Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 19, Number 2 (2023), pp. 579-594

https://dx.doi.org/10.37622/GJPAM/19.2.2023.579-594

Some Results on Uniqueness of Certain Type of Difference Differential Polynomials Sharing a Small Function

Harina P. Waghamore and Roopa M.

Department of Mathematics, Bangalore University Jnana Bharathi Campus, Bangalore 560056*,, India.*
E-mail IDs: harinapw@gmail.com (Harina P. Waghamore); mroopaprakash@gmail.com (Roopa M.)

Abstract

We demonstrate throughout the paper that uniqueness of entire functions with weakly weighted sharing and relaxed weighted sharing, whose differential-difference polynomials sharing a small function. We obtain some results of P. Sahoo, G. Biswas, which improve and extend the results.
Keywords : Uniqueness; Entire function; Small function; Weighted sharing; difference polynomial
MR(2020) Subject Classification: 30D35

1. INTRODUCTION AND DEFINITIONS

In this article, we consider a meromorphic function which always mean a meromorphic function in the complex plane $\mathbb{C}$. The basic fundamentals of Nevanlinna's value distribution theory(see [9, 13, 27]). A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$, if $T(r, \varphi(z)) = S(r, f)$, where $S(r, f)$ denotes any quantity which satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of the finite logarithmic measure. Let k be a positive integer or infinity and let $a \in \mathbb{C} \cup \{\infty\}$. We denote set $E(a, f) = \{z \mid f - a = 0\}$, is the counting zeros with multiplicity k is counted k times. If the zeros are counted only once, then the set is denoted by $\overline{E}(a, f)$. We state that f and g are two non-constant meromorphic functions that share the value of a CM(counting multiplicity), when $E(a, f) = E(a, g)$.

Suppose $\overline{E}(a,f)=\overline{E}(a,g)$, then f and g share the value of a IM(ignoring multiplicity). $E_{k)}(a,f)$ represents the collection of all a points in f whose multiplicity does not exceed k and are counted according to their multiplicity. $\overline{E}_{k)}(a,f)$ is a collection of all a points in f whose multiplicity doesn't exceed k. In this paper, $\rho(f)$ denotes the order of f (see [9, 13, 27]).
We require the following definitions.
Definition 1.1 ([11]) *Let $a\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|=1)$ the counting function of simple a-points of f. For a positive integer k, we denote by $N(r,a;f|\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r,a;f\geq k)$, we denote the corresponding reduced counting function. Analogously we can define $N(r,a;f\geq k)$ and $\overline{N}(r,a;f\geq k)$.*
Definition 1.2 ([12]) *Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. Then*

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f|\geq 2)+\ldots+\overline{N}(r,a;f\geq k);$$

Clearly,

$$N_1(r,a;f)=\overline{N}(r,a;f)$$

Definition 1.3 ([15]) *Let $a\in\mathbb{C}\cup\{\infty\}$. We denote by $N_E(r,a;f,g)$ $\left(\overline{N}_E(r,a;f,g)\right)$ by the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)$ $\left(\overline{N}_0(r,a;f,g)\right)$ the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If*

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share the value a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share the value a "IM".
Definition 1.4 ([15]) *Let f and g share the value a "IM" and k be a positive integer or infinity. Then $\overline{N}^E_{k)}(r,a;f,g)$ denotes the reduced counting function of those a–points of f whose multiplicities are equal to the corresponding a–points of g, and both of their multiplicities are not less than k, $\overline{N}^0_{(k}(r,a;f,g)$ denotes the reduced counting function*

those a-points of f which are a–points of g, and both of their multiplicities are no more than k.

Weakly weighted sharing is defined as a scaling between sharing IM and sharing CM as follows.

Definition 1.5 ([15]) *Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If*

$$\begin{aligned}
\overline{N}(r,a;f| \leq k) - \overline{N}^{E}_{k)}(r,a;f,g) &= S(r,f),\\
\overline{N}(r,a;g| \leq k) - \overline{N}^{E}_{k)}(r,a;f,g) &= S(r,g),\\
\overline{N}(r,a;f| \geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) &= S(r,f),\\
\overline{N}(r,a;g| \geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) &= S(r,g), if\ k=0\\
\overline{N}\left(r,\frac{1}{f-a}\right) - \overline{N}_0(r,a;f,g) &= S(r,f),\\
\overline{N}\left(r,\frac{1}{g-a}\right) - \overline{N}_0(r,a;f,g) &= S(r,g),
\end{aligned}$$

then we say that f and g share the value a weakly with weight k and we write f and g share "(a,k)".

