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Derivatives of Riemann's Zeta Function and Epsilon Function

Darrell Cox[1]

[1] *Department of Mathematics, Grayson County College, United States*
Email– darkencox273@gmail.com

Abstract

The derivatives of Riemann's zeta function and epsilon function are used to define a three-dimensional function that contains the usual zeta function (two-dimensional). Every complex number having a real component of 1/2 then has a zero. Another way to compute the derivative of the Riemann zeta function is introduced.
Keywords: Riemann zeta function, derivative of Riemann zeta function, logarithmic spirals

1. INTRODUCTION

The derivative of the Riemann zeta function is

$$\zeta'(z) = \frac{d\zeta}{dz} = -\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}. \tag{1}$$

2. RIEMANN'S EPSILON FUNCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{2}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.)
Equation (1) in section 1.8 of Edwards' book is

$$\epsilon(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s) \tag{3}$$

The equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{4}$$

can be substituted into the equation above. $\zeta(s)$ generates logarithmic spirals.

A transformation of $\epsilon(s)$ is

$$\zeta'(s) = \frac{\Pi(s/2)(s-1)\pi^{-s/2}\zeta(s)\Pi(s-1)}{2\pi} \tag{5}$$

A plot of $\zeta'(s)$ for the first non-trivial zeta function zero ($s = (0.5, 14.1347251417)$) and $n \leq 1000000$ is

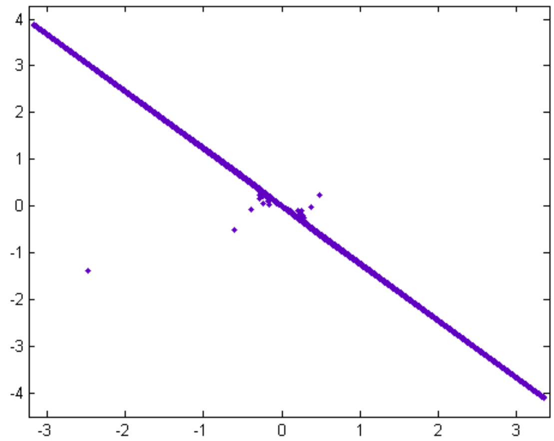

Figure 1:

A plot of the logarithms of the n values of the inflection points (where the curve decreases towards the x-axis and then increases) is

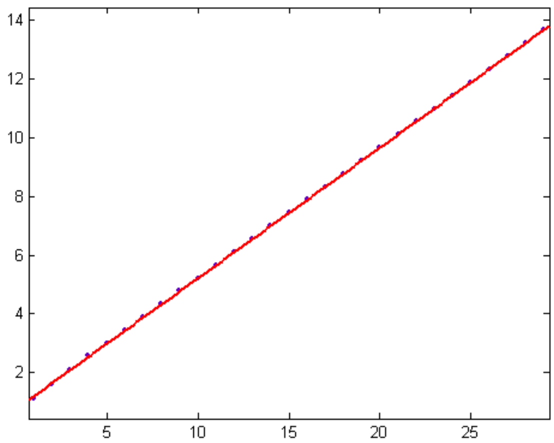

Figure 2:

For a linear least-squares fit of the curve, $p_1 = 0.446$ with a 95% confidence interval of (0.4449, 0.447), $p_2 = 0.7494$ with a 95% confidence interval of (0.7314, 0.7674), SSE=0.01432, R-squared=1.0, and RMSE=0.02303.
A plot of $\zeta(s)$ for the first zeta function zero is

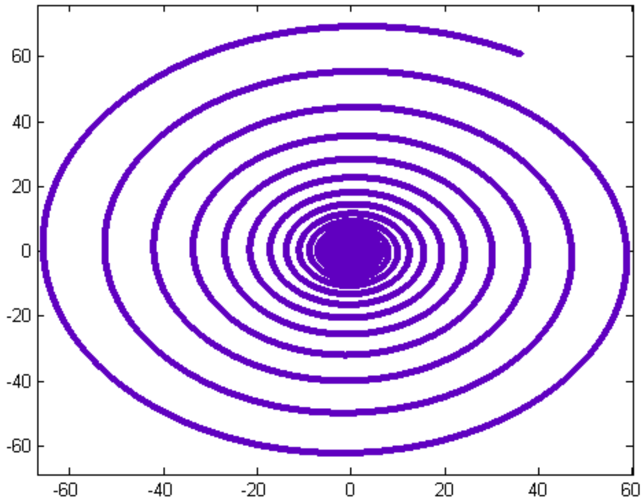

Figure 3:

