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A Comparison of Approaches to the Riemann Hypothesis

Darrell Cox[1] **and Deep Bhattacharjee**[2]

[1] *Department of Mathematics, Grayson County College, United States.*
[2] *Integrated Nanosciences Research, India.*

[1]*E-mail— darkencox273@gmail.com*

Abstract

The comparison involves Riemann's epsilon function, Barnes G-function, Selberg's zeta function, and Jacobi theta functions relevant to Brownian motion. A Selberg zeta function is shown to be equivalent to Riemann's epsilon function (by way of empirical evidence).

***Keywords* —** Riemann zeta function, gamma function, logarithmic spirals

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.)

Equation (1) in section 1.8 of Edwards' book is

$$\epsilon(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s) \tag{2}$$

The equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3}$$

can be substituted into the right-hand side of the above equation. A plot of $\epsilon(s)$ for

$s = (0.5, 14.13472514)$ (the first non-trivial zeta function zero) and $n = 1$ to 10000 is

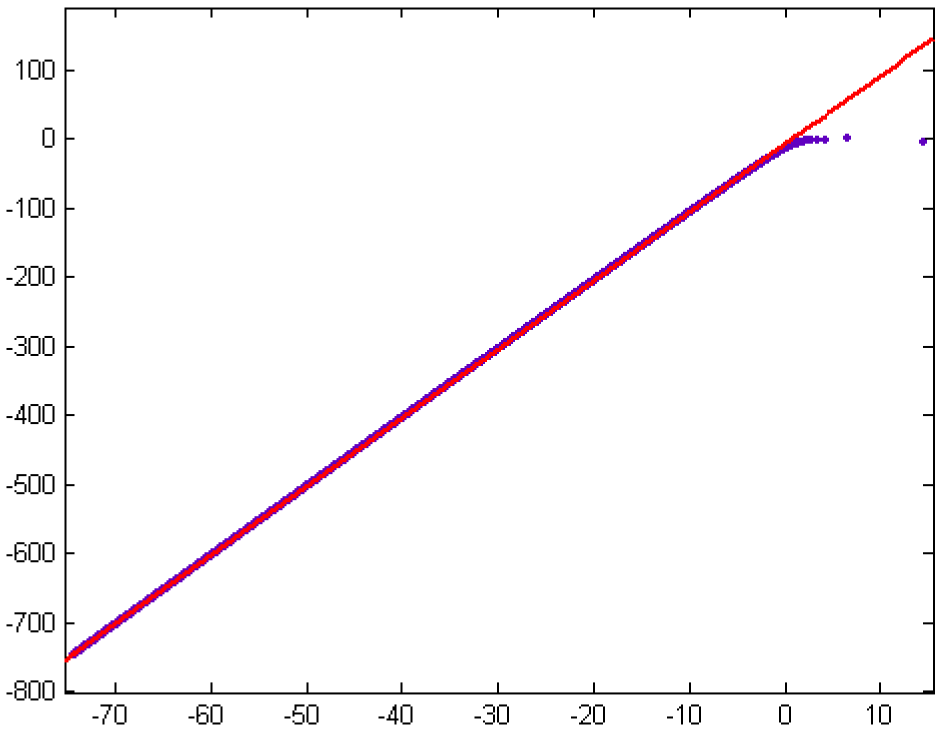

Figure 1

The slopes and intercepts for the first ten zeta function zeros are (9.912, -8.387), (14.34, 19.94), (-3.097, 1.78), (0.4184, -2.286), (-0.9089, -3.14), (0.3129, 2.809), (-2.586, 8.104), (1.143, -4.845), (-3.122, -11.28), and (1.879, 7.79). These "straight lines" are characteristic of the zeta function zeros.

These apparently random "straight lines" can be "normalized" by using the following function

$$\zeta'(s) = \frac{2\pi\zeta(s)}{\Pi(-s)} \tag{4}$$

A plot of $\zeta'(s)$ for the first zeta function zero is

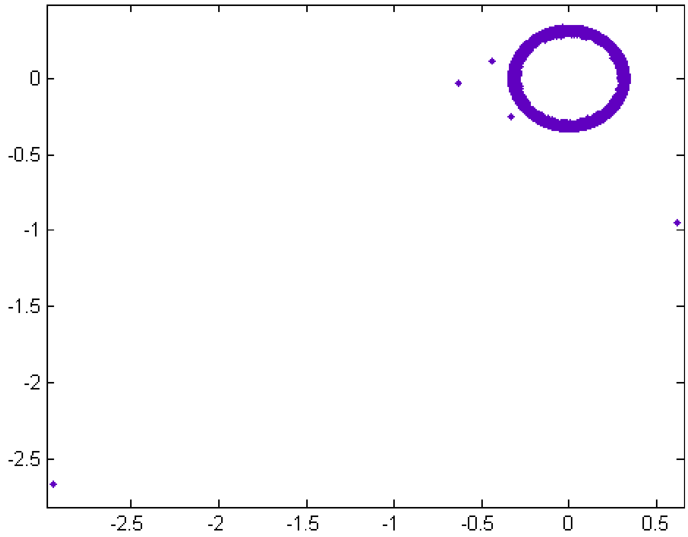

Figure 2

A plot of the logarithms of the n values of the inflection points is

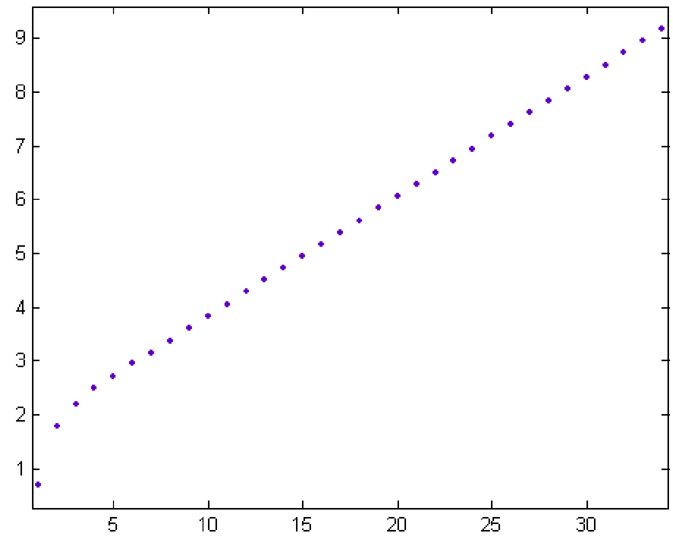

Figure 3

The slopes are half of what they are for the zeta function. A plot of the real and imaginary components versus n is