The term "relaxed weighted sharing", which is weaker than "weakly weighted sharing" was introduced by A. Banerjee et.al ([1]).

Definition 1.6 ([1]) *We denote by $\overline{N}(r,a;f| = p; g| = q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.*

Definition 1.7 ([1]) *Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a "IM". If for $p \neq q$,*

$$\sum_{p,q \leq p} \overline{N}(r,a;f| = p; g| = q) = S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner and in this case we write f and g share $(a,k)^*$.

Definition 1.8 ([28]) *Let f be an entire function and $a \in \mathbb{C} \cup \{\infty\}$. We define sum of linear difference differential polynomial as $L(z,f)$ in this paper.*

$$L(z,f) = a_1 f(z+c_1) + a_2 f(z+c_2), ..., +a_p f(z+c_p) = \sum_{j=1}^{p} a_j f(z+c_j),$$

where $a_1, a_2, ..., a_j \in S(f)/0$, p,j are positive integers(i.e. $j = 1,2,...,p$) and $c_1, c_2, ..., c_j$ are distinct complex numbers.

In the current study, many mathematicians have worked on the whole and meromorphic functions whose differential polynomials share a small function or fixed points (see [5], [6], [16], [21], [23], [26]). The value distribution theory in difference modulators has gained increasing attention from academics. By using difference operators, R.G.Halburd et.al [7] created a Nevanlinna theory in 2006 [7]. The research topics like difference modulators of Nevanlinna theory and the difference logarithmic derivative lemma were developed by R.G. Halburd et.al [8] in 2006 and Y.M. Chaing et.al [4] in 2008 plays an important role in considering the difference analogues of Nevanlinna theory. Distribution of zeros of various difference polynomial types changed with advancement of the difference counterpart of Nevanlinna theory.

I.Laine and Chung-Chun Yang [14] demonstrated the subsequent finding in 2007.

Theorem 1.9 ([14]) *Let $f(z)$ be a transcendental entire function of finite order and η be a nonzero complex constant. Then $n \geq 2$, $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely.*

We recall the following two examples.

Example 1.10 ([14]) Let $f(z) = 1 + e^z$. Then $f(z)f(z+\pi i) - 1 = -e^{2z}$ has no zeros. This shows that Theorem 1.9 [14] does not hold if $n = 1$.

Example 1.11 ([18]) Let $f(z) = e^{-e^z}$. Then $f^2(z)f(z+\eta) - 2 = -1$ and $\rho(f) = \infty$, where η is the nonzero constant satisfying $e^\eta = -2$. Evidently, $f^2(z)f(z+\eta) - 2$ has no zeros. This shows that Theorem 1.9 does not remain valid if f is of infinite order.

According to Theorem 1.9 ([14]), the following uniqueness result was established in 2010 by X.G.Qi, L.Z. Yang and K. Liu [20].

Thoerem 1.12 ([20]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and η be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

J.L. Zhang [29] by using the zeros of a various kind of difference polynomial established the following theorem.

Theorem 1.13 ([29]) *Let $f(z)$ be a transcendental entire function of finite order, $\varphi(z)(\not\equiv 0)$ be a small function with respect to $f(z)$ and η be a nonzero complex constant. If $n(\geq 2)$ is an integer then $f^n(z)(f(z)-1)f(z+\eta) - \varphi(z)$ has infinitely many zeros.*

The author J.L. Zhang [29] also supported the subsequent uniqueness finding in the same study.

Theorem 1.14 ([29]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share $\varphi(z)$ CM, then $f(z) \equiv g(z)$.*

The concept of weakly weighted sharing and relaxed weighted sharing was introduced by C. Meng [19] proved the following results which improve and extend Theorem 1.14 in different directions.

Theorem 1.15 ([19]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share "$(\varphi(z),2)$", then $f(z) \equiv g(z)$.*

Theorem 1.16 ([19]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geq 10$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share $(\varphi(z),2)^*$, then $f(z) \equiv g(z)$.*

Theorem 1.17 ([19]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geq 16$ is an integer. If $\overline{E}_{2)}(\varphi(z), f^n(z)(f(z)-1)f(z+\eta)) = \overline{E}_{2)}(\varphi(z), g^n(z)(g(z)-1)g(z+\eta))$, then $f(z) \equiv g(z)$.*

P. Sahoo [22] recently responded to the query and established the following findings, which generalize Theorems from 1.15 to 1.17.