A plot of the logarithms of n values of the inflection points is

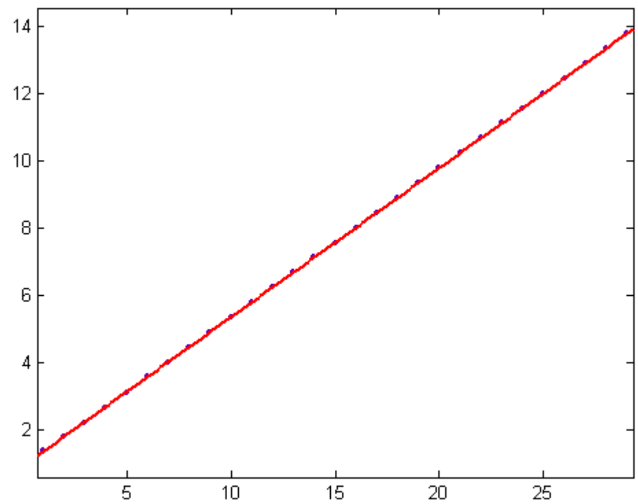

Figure 4:

For a linear least-squares fit of the curve, $p_1 = 0.4444$ with a 95% confidence interval of (0.4438, 0.445), $p_2 = 0.8882$ with a 95% confidence interval of (0.8777, 0.8988), SSE=0.004894, R-squared=1.0, and RMSE=0.01346. The above transformation just makes the logarithmic spirals edge-on.

The slopes and intercepts of the logarithms of the n values of the inflection points for the first twenty zeta function zeros are (0.4444, 0.8882), (0.2988, 1.496), (0.2512, 1.759), (0.2064, 1.862), (0.1909, 1.903), (0.1673, 2.167), (0.1535, 2.151), (0.1450, 2.176), (0.1309, 2.226), (0.1263, 2.397), (0.1186, 2.491), (0.1113, 2.559), (0.1059, 2.542), (0.1033, 2.582), (0.0965, 2.509), (0.9367, 2.716), (0.09033, 2.712), (0.08719, 2.789), (0.08301, 2.821), and (0.08145, 2.769). A plot of the slopes versus $\log(x), x = 1, 2, 3, ..., 20$ is

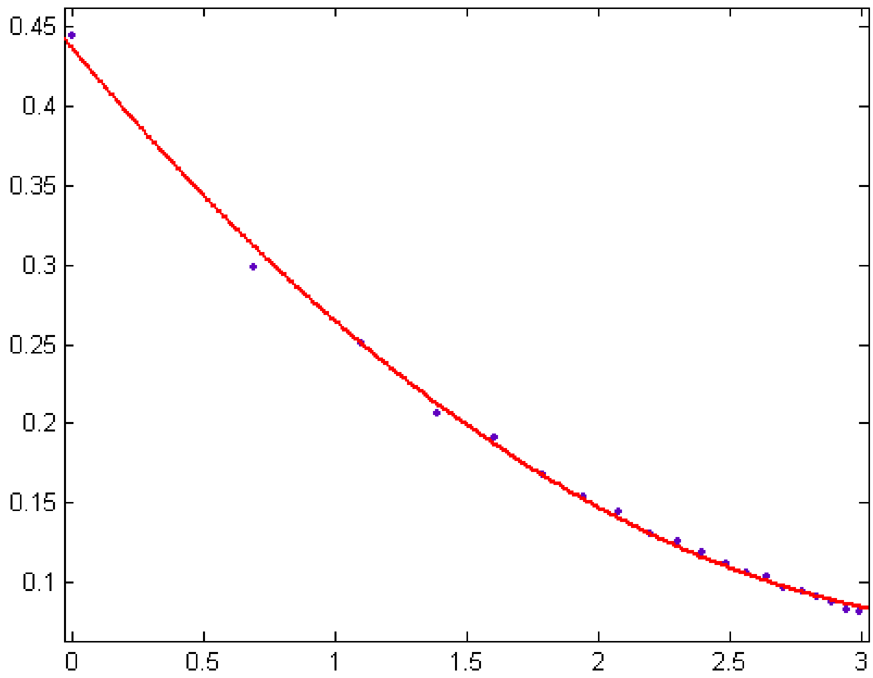

Figure 5:

For a quadratic least-squares fit of the curve, $p_1 = 0.02734$ with a 95% confidence interval of (0.02425, 0.03042), $p_2 = -0.1995$ with a 95% confidence interval of (-0.2102, -0.1888, $p_3 = 0.4369$ with a 95% confidence interval of (0.4282, 0.4457), SSE=0.0003655, R-squared=0.9976, and RMSE=0.004637.