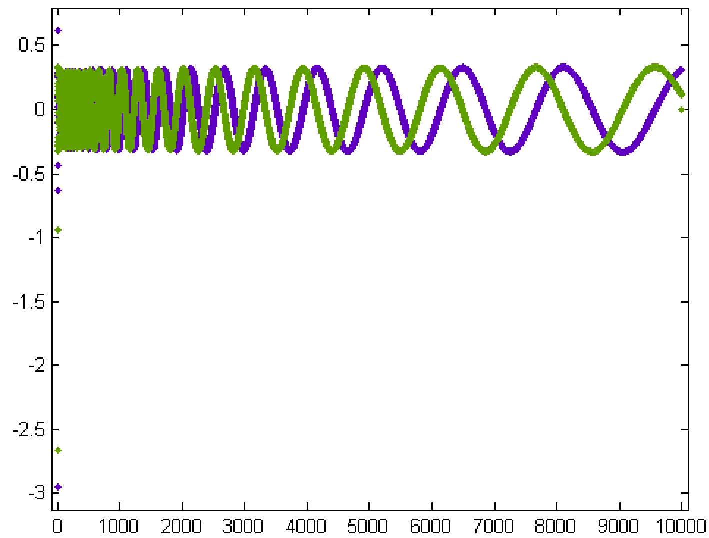

Figure 4

Note that the amplitude is less than 1/2 after the first few n values.

A plot of $\epsilon(s)$ when $\zeta'(s)$ is substituted for $\zeta(s)$ is

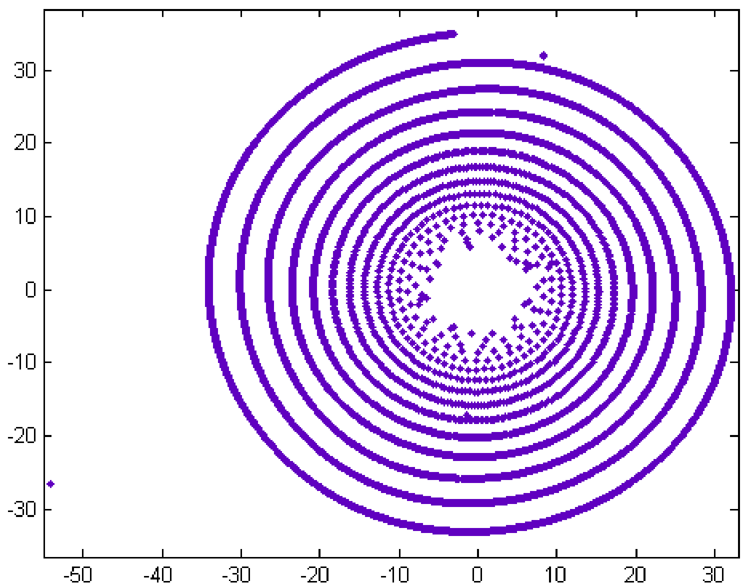

Figure 5

This logarithmic spiral is essentially the same as that for the zeta function.

2. BARNES G-FUNCTION AND THE RIEMANN HYPOTHESIS

An equation given in Kargin's [2] article relating $\zeta(s)$ to the series $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ is

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-n^2\pi x} dx = \int_0^{\infty} x^{s/2-1}\theta(x)dx \tag{5}$$

In the following, the partition function is $\{\lambda_k\}$=$\{n$ with multiplicity $n\}$, n=1, 2, 3,.... Equation A.16 in Vores [3] article is

$$\Theta^B(t) = \sum_{n=1}^{\infty} ne^{-tn} = -\frac{d}{dt}(\frac{1}{e^t - 1}) \tag{6}$$

Equation A.17 (the reflection formula for $\Gamma(z)$) is

$$Z^B(s) = \sum_{n=1}^{\infty} n \cdot n^{-s}\zeta(s-1) \tag{7}$$

In the following, the denominator of $\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}}$ is deleted to be consistent with the partition function. This leads to

$$\Pi(\frac{s}{2})\Theta^B(s)\zeta(s) \tag{8}$$

A plot of these values for $s = (0.5, 14.13472514)$ and $n <= 1000$ is

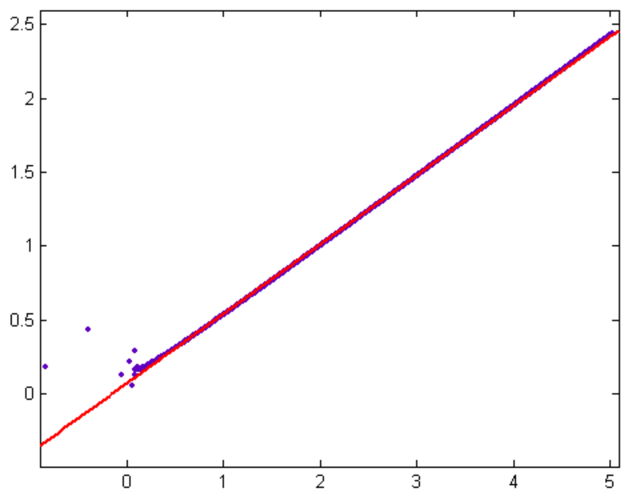

Figure 6

For the first 10 zeta function zeros, the slopes and intercepts of the lines are (0.4678, -0.07512), (0.2227, 0.05083), (0.4881, -0.6597), (-1.201, -0.194), (0.5112, 0.08258), (0.4429, -0.7009), (-0.05459, 0.03992), (-3.224, 0.6897), (-0.2667, 0.04443), and (0.3, $-\infty$). The line corresponding to the tenth zeta function zero is almost vertical.

These apparently random "straight lines" can be normalized by using the following function