Theorem 1.18 ([22]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n \geq m+6$. If $f^n(z)\left(f^m(z)-1\right)f(z+\eta)$ and $g^n(z)\left(g^m(z)-1\right)g(z+\eta)$ share "$(\varphi(z),2)$", then $f(z) \equiv \mathrm{tg}(z)$ where $t^m=1$.*

Theorem 1.19 ([22]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n \geq 2m+8$. If $f^n(z)\left(f^m(z)-1\right)f(z+\eta)$ and $g^n(z)\left(g^m(z)-1\right)g(z+\eta)$ share $(\varphi(z),2)^*$, then $f(z) \equiv tg(z)$ where $t^m=1$.*

Theorem 1.20 ([22]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n \geq 4m+12$. If $\overline{E}_{2)}(\varphi(z), f^n(z)(f(z)-1)f(z+\eta)) = \overline{E}_{2)}(\varphi(z), g^n(z)(g(z)-1)g(z+\eta))$, then $f(z) \equiv tg(z)$, where $t^m=1$.*

The conclusions of the work improve and extend the Theorems from 1.18 to 1.20 by the author P. Sahoo [24] in responsive above mentioned question.

Theorem 1.21 ([24]) *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constant, $n, k(\geq 0), m(\geq 1)$*

are integers such that $n \geq 2k + m + 6$. *If* $(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)} = (g^n(z)(g^m(z)-1)g(z+\eta))^{(k)}$ *share* "$(\varphi(z), 2)$", *then* $f \equiv tg$ *where* $t^m = 1$.

Theorem 1.22 ([24]) *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order and* $\varphi(z)(\not\equiv 0, \infty)$ *be a small function of both* $f(z)$ *and* $g(z)$ *with finitely many zeros. Suppose that* η *is a nonzero complex constant,* $n, k(\geq 0), m(\geq 1)$ *are integers such that* $n \geq 3k + 2m + 8$. *If* $(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)} = (g^n(z)(g^m(z)-1)g(z+\eta))^{(k)}$ *share* $(\varphi(z), 2)^*$, *then* $f \equiv tg$, *where* $t^m = 1$.

Theorem 1.23 ([24]) *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order and* $\varphi(z)(\not\equiv 0, \infty)$ *be a small function of both* $f(z)$ *and* $g(z)$ *with finitely many zeros. Suppose that* η *is a nonzero complex constant,* $n, k(\geq 0), m(\geq 1)$ *are integers such that* $n \geq 5k + 4m + 12$. *If*
$\overline{E}_{2)}\left(\varphi(z), (f^n(z)(f(z)-1)f(z+\eta))^{(k)}\right) = \overline{E}_{2)}\left(\varphi(z), (g^n(z)(g(z)-1)g(z+\eta))^{(k)}\right)$, *then* $f(z) \equiv tg(z)$, *where* $t^m = 1$.

Note: Though the authors P. Sahoo and Gurudas Biswas [24] claimed that the result holds for $n \geq m + 6$, from the proof it is easily seen that the result holds if $n \geq m + 5$.

The motivation for the current study can be found in the following question with regard to Theorems 1.21 to Theorem 1.23.

Question: What can be said about the entire functions f and g, if we consider the sum of linear difference differential polynomial of the form $\left(f^n(z)P(f)\sum_{j=1}^{p} a_j f(z+c_j)\right)^{(k)}$, where $k(\geq 0)$ is an integer.

LEMMAS

We state some lemmas which will be needed in the sequel. We denote H the following

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are non-constant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 1.24 ([4]) *Let* $f(z)$ *be a meromorphic function of order* $\rho(f) < \infty$, *and let* η *be a nonzero complex constant. Then for each* $\varepsilon > 0$, *we have*

$$T(r, f(z+\eta)) = T(r, f) + O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\{\log r\}.$$

Lemma 1.25 ([3]) *Let* $f(z)$ *be an entire function of order* $\rho(f) < \infty$, *and* $F = \left(f^n(z)P(f)\sum_{j=1}^{p} a_j f(z+c_j)\right)^{(k)}$. *Then*

$$T(r, F) = (n + m + p)T(r, f) + O\left\{r^{\rho(f)-1+\varepsilon}\right\} + S(r, f).$$

Lemma 1.26 ([30]) *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,F^{(k)}\right) - T(r,f) + N_{p+k}(r,0,f) + S(r,f), \quad (2.1)$$
$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty,f) + N_{p+k}(r,0,f) + S(r,f). \quad (2.2)$$