A plot of the intercepts versus $\log(x)$ is

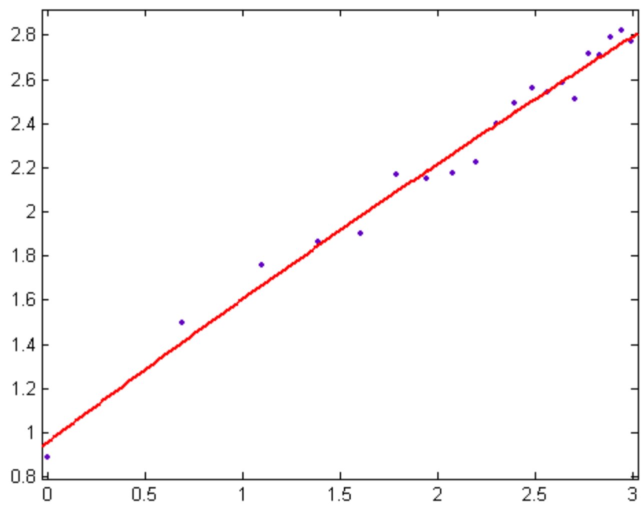

Figure 6:

Sometimes it is necessary to disregard the first few n values to accurately compute the slope. This makes the intercepts less accurate.
A variant of $\epsilon(s)$ is

$$\epsilon_1(s) = \Pi(\frac{s}{2})\Pi(s)(s-1)\zeta(s)^2 \tag{6}$$

A plot of $\epsilon_1(s)$ for the first zeta function zero and $n \leq 10000$ is

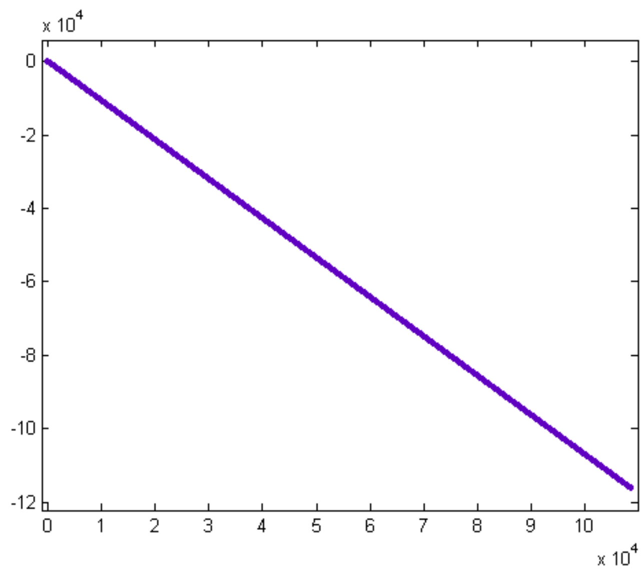

Figure 7:

For a linear least-squares fit of the curve, $p_1 = -1.073$ with a 95% confidence interval of $(-1.073, -1.073)$, $p_2 = 411.6$ with a 95% confidence interval of (411.4, 412.9), SSE=$1.531 \cdot 10^9$, R-squared=1, and RMSE=123.7.

The slopes and intercepts for the first ten zeta function zeros are $(-1.071, 73.86)$, $(-0.1045, 76.4$, $(-1.037, 77.21)$, $(-1.028, 77.94)$, $(-1.025, 78.2)$, $(-1.021, 78.55)$, $(-1.018, 78.79)$, $(-1.016, 78.87)$, $(-1.013, 79.11)$, and $(-1.012, 79.18)$. A plot of the slopes versus $\log(x), x = 1, 2, 3, ..., 10$ is

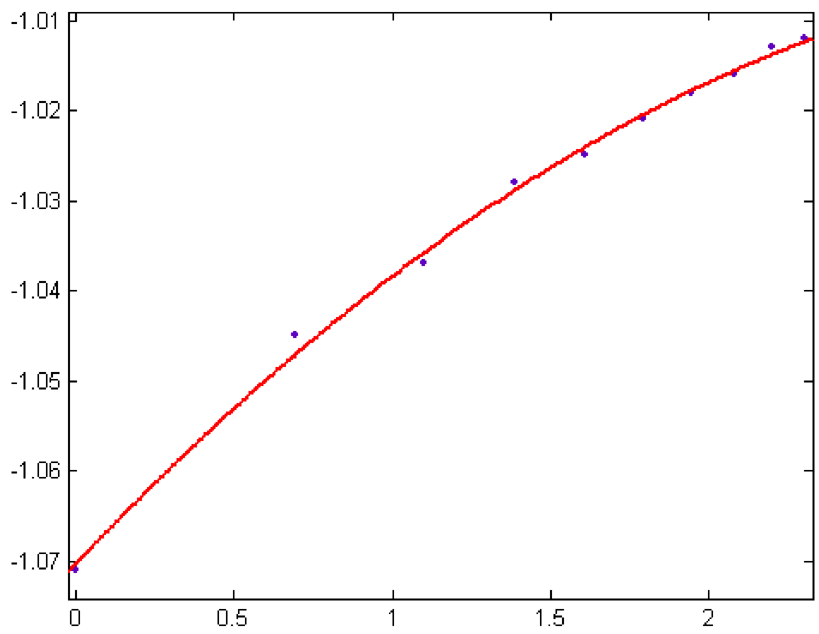

Figure 8:

For a quadratic least-squares fit of the curve, $p_1 = -0.005196$ with a 95% confidence interval of $(-0.006984, -0.003407)$, $p_2 = 0.03713$ with a 95% confidence interval of (0.03264, 0.04161), $p_3 = -1.07$ with a 95% confidence interval of $(-1.073, -1.068$, SSE=$9.218 \cdot 10^{-6}$, R-squared=0.9969, and RMSE=0.001148.