$$\zeta'(s) = \frac{2\pi\zeta(s)}{\Pi(1-s)Z^B(s)} \tag{9}$$

A plot of $\zeta'(s)$ for the first zeta function zero and $n = 10$ to 1000 is

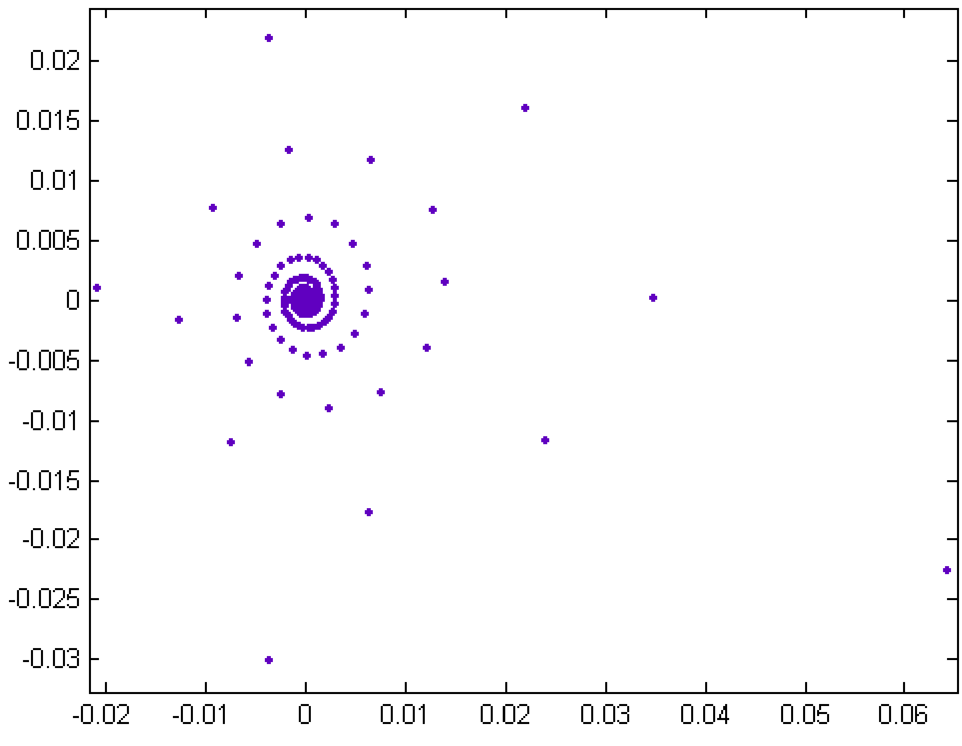

Figure 7

A plot of the logarithms of the n values ($n = 1$ to 1000) of the inflection points is

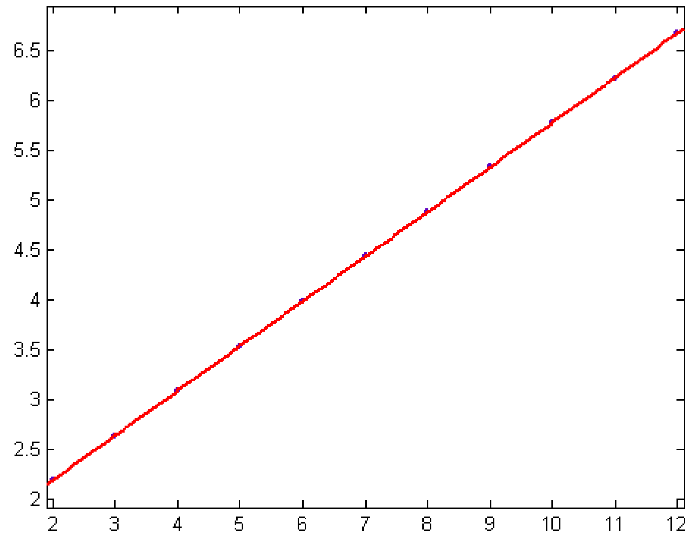

Figure 8

The slope and intercept is (0.4482, 1.298), approximately equal to that for the usual zeta function. A plot of the real components versus n is

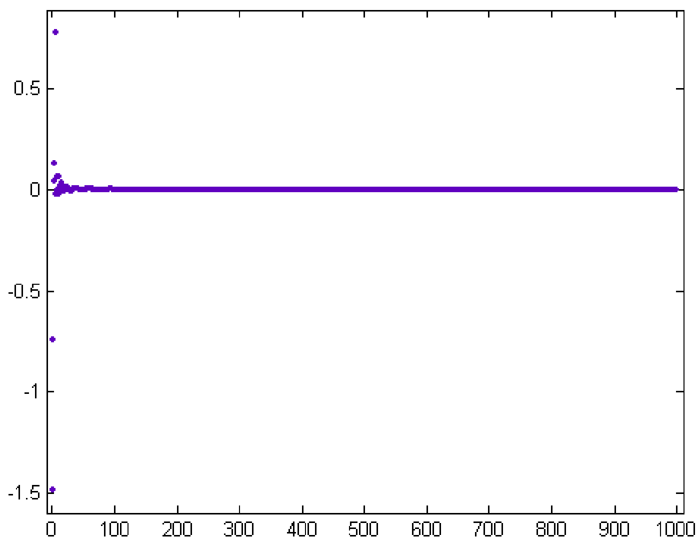

Figure 9

A plot of the imaginary components versus n is

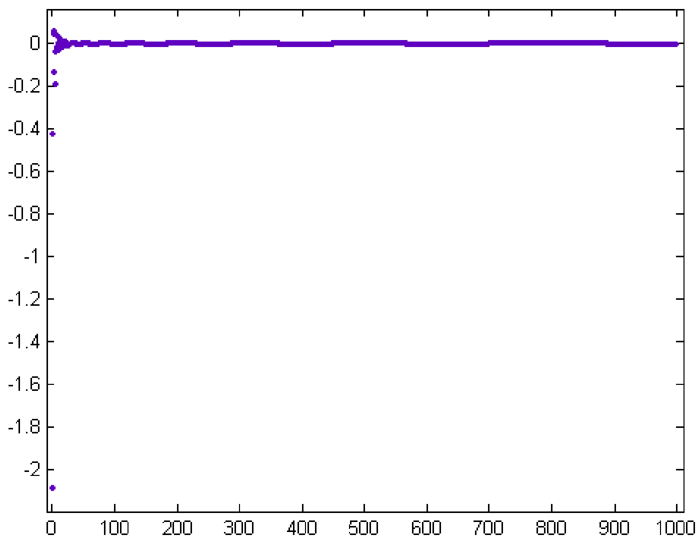

Figure 10

A plot of $\Pi(\frac{1}{2}s)\Theta^B(s)\zeta'(s)$ for $n = 10$ to 1000 is

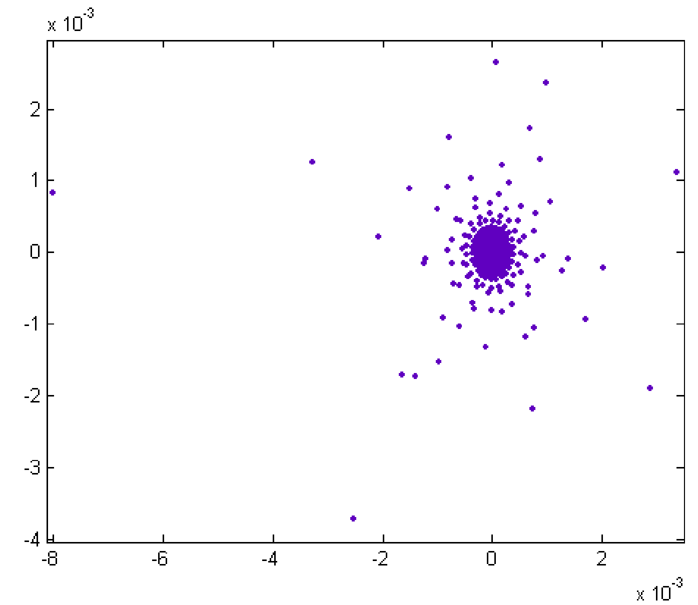

Figure 11

A plot of the logarithms of the n values of the inflection points (for $n = 1$ to 1000) is