Lemma 1.27 ([1]) *Let F and G be two nonconstant meromorphic functions that share "$(1,2)$" and $H \not\equiv 0$. Then,*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) - \sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{G'}{G} \mid\geq p\right) + S(r,F) + S(r,G),$$

and same inequality holds for $T(r,G)$.
Lemma 1.28 ([1]) *Let F and G be two nonconstant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty,F) - m(r,1;G) + S(r,F) + S(r,G),$$

and the same inequality is true for $T(r,G)$.
Lemma 1.29 ([17]) *Let F and G be two nonconstant entire functions and $p \geq 2$ be an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G)$$

and the same inequality holds for $T(r,G)$.
Lemma 1.30 *Let f and g be two entire functions, $n(\geq 1), m(\geq 1), k(\geq 0), j \geq 1$ be integers and $a_j (j = 1, 2, \ldots, p), c$ are nonzero constants, and let*

$$F = \left(f^n P(f) \sum_{j=1}^{p} a_j f\left(z + c_j\right)\right)^{(k)}, G = \left(g^n P(g) \sum_{j=1}^{p} a_j g\left(z + c_j\right)\right)^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that

$$\overline{N}\left(r,c_1;F\right) = \bar{N}(r,0;G) \; and \; \overline{N}\left(r,c_2;G\right) = \overline{N}(r,0;F),$$

then $n \leq 2k + m + p + 2$.

Proof We consider the functions as

$$F_1 = \left(f^n P(f) \sum_{j=1}^{p} a_j f\left(z + c_j\right)\right)^{(k)}, G_1 = \left(g^n P(g) \sum_{j=1}^{p} a_j g\left(z + c_j\right)\right)^{(k)}.$$

By the second fundamental theorem of Nevanlinna we have,

$$T(r,f) \leq \overline{N}(r,0;F) + \overline{N}\left(r,c_1;F\right) + S(r,F) \leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \tag{2.3}$$

Using ((2.1)), ((2.2)), ((2.3)) and Lemma 1.24 and Lemma 1.25 we obtain,

$$\begin{aligned}(n+m+p)T(r,f) \leq & T(r,F) - \overline{N}(r,0;F) + N_{k+1}\left(r,0;F_1\right) + S(r,f),\\ \leq & N\left(r,0;G\right) + N_{k+1}\left(r,0;F_1\right) + S(r,f),\\ \leq & N_{k+1}\left(r,0;F_1\right) + N_{k+1}\left(r,0;G_1\right) + S(r,f) + S(r,g),\\ \leq & (k+1)((\overline{N}(r,0;f)) + \overline{N}(r,0;g)) + N\left(r,1;f^m\right) + N\left(r,1;g^m\right)\\ & + N(r,0;f(z+\eta)) + N(r,0;g(z+\eta)) + S(r,f) + S(r,g),\\ \leq & (k+m+p+1)(T(r,f)+T(r,g)) + O\left\{r^{\rho(f)-1+\varepsilon}\right\} +\\ & O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,f). \end{aligned} \tag{2.4}$$

Similarly

$$\begin{aligned}(n+m+p)T(r,g) \leq (k+m+p+1)(T(r,f)+T(r,g)) + O\left\{r^{\rho(f)-1+\varepsilon}\right\} +\\ O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g).\end{aligned} \tag{2.5}$$

Combining ((2.4)) and ((2.5)) we obtain,

$$(n-2k-m-p-2)(T(r,f)+T(r,g)) \leq O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g),$$

which gives $n \leq 2k+m+p+2$. This proves the lemma.

Lemma 1.31 ([2]) *Let f and g be two transcendental entire functions, $n(\geq 1), m(\geq 1), k(\geq 0), j(\geq 1)$ and p are integers, where $a_j (j = 1, 2, \ldots, p), c$ are constants and n,m be positive integers such that $n \geq m+5$. If*

$$f^n P(f) \sum_{j=1}^{p} a_j f\left(z+c_j\right) \equiv g^n P(g) \sum_{j=1}^{p} a_j g\left(z+c_j\right),$$

then $f \equiv tg$ for some constant t such that $t^m = 1$.

MAIN RESULTS

Theorem 1.32 *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that c is a nonzero complex constant, $n, k(\geq 0), m(\geq 1)$ and $j(\geq 1)$ are integers such that $n \geq 2k+m+p+5$. If*

$$\left(f^n(z)P(f)\sum_{j=1}^{p} a_j f\left(z+c_j\right)\right)^{(k)} \text{ and } \left(g^n P(g)\sum_{j=1}^{p} a_j g\left(z+c_j\right)\right)^{(k)}$$

share "$(\varphi(z), 2)$", *then* $f \equiv tg$, *where* $t^m = 1$.