A plot of the intercepts versus $\log(x), x = 1, 2, 3, ..., 10$ is

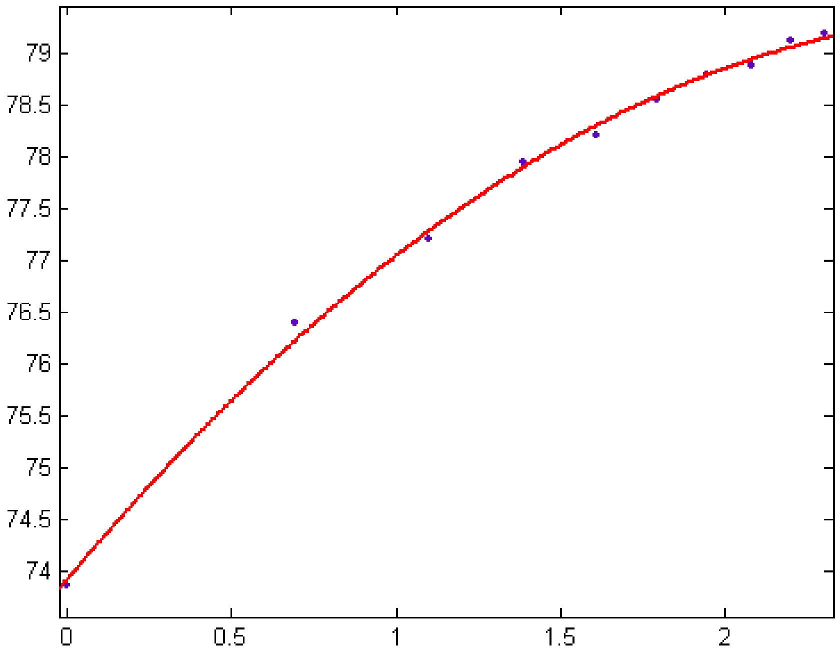

Figure 9:

For a quadratic least-squares fit of the curve, $p_1 = -0.6599$ with a 95% confidence interval of (-0.8067, -0.5132), $p_2 = 3.783$ with a 95% confidence interval of (3.416, 4.151), $p_3 = 73.92$ with a 95% confidence interval of (73.71, 74.13), SSE=0.06204, R-squared=0.9974, and RMSE=0.09414.

For $n \leq 100000$, the slopes and intercepts are (-1.073, 411.6), (-1.048, 427.6), (-1.04, 432.7), (-1.033, 437.5), (-1.03, 439.1), (-1.026, 441.6), (-1.024, 443), (-1.023, 443.9), ($-.102$, 445.4), and (-1.019, 445.9). The curves of slopes and intercepts versus $\log(x), x = 1, 2, 3, ..., 10$ are still quadratic but the parameters are different than for $n \leq 10000$.

A plot of $\epsilon_1(s)^2$ for the first zeta function zero and $n \leq 100000$ is

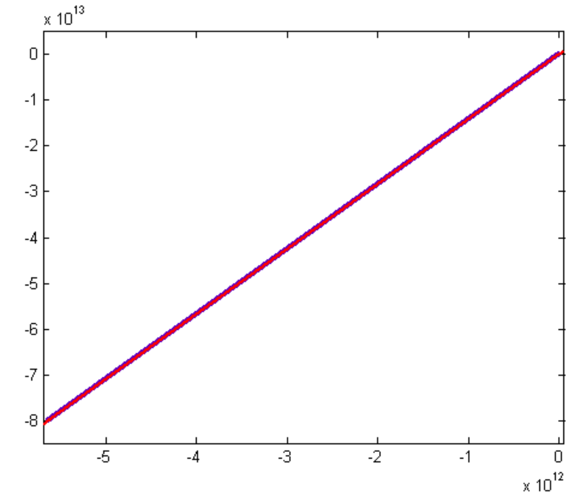

Figure 10:

The slope is 14.17, approximately equal to $\zeta(s)$. $\int x^n dx = \frac{x^{n+1}}{n+1}$ except when $n = -1$, so $\epsilon_1(s)$ is twice the derivative of $\zeta(s)$.