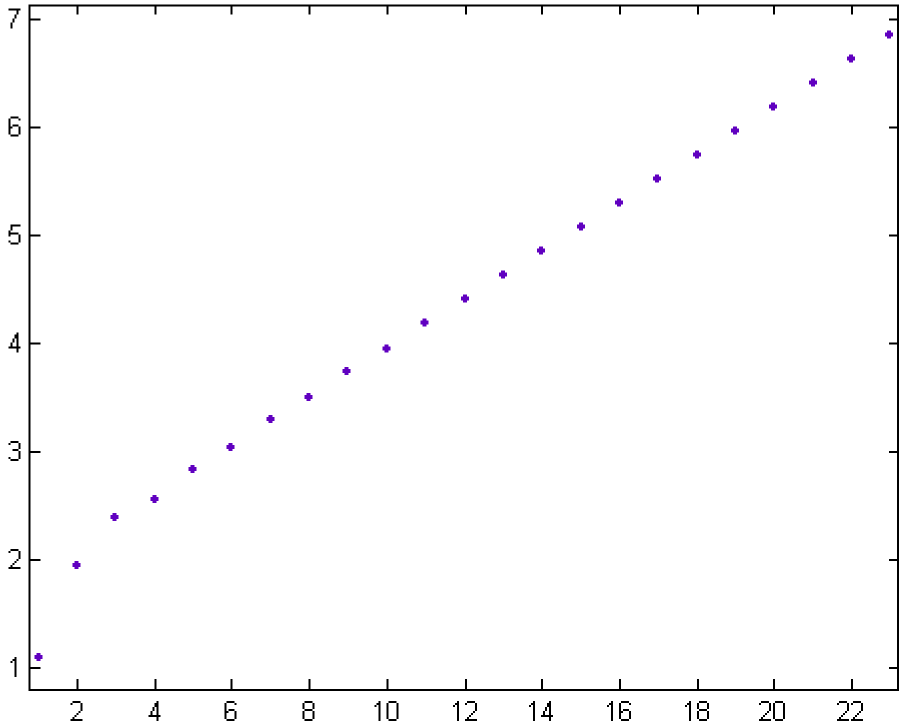

Figure 12

The slope and intercept is (0.2237, 1.73). The slope is about half that of the usual zeta function.

A smaller maximum n value (1000) was chosen to avoid overflow of the exponential function. Also, there are far fewer inflection points than for the zeta function. Sometimes, the first few n values have to be omitted to get an accurate slope (which makes the intercept less accurate). This accounts for some of the differences.

3. SELBERG'S ZETA FUNCTION AND THE RIEMANN HYPOTHESIS

An equation given in Kargin's [2] article relating $\zeta(s)$ to the series $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$ is

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-n^2 \pi x} dx = \int_0^{\infty} x^{s/2-1} \theta(x) dx \tag{10}$$

A plot of $\frac{\Pi(\frac{s}{2})\zeta(s)}{\pi^{s/2}}$ for $s = (0.5, 14.13472514)$ and $n <= 10000$ is

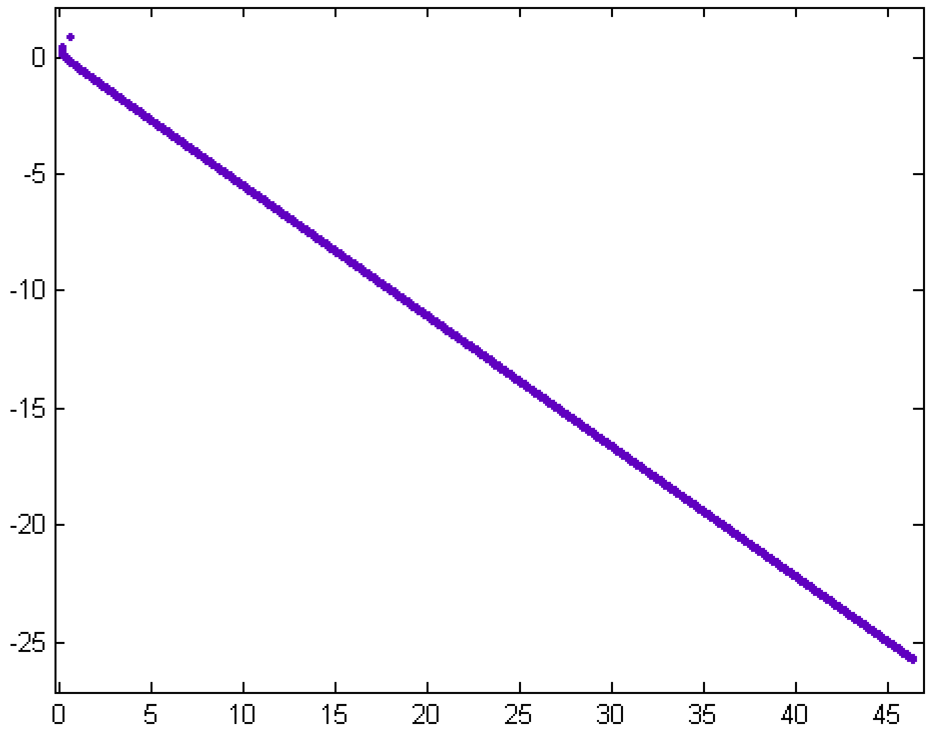

Figure 13

The slopes and intercepts for the first 10 zeta function zeros are ($-.5575$, 0.06781), (1.716, -0.1299), (-0.3969, 0.07313), (-0.2973, 0.07223), (-0.01981, -0.07067), (1.371, -0.1215), (0.2749, 0.07461), (0.7656, -0.09264), (-72.03, 3.749), and (-0.4733, -0.08077). These "straight lines" are apparently random.