Proof Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$, where $F_1 = f^n P(f) \sum_{j=1}^{p} a_j f(z + c_j)$ and $G_1 = g^n P(g) \sum_{j=1}^{p} a_j g(z + c_j)$. Then F and G are transcendental meromorphic functions that share "$(1, 2)$" except the zeros and poles of $\alpha(z)$. From Lemma 1.25, we see that

$$T(r, F_1) = (n + m + p)T(r, f) + O\left\{r^{\rho(f)-1+\varepsilon}\right\} + S(r, f), \tag{2.6}$$

$$T(r, G_1) = (n + m + p)T(r, g) + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r, g). \tag{2.7}$$

If possible, we may assume that $H \not\equiv 0$. Using ((2.1)),((2.6)) and Lemma 1.25, we get

$$\begin{aligned} N_2(r, 0; F) &\leq N_2\left(r, 0; F_1^{(k)}\right) + S(r, f), \\ &\leq N_{k+2}(r, 0; F_1) + S(r, f), \\ &\leq T(r, F) - (n + m + p)T(r, f) + N_{k+2}(r, 0, F_1) + S(r, f). \end{aligned}$$

From this we get,

$$(n + m + p)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \tag{2.8}$$

Also, by ((2.2)) we obtain,

$$\begin{aligned} N_2(r, 0; F) \leq& N_2\left(r, 0; (F_1)^k\right) + S(r, f), \\ \leq& N_{k+2}(r, 0; F_1) + S(r, f). \end{aligned} \tag{2.9}$$

Similarly,

$$N_2(r, 0; G) \leq N_2(r, 0; G_1) + S(r, g). \tag{3.0}$$

Using ((2.9)), ((3.0)) and Lemmas 1.24 and 1.27, we obtain from ((2.8)),

$$\begin{aligned} (n + m + p)T(r, f) \leq& N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) + \\ & S(r, f) + S(r, g), \\ \leq& N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g), \\ \leq& (k + m + p + 2)\{T(r, f) + T(r, g)\} + O\left\{r^{\rho(f)-1+\varepsilon}\right\} + \\ & O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

In a similar manner we obtain,

$$(n + m + p)T(r, g) \leq (k + m + p + 2)\{T(r, f) + T(r, g)\}+$$

$$O\left\{r^{\rho(f)-1+\varepsilon}\right\}+O\left\{r^{\rho(g)-1+\varepsilon}\right\}S(r,f)+S(r,g). \tag{3.2}$$

((3.1)) and ((3.2)) together yields,

$$(n-2k-m-p-4)\{T(r,f)+T(r,g)\}\leq O\left\{r^{\rho(f)-1+\varepsilon}\right\}+O\left\{r^{\rho(g)-1+\varepsilon}\right\}+ S(r,f)+S(r,g),$$

a contradiction with the assumption that $n\geq 2k+m+p+5$. Therefore we must have $H=0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both side twice we get from above,

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.3}$$

where $A\neq 0$ and B are constants. From ((3.3)) it is clear that F,G share 1 CM and hence they share "$(1,2)$". Therefore $n\geq 2k+m+p+5$. We now discuss the following three cases separately. **Case 1.** Suppose that $B\neq 0$ and $A=B$. Then from ((3.3)) we obtain,

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{3.4}$$

If $B=-1$, then from ((3.4)), we obtain $FG\equiv 1$. Then

$$\left[f^nP(f)\sum_{j=1}^{p}a_jf\left(z+c_j\right)\right]^{(k)}\left[g^nP(g)\sum_{j=1}^{p}a_jg\left(z+c_j\right)\right]^{(k)}\equiv\varphi^2.$$