3. ANOTHER VARIANT OF THE EPSILON FUNCTION

A plot of $\epsilon(s)$ for the first zeta function zero and $n \leq 10000$ is

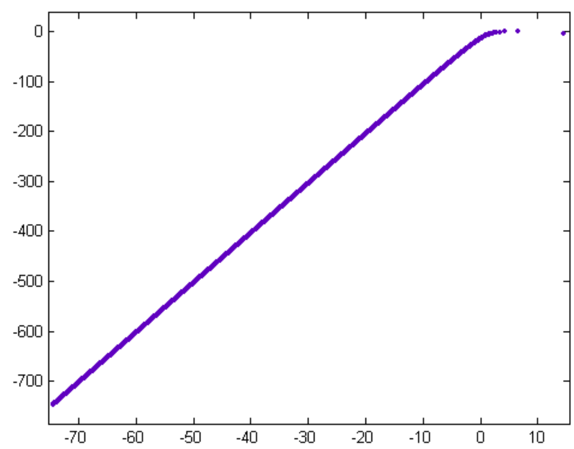

Figure 11:

A variant of $\epsilon(s)$ is

$$\epsilon_2(s) = \Pi(-s)\pi^{s/2}\zeta(-s) \tag{7}$$

This variant generates straight lines for any s value. A plot of $\epsilon_2(s)$ for the first zeta function zero and $n \leq 10000$ is

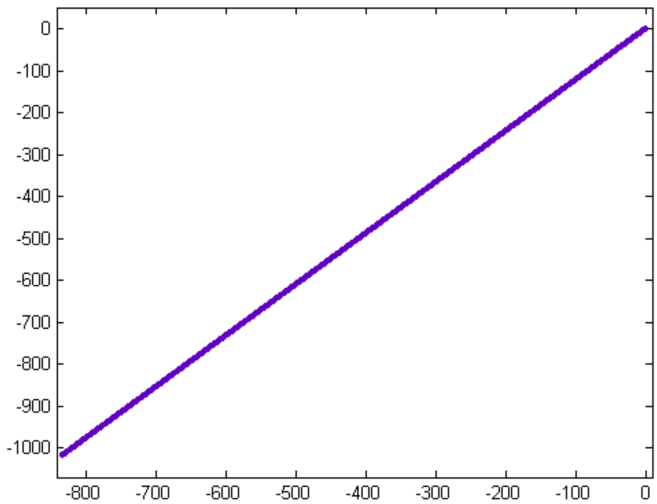

Figure 12:

A plot of the superimposed curves for the second zeta function zero is

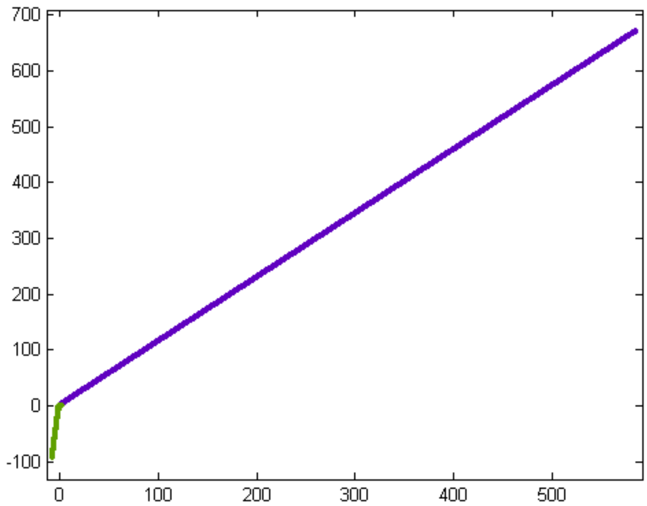

Figure 13:

The function $\Pi((s/2)-1)\pi^{-s/2}\zeta(s)$, which occurs in the symmetrical form of the functional equation, has poles at $s=0$ and $s=1$. Riemann multiplies it by $s(s-1)/2$

and defines

$$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s). \tag{8}$$

Similarly, multiplying $\epsilon_2(s)$ by $-s(-s-1)/2$ and adding π gives $\xi_1(s)$. A plot of $\xi(s)$ and $\xi_1(s)$ for the first zeta function zero and $n \leq 100$ is

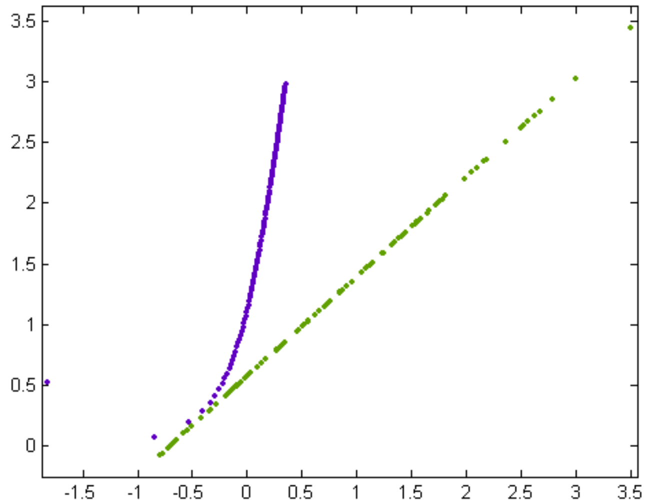

Figure 14:

$\xi_1(s)$ can be interpreted as being the derivative of $\xi(s)$.