The partition function is related to a Jacobi theta function. Equation 6.1 in Vores [3] article is

$$\Theta(t) = \sum_{1}^{\infty} e^{-tk^2} = (\Theta_3(0|\frac{it}{\pi}) - 1)/2 \tag{11}$$

The definition of $\Theta_3(z|\tau)$ is

$$\Theta_3(z|\tau) = (-i\tau)^{-1/2} e^{z^2}/i\pi\tau\Theta_3(\frac{z}{i}| - \frac{1}{\tau}) \tag{12}$$

Equation 6.5 (the reflection formula for $\Gamma(z)$) is

$$Z(s) = \sum_{k=1}^{\infty} k^{-2s}\zeta(2s) \tag{13}$$

In the following, the denominator of $\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}}$ is deleted to be consistent with the partition function. This leads to the expression

$$\Pi(\frac{s}{2})\zeta(s) \tag{14}$$

$\frac{\Pi(-s)}{2\pi}$ is multiplied by the above expression to normalize the above "straight lines". This gives the expression

$$\zeta'(s) = \frac{2\pi\Pi(\frac{s}{2})\zeta(s)}{\Pi(-s)} \tag{15}$$

A plot of this function for the first zeta function zero is

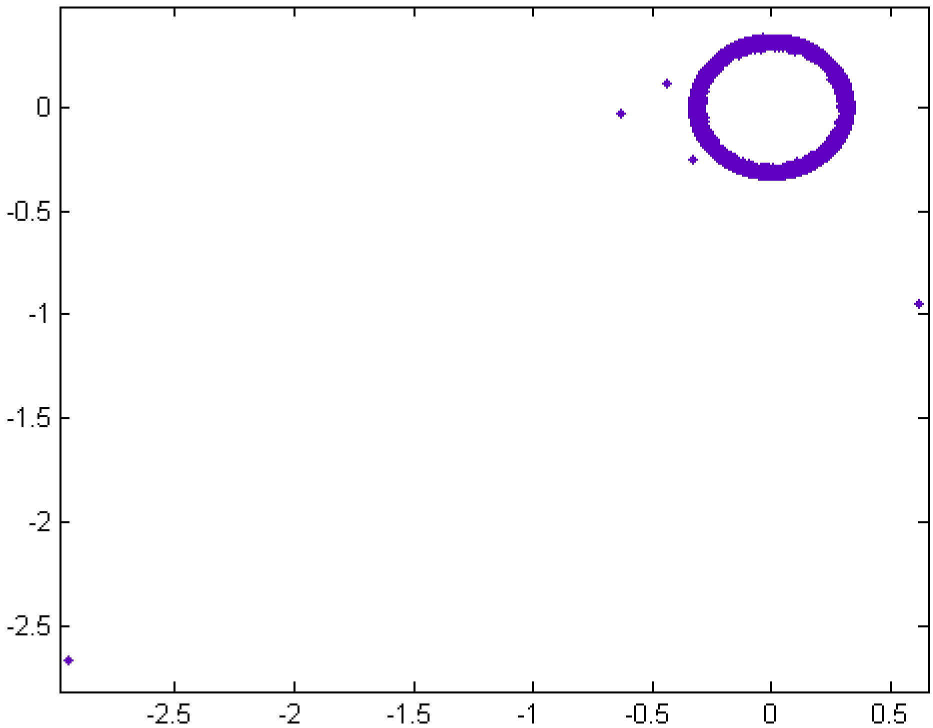

Figure 14

Note that this plot is identical to the plot given by the Riemann ϵ function (figure (2)). A plot of the logarithms of the n values of the inflection points is

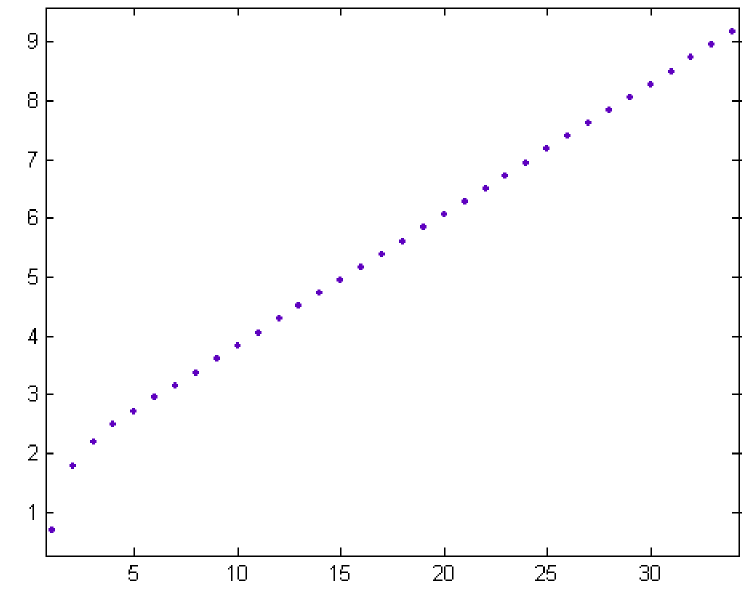

Figure 15

The slope and intercept is (0.2227, 1.598). This is about half that of the usual zeta function. A plot of the real and imaginary components versus n is