Since the number of zeros of $\varphi(z)$ is finite, it follows that f as well as g has finitely many zeros. We put $f(z)=h(z)e^{\beta(z)}$, where $h(z)$ is a nonzero polynomial and $\beta(z)$ is a nonconstant polynomial. Now replacing $\beta(z+c)$ by $\gamma(z)$ and $h(z+c)$ by $\mu(z)$ we deduce that,

$$\begin{aligned}\left(f^n(z)\left(f^{m(z)}-1\right)f(z+c)\right)^{(k)}&=\left(h^n(z)e^{n\beta(z)}\left(h^m(z)e^{m\beta(z)}-1\right)h(z+c)e^{\beta(z+c)}\right)^{(k)}\\&=\left(h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)}\left(h^m(z)e^{m\beta(z)}-1\right)\right)^{(k)}\\&=\left(h^{n+m}(z)\mu(z)e^{(n+m)\beta(z)+\gamma(z)}-h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)}\right)^{(k)}\\&=e^{(n+m)\beta(z)+\gamma(z)}P_1(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\beta^{(k)}(z),\gamma^{(k)},\\&\quad h^{(k)}(z),\mu^{(k)}(z))-e^{n\beta(z)+\gamma(z)}P_2(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\\&\quad \beta^{(k)}(z),\gamma^{(k)},h^{(k)}(z),\mu^{(k)}(z))\\&=e^{n\beta(z)+\gamma(z)}\left(P_1e^{m\beta(z)}-P_2\right).\end{aligned}$$

Obviously $P_1e^{m\beta(z)} - P_2$ has infinite number of zeros, which contradicts with the fact that g is an entire function.
If $B \neq -1$, from ((3.4)) we have, $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F)$.
Using ((2.1)), ((2.2)), ((2.7)) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G)\\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) - (n+m+p)T(r,g) + S(r,g).\end{aligned}$$

This gives,

$$\begin{aligned}(n+m+p)T(r,g) \leq &(k+m+p+1)\{T(r,f)+T(r,g)\} + O\left\{r^{\rho(f)-1+\varepsilon}\right\} +\\ &O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,g).\end{aligned}$$

Thus we obtain,

$$\begin{aligned}(n-2k-m-p-2)\{T(r,f)+T(r,g)\} \leq O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} +\\ S(r,f) + S(r,g),\end{aligned}$$

a contradiction since $n \geq 2k+m+6$.
Case 2. Let $B \neq 0$ and $A \neq B$. Then from equation ((3.3)) we get,
$F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}\left(r, \frac{B-A+1}{B+1}; G\right) = \overline{N}(r,0;F)$. Arguing similarly as in Case 1, we arrive at a contradiction.
Case 3. If $B = 0$ and $A \neq 0$. Then from (3.3) we get, $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, it follows that $\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r,0;G)$ and $N(r, 1-A; G) = \overline{N}(r,0;F)$.

Now applying Lemma 1.30, it can be shown that $n \leq 2k+m+p+2$, a contradiction. Thus $A = 1$ and then $F \equiv G$. Then,

$$\left[f^nP(f)\sum_{j=1}^{p}a_jf(z+c_j)\right]^{(k)} \equiv \left[g^nP(g)\sum_{j=1}^{p}a_jg(z+c_j)\right]^{(k)}.$$

Integrating once we obtain,

$$\left[f^nP(f)\sum_{j=1}^{p}a_jf(z+c_j)\right]^{(k-1)} \equiv \left[g^nP(g)\sum_{j=1}^{p}a_jg(z+c_j)\right]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 7, it follows that $n \leq 2k+m+p+2$, a contradiction. Hence $c_{k-1} = 0$. Repeating the above process k times, we deduce that

$$\left[f^n P(f) \sum_{j=1}^{p} a_j f\left(z+c_j\right)\right] \equiv \left[g^n P(g) \sum_{j=1}^{p} a_j g\left(z+c_j\right)\right],$$

which by Lemma 8 gives $f = tg$, where t is a constant satisfying $t^m = 1$. This completes the proof of Theorem 1.32.

Theorem 1.33 *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that c is a nonzero complex constant, $n, k(\geq 0), m(\geq 1)$ and $j(\geq 1)$ are integers such that $n \geq 3k+2m+2p+6$. If*

$$\left(f^n P(f) \sum_{j=1}^{p} a_j f\left(z+c_j\right)\right)^{(k)} \text{ and } \left(g^n(z) P(g) \sum_{j=1}^{p} a_j g\left(z+c_j\right)\right)^{(k)},$$

, share $(\varphi(z), 2)^*$ then $f \equiv tg$, where $t^m = 1$.

