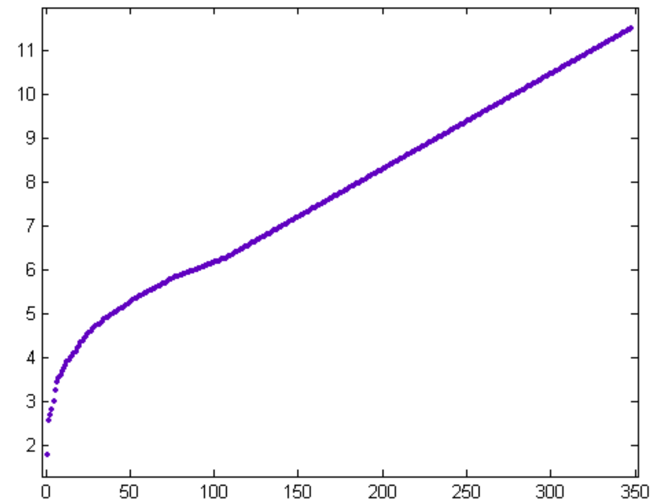

Figure 23:

The first 107 n values increase quadratically, not exponentially. A plot of the values is

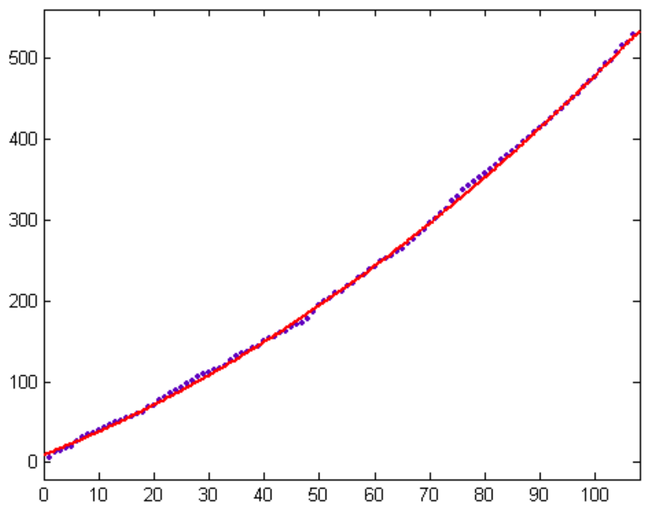

Figure 24:

For a quadratic least-squares fit of the curve, $p_1 = 0.01989$ with a 95% confidence interval of (0.01925, 0.02054), $p_2 = 2.706$ with a 95% confidence interval of (2.634, 2.778), $p_3 = 9.471$ with a 95% confidence interval of (7.782, 11.16), SSE=863.9, R-squared=0.9996, and RMSE=2.882.

For a linear least-squares fit of the logarithms of the n values of the inflection points (for the last 241 n values), $p_1 = 0.02174$ with a 95% confidence interval of (0.02173, 0.02174), $p_2 = 3.946$ with a 95% confidence interval of (3.946, 3.947), SSE=0.0001308, R-squared=1, and RMSE=0.0007398. The slope is about half of that for the forty-ninth zeta function zero.

The number of inflection points can be used to detect Riemann zeta function zeros. The number of inflections points for imaginary components of 15.0, 15.5, 16.0,...,31.5 are 21, 25, 27, 29, 31, 32, 32, 35, 37, 37, 38, 34, 11, 31, 37, 38, 39, 39, 38, 29, 5, 34, 48, 58, 65, 79, 82, 70, 63, 54, 37, 4, 38, and 44 respectively. A plot of the logarithms of these values versus the imaginary components is

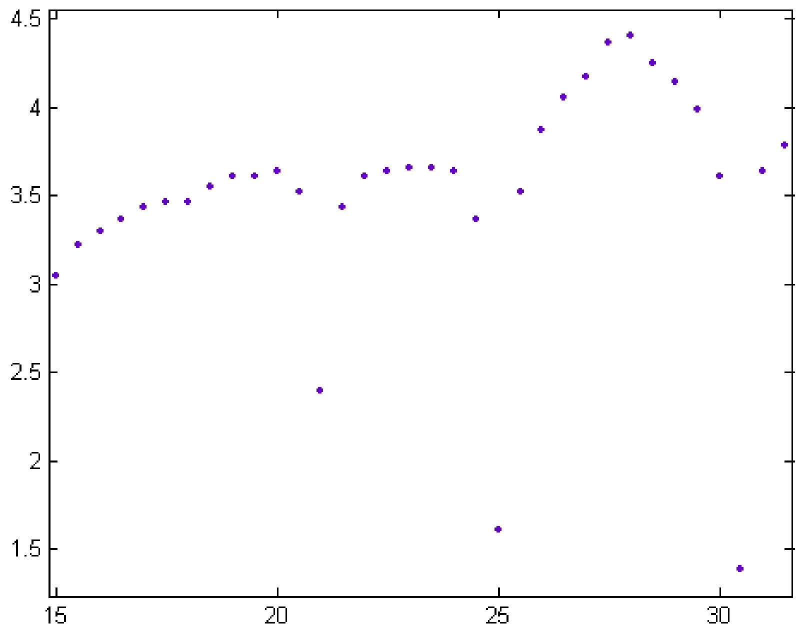

Figure 25:

There are then zeta function zeros close to 21.0 ($s = (0.5, 21.02203963877156)$), 25.0 ($s = (0.5, 25.01085758014569)$, and 30.5 ($s = (0.5, 30.42487612585951)$).