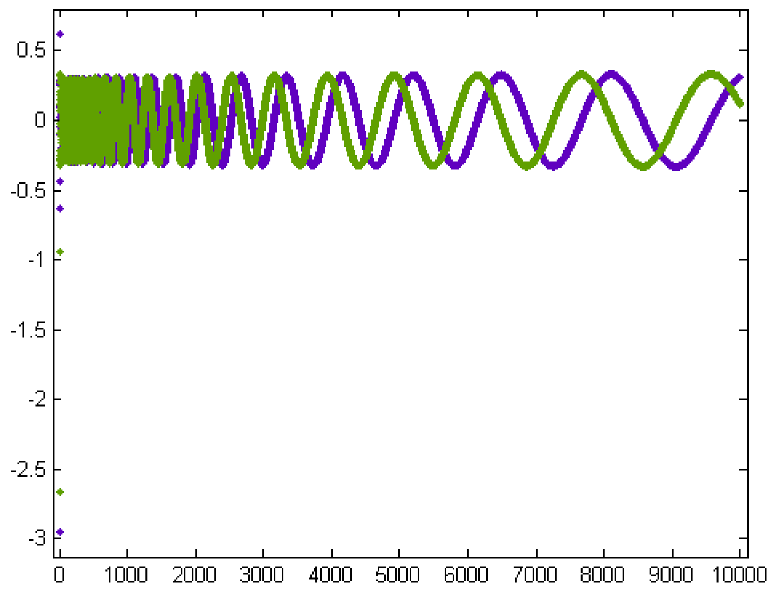

Figure 16

A plot of $\Pi(\frac{s}{2})\zeta'(s)$ for the first zeta function zero is

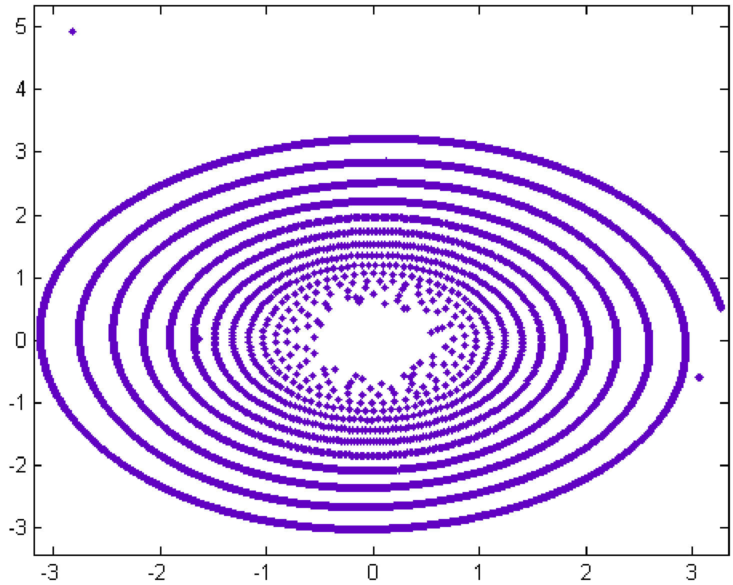

Figure 17

A plot of the logarithms of the n values of the inflection points is

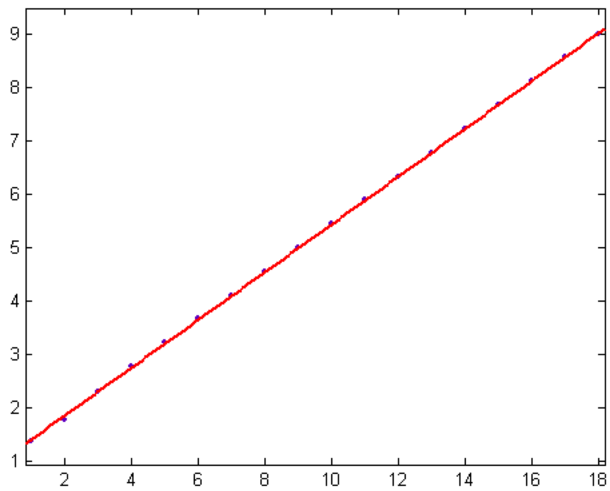

Figure 18

The slope and intercept is (0.4474, 0.9601). The is the about the same as for the usual zeta function.

A plot of $\zeta'(s)$ for the tenth zeta function zero is

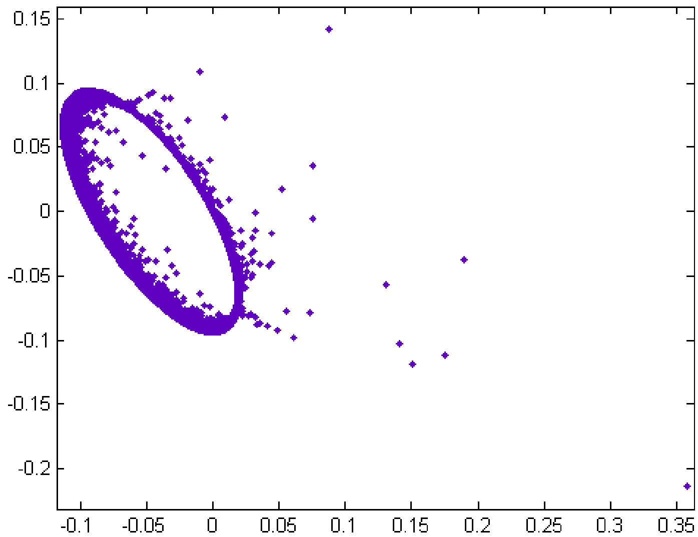

Figure 19

A plot of the logarithms of the n values of the inflection points is

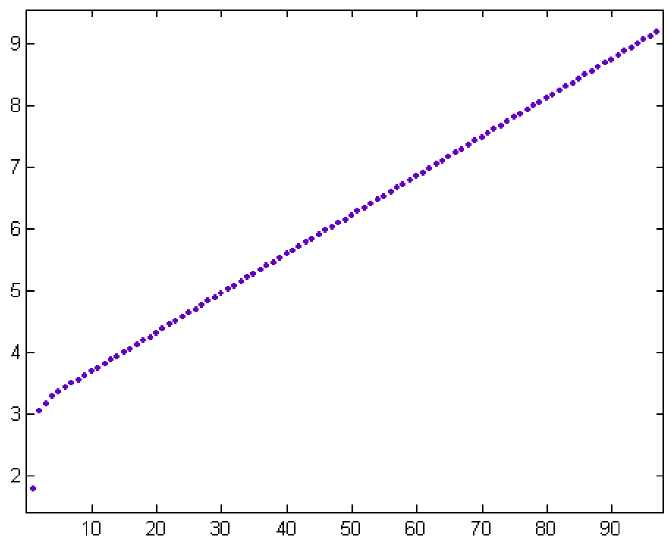

Figure 20

The slope and intercept is (0.06323, 3.053). This is about the same as for the usual zeta function.

A plot of $\Pi(\frac{s}{2})\zeta'(s)$ for the tenth zeta function zero is

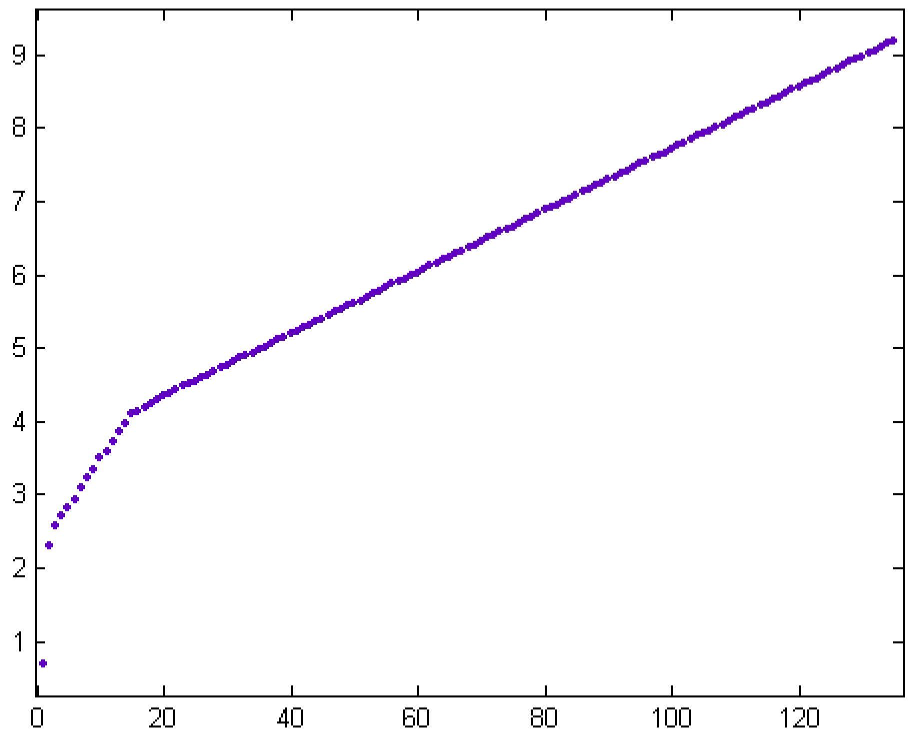

Figure 21

The slope and intercept of the first line segment is (0.1286, 2.185). The slope and intercept of the second line segment is (0.04221, 3.495). The slope of the first line segment is about the same for the usual zeta function.