4. RELATIONSHIP WITH THE PRIMES

Taking the logarithm of $\zeta(s)$ using Euler's infinite product formulation gives

$$\log\zeta(s) = \sum_p -\log(1-p^{-s}) \tag{9}$$

where p denotes a prime. Then taking the derivative gives

$$\frac{d\zeta}{ds}(s)/\zeta(s) = -\sum_{n=1}^{\infty}\Lambda(n)n^{-s} \tag{10}$$

where $\Lambda(n)$ denotes the Mangoldt function. $\Lambda(n)$ equals $\log p$ if $n = p^k$ for some prime p and integer $k \geq 1$ or 0 otherwise. The Fourier transform of the Mangoldt function gives a spectrum with spikes at ordinates equal to the imaginary parts of the Riemann zeta function zeros. See Mazur and Stein [2] for details.

A plot of $\log\zeta(s)$ for the first zeta function zero and the first 300000 primes is

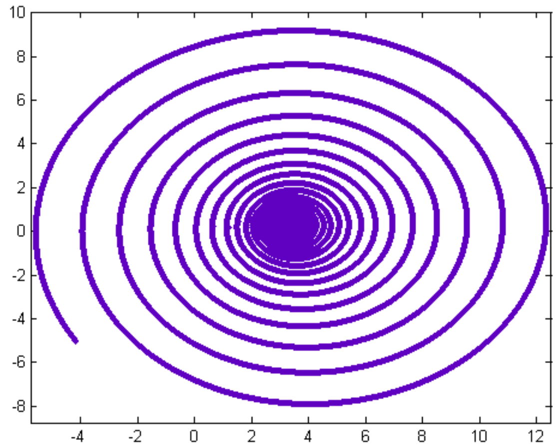

Figure 15:

A plot of the logarithms of the n values of the inflection points is

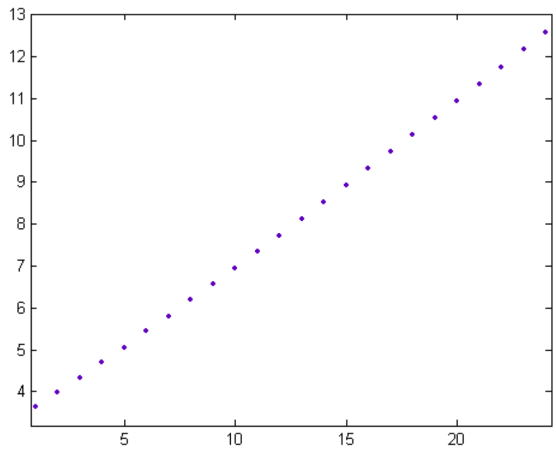

Figure 16:

The slopes and intercepts for the first ten zeta function zeros are (0.3906, 3.104), (0.2724, 6.099), (0.2253, 5.241), (0.1912, 9.225), (0.1738, 8.823), (0.1506, 4.282), (0.1393, 3.232), (0.1328, 9.956), (0.1221, 10.85), and (0.1151, 8.381). A plot of the slopes versus $\log(x), x = 1, 2, 3, ..., 10$ is

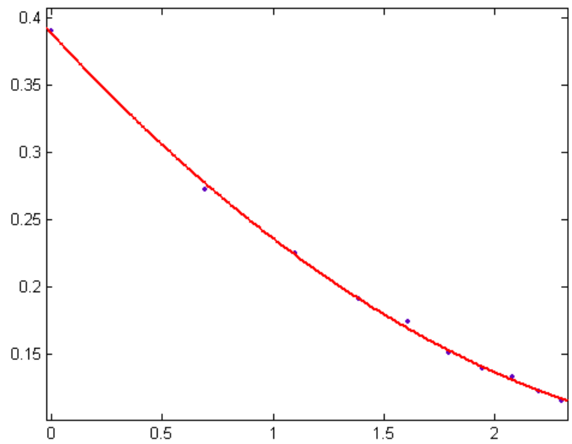

Figure 17:

For a quadratic least-squares fit of the curve, $p_1 = 0.0265$ with a 95% confidence interval of (0.02173, 0.03127), $p_2 = -0.1793$ with a 95% confidence interval of $(-0.1913, -0.1674)$, $p_3 = 0.3889$ with a 95% confidence interval of (0.382, 0.3958), SSE=$6.556 \cdot 10^{-5}$, R-squared=0.999, and RMSE=0.00306. These parameters are almost the same as those for $\zeta(s)$.
A plot of $\frac{\log \zeta(s)}{\epsilon_1(s)}$ for the first zeta function zero and $n \leq 300000$ is