The derivative of the integral of the theta function has thus been transformed into a derivative resembling that of the epsilon function. There are probably other ways to do this since $\epsilon(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s)$ and the former function equals $\Pi(s/2)\pi^{-s/2}\zeta(s)$.

4. CONCLUSION

These are simple empirical derivations of three-dimensional functions where the Riemann Hypothesis appears to be true. The empirically derived functions were devised so that they have mostly straight lines for the Riemann zeta function zeros.

5. METHODS

The output of the first C program is divided by the output of the second C program to give the two and three-dimensional logarithmic spirals. Set "noadd" to 1 to use the derivative of the integral of the theta function. The prime look-up table contains the primes less than 1500000.

```
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
//
// Π(−s/2)π^{−s/2}ζ(−s/2)
//
unsigned int max=1000000;
double s=.50; // usually set to .50
double t=13.5;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
```

```
double pi=3.14159265359;
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int noadd=0; // if set, don't multiply by s(s-1)/2 and add pi
unsigned int tsize=114155; // size of prime look-up table
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
FILE *Outfp;
Outfp = fopen("transe7b.dat","w");
if (max>1500000) {
    printf("max too large \n");
    return;
    }
if (s>=0.0)
    temp1=pow(pi,s/2.0);
else {
    temp1=pow(pi,-s/2.0);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(pi/2.0)));
tempt=temp1*(sin(t*log(pi/2.0)));
a=temps*temps+tempt*tempt;
e=temps/a;
f=-tempt/a;
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    a=-s/2.0;
    if (a>=0.0)
        temp1=pow((double)(x+1),a);
    else {
        temp1=pow((double)(x+1),-a);
        temp1=1.0/temp1;
```

```
    }
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
c=-s/2.0;
if (c>=0.0)
    temp1=pow((double)x,c);
else {
    temp1=pow((double)x,-c);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;
temps=a*sums-b*sumt;
tempt=a*sumt+b*sums;
a=temps;
b=temps;
temps=a*e-b*f;
tempt=a*f+b*e;
if (noadd==0) {
    a=-s*(-s-1.0)/2.0;
    temps=temps*a;
    tempt=tempt*a;
    temps=temps+pi;
    tempt=tempt+pi;
    }
if (x>xmin) {
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
```

```
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
//
// Epsilon function variant
//
unsigned int max=1000000;
double s=0.50; // usually set to 0.50
double t=13.5;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int tsize=114155; // size of prime look-up table
double pi=3.14159265359;
```

```
double *rsave,*isave;
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
double sumr,sumi,I,R;
double newprod,newtemps,newtempt,newa,newb;
FILE *Outfp;
Outfp = fopen(”transe2.dat”,”w”);
rsave=(double*) malloc(16000004);
if (rsave==NULL)
    return;
isave=(double*) malloc(16000004);
if (isave==NULL)
    return;
if (max>1500000) {
    printf(”max too large \n”);
    return;
    }
//
```

// $\frac{\Pi(s/2)(n+1)^{s}\zeta(s)(s-1)}{2\pi}$

```
//
f=2.0*pi;
temp1=f*f;
e=0.0;
f=-f/temp1;
prods=1.0;
newprod=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s/2.0);
    else {
        temp1=pow((double)(x+1),-s/2.0);
        temp1=1.0/temp1;
        }
```

```
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
if (s>=0.0)
    newtemp1=pow((double)(x+1),s);
else {
    newtemp1=pow((double)(x+1),-s);
    newtemp1=1.0/newtemp1;
    }
newtemps=newtemp1*(cos(t*log(x+1)));
newtempt=newtemp1*(sin(t*log(x+1)));
newa=newprod*newtemps-newtempt;
newb=newprod*newtempt+newtemps;
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;
temps=a*sums-b*sumt;
tempt=a*sumt+b*sums;
c=temps*e-tempt*f;
d=temps*f+tempt*e;
temps=c*(s-1.0)-d*t;
tempt=c*t+d*(s-1.0);
a=temps*newa-tempt*newb;
b=temps*newb+tempt*newa;
temps=a;
tempt=b;
rsave[x-1]=temps;
```

```
    isave[x-1]=tempt;
    if (x==(x/1000)*1000)
    printf(" %d \n",x);
    }
//
// zeta(s)
//
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    temp1=pow((double)temp,s);
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    a=rsave[x-1];
    b=isave[x-1];
    temps=a*sumr-b*sumi;
    tempt=a*sumi+b*sumr;
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        olds=temps;
        oldt=tempt;
        }
    }
```

```
fclose(Outfp);
return;
}
```

REFERENCES

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[2] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)

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