The above random "straight lines" can also be normalized by using the following function involving the reflection formula

$$\zeta'(s) = \frac{2\pi\zeta(s)}{\Pi(1-s)Z(1-s)} \tag{16}$$

A plot of this expression for the first zeta function zero and $n \leq 10000$ is

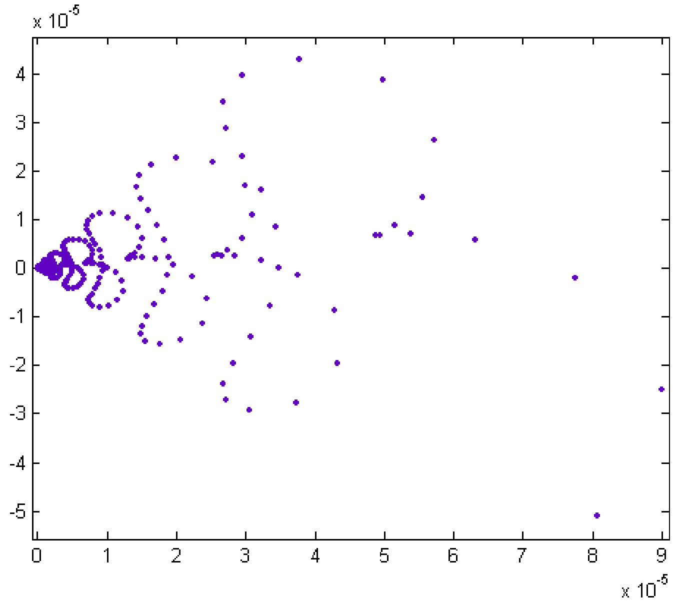

Figure 22

A plot of the logarithms of the n values of the inflection points is

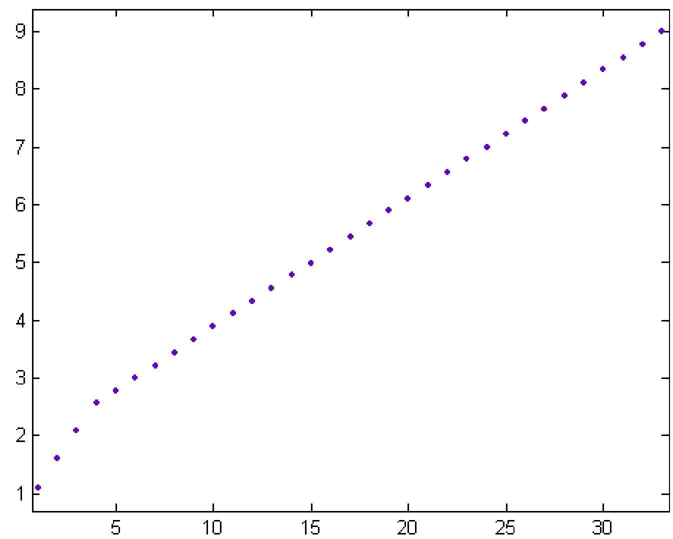

Figure 23

The slope and intercept is (0.2224, 1.662) (starting with the fourth n value). The slope is about half of that for the usual zeta function.

A plot of $\Pi(\frac{s}{2})\zeta'(s)$ is

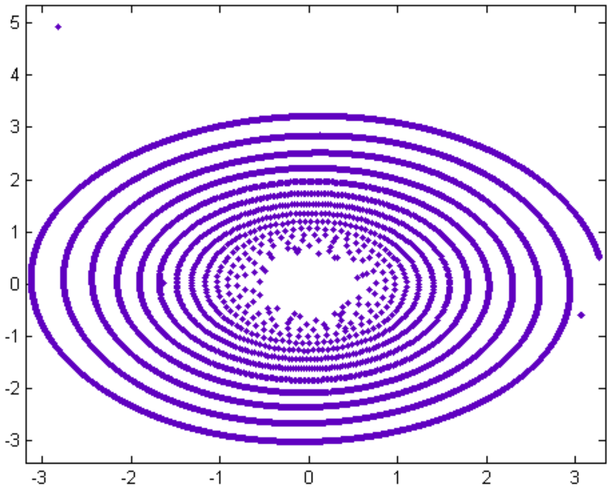

Figure 24

A plot of the logarithms of the n values of the inflection points is

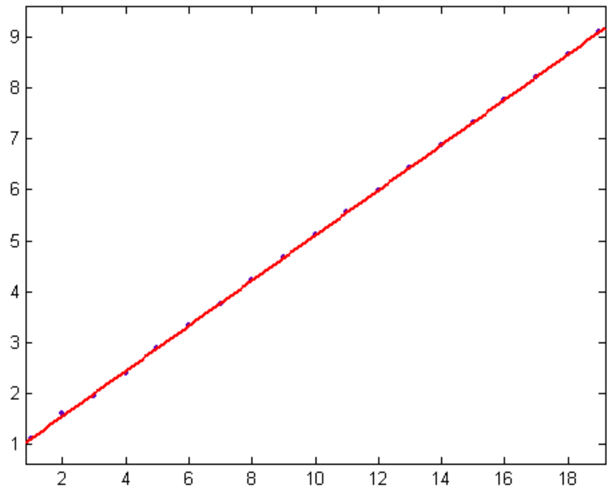

Figure 25

The slope and intercept is (0.445, 0.6557). This is about equal to that for the usual zeta function.