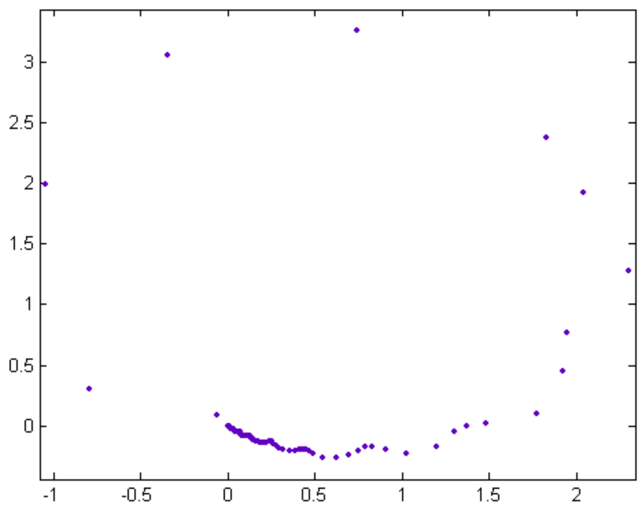

Figure 18:

A plot of the logarithms of the n values of the inflection points (taken to be on the right-hand side of the curves) is

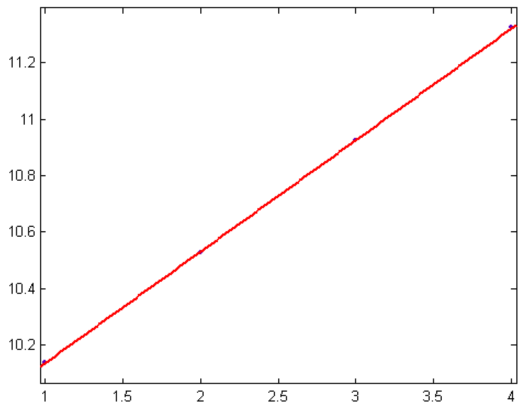

Figure 19:

For a linear least-squares fit of the curve, $p_1 = 0.3951$ with a 95% confidence interval of (0.3869, 0.4032), $p_2 = 9.741$ with a 95% confidence interval of (9.718, 9.763), SSE=$3.611 \cdot 10^{-5}$, R-squared=1.0, and RMSE=0.004249. These parameters are almost the same as for $\log \zeta(s)$.

5. AN ALTERNATE DEFINITION OF THE ZETA FUNCTION

A plot of $\frac{\zeta_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 13.5)$ and $100 \leq n \leq 50000$ is

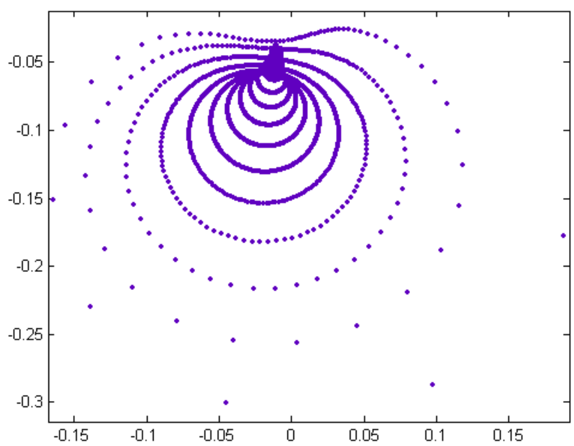

Figure 20:

It appears that $\frac{\zeta_1(s)}{\epsilon_1(s)}$ for s values with imaginary components other than those of the

zeros will eventually converge to zero.
A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 13.5)$ and $200 \leq n \leq 2000000$ is

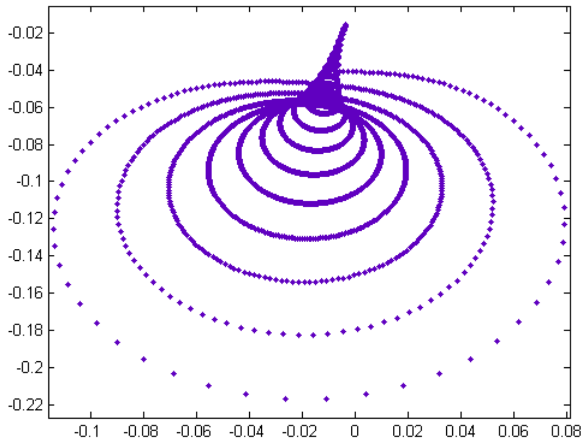

Figure 21:

A plot of $\frac{\epsilon_1(s)}{\xi_1(s)}$ (the reciprocal of the above) for $s = (0.5, 49.7738324776)$ (the tenth zeta function zero) and $1000 \leq n \leq 100000$ is

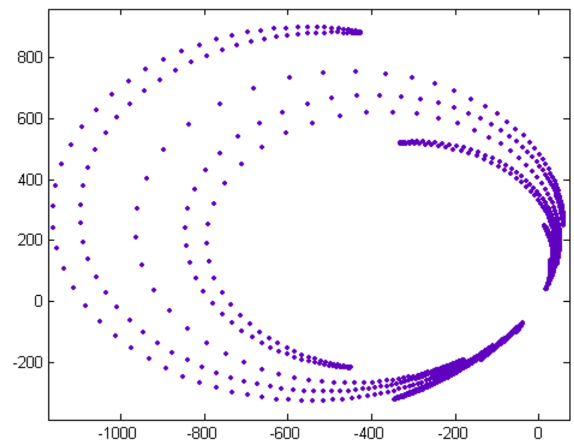

Figure 22:

A plot of $\frac{\xi_1(s)}{\epsilon_1(s)}$ for $s = (0.5, 49.7738324776)$ and $10 \leq n \leq 100000$ is

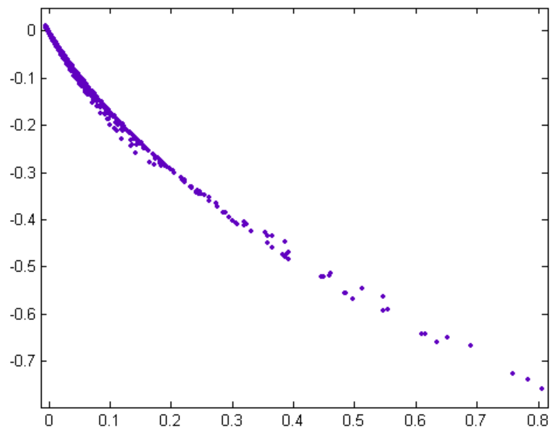

Figure 23:

Both are "two-dimensional".

C code for computing the derivatives of the epsilon and zeta functions is as follows. The output of the first program is then divided by the output of the second function. The prime look-up table contains the primes less than 1500000.

```
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
//
// Derivative of epsilon function
//
unsigned int max=100000;
double s=-.50; // usually set to -.50
//double t=13.5;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
double t=49.77383247767230;
```

```
double pi=3.14159265359;
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int tsize=114155; // size of prime look-up table
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
FILE *Outfp;
Outfp = fopen("transe6x.dat","w");
if (max>1500000) {
    printf("max too large \n");
    return;
    }
if (s>=0.0)
    temp1=pow(pi,s/2.0);
else {
    temp1=pow(pi,-s/2.0);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(pi/2.0)));
tempt=temp1*(sin(t*log(pi/2.0)));
a=temps*temps+tempt*tempt;
e=temps/a;
f=-tempt/a;
printf(" %.10llf %.10llf \n",e,f);
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s);
    else {
        temp1=pow((double)(x+1),-s);
        temp1=1.0/temp1;
        }
```

```
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;
temps=a*sums-b*sumt;
tempt=a*sumt+b*sums;
a=temps;
b=temps;
temps=a*e-b*f;
tempt=a*f+b*e;
a=s;
a=a*(a-1)/2;
temps=temps*a;
tempt=tempt*a;
temps=temps+pi;
tempt=tempt+pi;
if (x>xmin) {
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
```

```
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
extern char *malloc();
//
// Derivative of zeta function
//
unsigned int max=100000;
double s=0.50; // set to 0.50
//double t=13.5;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
double t=49.77383247767230;
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int tsize=114155; // size of prime look-up table
double pi=3.14159265359;
double *rsave,*isave;
void main() {
unsigned int temp,x;
```

```
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
double sumr,sumi,I,R;
double newprod,newtemp1,newtemps,newtempt,newa,newb;
FILE *Outfp;
Outfp = fopen("transe2x.dat","w");
rsave=(double*) malloc(16000004);
if (rsave==NULL)
    return;
isave=(double*) malloc(16000004);
if (isave==NULL)
    return;
if (max>1500000) {
    printf("max too large \n");
    return;
    }
//
// Pi(s/2)/2pi*Pi(s)*zeta(s)(s-1)
//
f=2.0*pi;
temp1=f*f;
e=0.0;
f=-f/temp1;
prods=1.0;
newprod=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s/2.0);
    else {
        temp1=pow((double)(x+1),-s/2.0);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
```

```
b=prods*tempt+temps;
if (s>=0.0)
    newtemp1=pow((double)(x+1),s);
else {
    newtemp1=pow((double)(x+1),-s);
    newtemp1=1.0/newtemp1;
    }
newtemps=newtemp1*(cos(t*log(x+1)));
newtempt=newtemp1*(sin(t*log(x+1)));
newa=newprod*newtemps-newtempt;
newb=newprod*newtempt+newtemps;
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;
temps=a*sums-b*sumt;
tempt=a*sumt+b*sums;
c=temps*e-tempt*f;
d=temps*f+tempt*e;
temps=c*(s-1.0)-d*t;
tempt=c*t+d*(s-1.0);
a=temps*newa-tempt*newb;
b=temps*newb+tempt*newa;
temps=a;
tempt=b;
rsave[x-1]=temps;
isave[x-1]=tempt;
if (x==(x/1000)*1000)
    printf(" %d \n",x);
```

```
        }
//
// zeta(s)
//
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    temp1=pow((double)temp,s);
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    a=rsave[x-1];
    b=isave[x-1];
    temps=a*sumr-b*sumi;
    tempt=a*sumi+b*sumr;
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %.10llf \n",log(x));
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)

[2] B. Mazur and W. Stein, *Prime Numbers and the Riemann Hypothesis*, Cambridge University Press, (2016)