Proof Let F, G, F_1 and G_1 be defined as in Theorem 1. Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\varphi(z)$. Let $H \not\equiv 0$. Then using ((2.2)) for $p = 1$, ((3.0)) and Lemmas 1.24 and 1.28 we obtain from ((2.8)),

$$\begin{aligned}(n+m+p)T(r,f) \leq & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + N_{k+2}\left(r,0;F_1\right) + S(r,f) + S(r,g), \\ \leq & N_{k+2}\left(r,0;F_1\right) + N_{k+2}\left(r,0;G_1\right) + N_{k+1}\left(r,0;F_1\right) + S(r,f) + S(r,g), \\ \leq & (2k+2m+2p+3)T(r,f) + (k+m+p+2)T(r,g) + \\ & O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g). \quad (3.5)\end{aligned}$$

In similar manner we obtain,

$$\begin{aligned}(n+m+p)T(r,g) \leq & (2k+2m+2p+3)T(r,g) + (k+m+p+2)T(r,f) + \\ & O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g). \quad (3.6)\end{aligned}$$

From ((3.5)) and ((3.6)), we get

$$\begin{aligned}(n-3k-2m-2p-5)\{T(r,f)+T(r,g)\} \leq & O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + \\ & S(r,f) + S(r,g),\end{aligned}$$

contradicting with the fact that $n \geq 3k+2m+2p+6$. Thus we must have $H = 0$. Then the result follows from the proof of Theorem 1.32 . This completes the proof of

Theorem 1.33.

Theorem 1.34 *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\varphi(z)(\not\equiv 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that c is a nonzero complex constant, $n, k(\geq 0), m(\geq 1)$ and $j(\geq 1)$ are integers such that $n \geq 5k + 4m + 4p + 8$. If*

$$\overline{E}_{2)}\left(\varphi(z), \left(f^n(z)P(f)\sum_{j=1}^{p} a_j f\left(z + c_j\right)\right)^{(k)}\right) = \overline{E}_{2)}\left(\varphi(z), \left(g^n(z)P(g)\sum_{j=1}^{p} a_j g\left(z + c_j\right)\right)^{(k)}\right),$$

then $f(z) \equiv tg(z)$, where $t^m = 1$.

Proof Let F, G, F_1 and G_1 be similar as in Theorem 1. Then F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1; F) = \overline{E}_{2)}(1; G)$ except the zeros and poles of $\varphi(z)$. Let $H \not\equiv 0$. Then by using ((2.3)), ((3.0)) and Lemmas 1.24 and 1.29 we obtain from ((2.8)),

$$\begin{aligned}(n+m+p)T(r,f) \leq\ & N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + N_{k+2}\left(r,0;F_1\right) + \\ & S(r,f) + S(r,g), \\ & \leq N_{k+2}\left(r0;F_1\right) + N_{k+2}\left(r,0;G_1\right) + 2N_{k+1}\left(r,0;F_1\right) + N_{k+1}\left(r,0;G_1\right) + \\ & S(r,f) + S(r,g), \\ & \leq (3k+3m+4p+4)T(r,f) + (2k+2m+2p+3)T(r,g) + \\ & O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g). \qquad (3.7)\end{aligned}$$

Similarly,

$$\begin{aligned}(n+m+p)T(r,g) \leq & (3k+3m+4p+4)T(r,g) + (2k+2m+2p+3)T(r,f) + \\ & O\left\{r^{\rho(f)-1+\varepsilon}\right\} + O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g). \qquad (3.8)\end{aligned}$$

Combining ((3.7)) and ((3.8)) we obtain,

$$(n-5k-4m-4p-7)(T(r,f)+T(r,g)) \leq O\left\{r^{\rho(f)-1+\varepsilon}\right\}$$

$$+O\left\{r^{\rho(g)-1+\varepsilon}\right\} + S(r,f) + S(r,g),$$

a contradiction with assumption that $n \geq 5k + 4m + 4p + 8$. Thus $H \equiv 0$ and the rest of the theorems follows from the proof of Theorem 1.32. This completes the proof of Theorem 1.34.

ACKNOWLEDGEMENTS

We thank the referees for their time and comments.

REFERENCES

[1] A. Banerjee, S. Mukherjee. *Uniqueness of meromorphic functions concerning differential monomials sharing the same value*. Bull. Math. Soc. Sci. Math. Roumanie (N.S.)., 2007, **50/98**(3): 191–206.

[2] Bhoosnurmath, Subhas S, Kabbur, Smita R. *Value distribution and uniqueness theorems for difference of entire and meromorphic functions*. International Journal of Analysis and Applications., 2013, **2**(2): 124–136.

[3] M. R. Chen, Z. X. Chen. *Properties of difference polynomials of entire functions with finite order*. Chinese Ann. Math. Ser. A., 2012, **33**(3): 359–374.

[4] Y.-M. Chiang, S. Feng.*On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*. Ramanujan J., 2008, **16**(1): 105–129.

[5] M. Fang, W. Hong.*A unicity theorem for entire functions concerning differential polynomials*. Indian J. Pure Appl. Math., 2001, **32**(9): 1343–1348.

[6] M. Fang, X. H. Hua. *Entire functions that share one value*. Nanjing Daxue Xuebao Shuxue Bannian Kan., 1996, **13**(1): 44–48.

[7] R. G. Halburd, R. Korhonen.*Nevanlinna theory for the difference operator*. Ann. Acad. Sci. Fenn. Math., 2006, **31**(2): 463–478.

[8] R. G. Halburd, R. Korhonen.*Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*. J. Math. Anal. Appl., 2006, **314**(2): 477–487.

[9] W. K. Hayman. *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[10] I. Lahiri.*Value distribution of certain differential polynomials*. Int. J. Math. Math. Sci., 2001, **28**(2): 83–91.

[11] I. Lahiri.*Weighted value sharing and uniqueness of meromorphic functions*. Complex Variables Theory Appl., 2001, **46**(3): 241–253.

[12] I. Laine. *Nevanlinna theory and complex differential equations*. De Gruyter Studies in Mathematics, de Gruyter, Berlin., 1993, **15**.

[13] I. Laine, C.-C. Yang. *Value distribution of difference polynomials*. Proc. Japan Acad. Ser. A Math. Sci., 2007, **83**(8): 148–151.

[14] S. H. Lin, W. C. Lin. *Uniqueness of meromorphic functions concerning weakly weighted-sharing*. Kodai Math. J., 2006, **29**(2): 269–280.

[15] W. C. Lin, H.-X. Yi.*Uniqueness theorems for meromorphic functions concerning fixed-points*. Complex Var. Theory Appl., 2004, **49**(11): 793–806.

[16] X. Q. Lin, W. C. Lin. *Uniqueness of entire functions sharing one value*. Acta Math. Sci. Ser. B (Engl. Ed.)., 2011, **31**(3): 1062–1076.

[17] K. Liu, L. Yang. *Value distribution of the difference operator*. Arch. Math. (Basel)., 2009, **92**(3): 270–278.

[18] C. Meng. *Uniqueness of entire functions concerning difference polynomials*. Math. Bohem., 2014, **139**(1): 89–97.

[19] X.G. Qi, L. Yang, K. Liu. *Uniqueness and periodicity of meromorphic functions concerning the difference operator*. Comput. Math. Appl., 2010, **60** (6): 1739–1746.

[20] P. Sahoo. *Unicity theorem for entire functions sharing one value*. Filomat., 2013, **27**(5): 797–809.

[21] P. Sahoo. *Uniqueness of entire functions related to difference polynomials*. Commun. Math. Stat., 2015, **3**(2): 227–238.

[22] P. Sahoo, S. Seikh. *Meromorphic functions whose certain differential polynomials share a small function with finite weight*. Analysis (Munich)., 2013, **33**(2): 143–157.

[23] P. Sahoo, G. Biswas. *Some results on uniqueness of entire functions concerning difference polynomials*. Tamkang J. Math., 2018, **49**(2): 85–97.

[24] P. Sahoo, H. Karmakar. *Results on uniqueness of entire functions whose certain difference polynomials share a small function*. Anal. Math., 2015, **41**(4): 257–272.

[25] H. P. Waghamore, S. B. Vijaylaxmi. *Weighted sharing of a set and uniqueness of meromorphic functions with derivatives*. Tbilisi Math. J., 2021, **14**(3): 1–15.

[26] C.-C. Yang, X. H. Hua, *Uniqueness and value-sharing of meromorphic functions*. Ann. Acad. Sci. Fenn. Math., 1997, **22**(2): 395–406.

[27] C.-C. Yang, H.-X. Yi. *Uniqueness theory of meromorphic functions*. Mathematics and its Applications. Kluwer Acad. Publ., Dordrecht., 2003, **557**.

[28] R. R. Zhang, C. X. Chen, Z. B. Huang.*Uniqueness on linear difference polynomials of meromorphic functions*. AIMS Math., 2021, **6**(4): 3874–3888.

[29] J. Zhang. *Value distribution and shared sets of differences of meromorphic functions.* J. Math. Anal. Appl., 2010, **367**(2): 401–408.

[30] J. Zhang, L. Yang. *Some results related to a conjecture of R. Brück.* JIPAM. J. Inequal. Pure Appl. Math., 2007, **8**(1): 18, 11 pp.