4. BROWNIAN MOTION AND THE RIEMANN HYPOTHESIS

An equation given in Kargin's [2] article relating $\zeta(s)$ to the series $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ is

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty}\int_0^{\infty} x^{s/2-1}e^{-n^2\pi x}dx = \int_0^{\infty} x^{s/2-1}\theta(x)dx \tag{17}$$

A plot of $\frac{\Pi(\frac{s}{2})\zeta(s)}{\pi^{s/2}}$ for $s = (0.5, 14.13472514)$ and $n <= 10000$ is

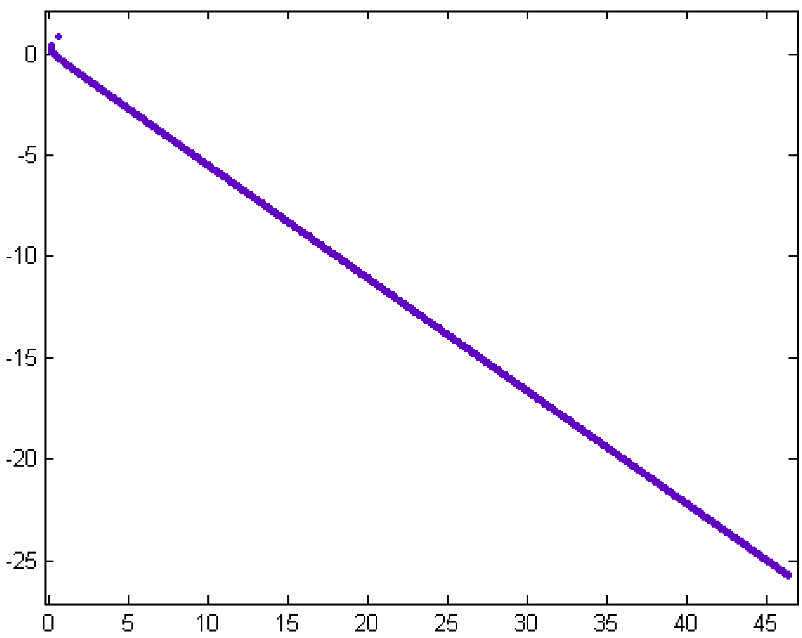

Figure 26

This representation can be used as a starting point for a connection between the Riemann zeta function and Brownian motion and Bessel processes. See Biane, et. al. [4] for details. The slopes and intercepts for the first 10 zeta function zeros are ($-.5575$, 0.06781), (1.716, -0.1299), (-0.3969, 0.07313), (-0.2973, 0.07223), (-0.01981, -0.07067), (1.371, -0.1215), (0.2749, 0.07461), (0.7656, -0.09264), (-72.03, 3.749), and (-0.4733, -0.08077).

The apparently random " straight lines" associated with the Jacobi theta function can be explained by multiplying by $\frac{\Pi(-s/2)}{2\pi i}$. This gives

$$\zeta'(s) = \frac{2\pi\zeta(s)}{\Pi'(-s/2)} \tag{18}$$

A plot of this for the first zeta function zero is

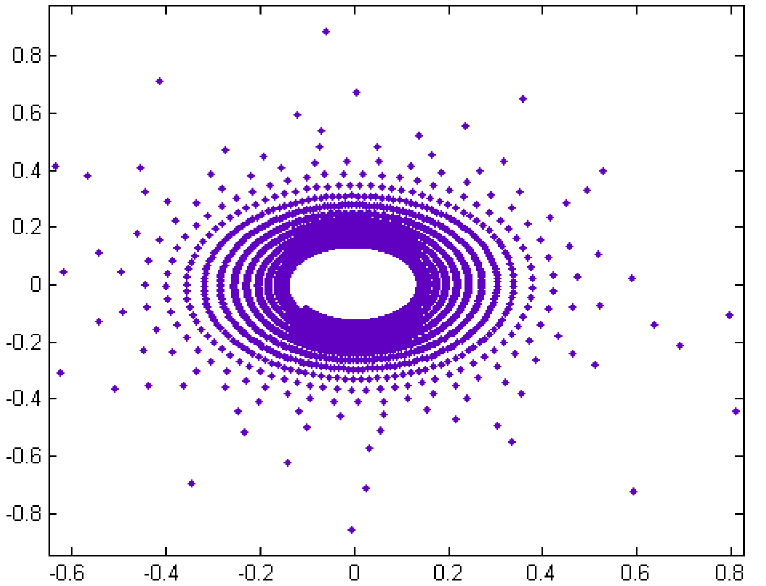

Figure 27

A plot of the real and imaginary components versus n is

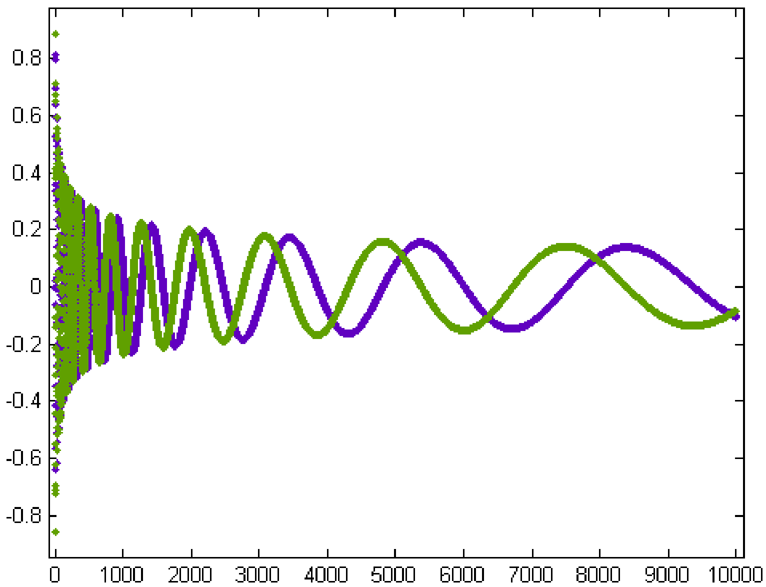

Figure 28

A plot of the logarithms of the n values of the inflection points is

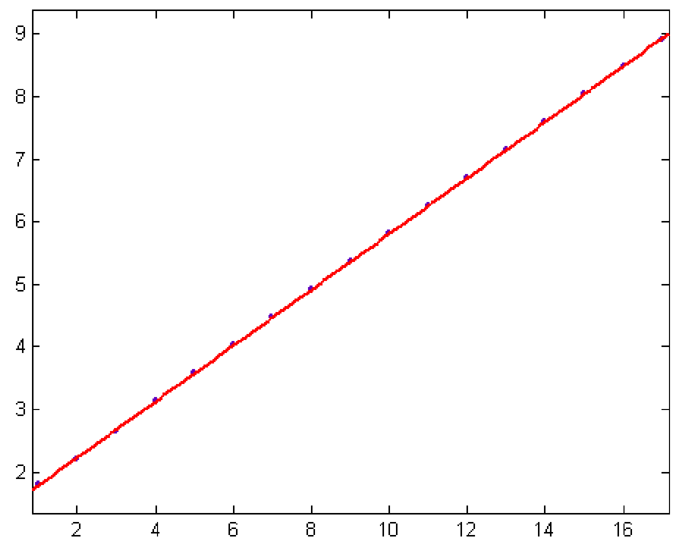

Figure 29

The slope and intercept is (0.4471, 1.334).

A plot of $\frac{\Pi(\frac{1}{2}s)\zeta'(s)}{\pi^{s/2}}$ is

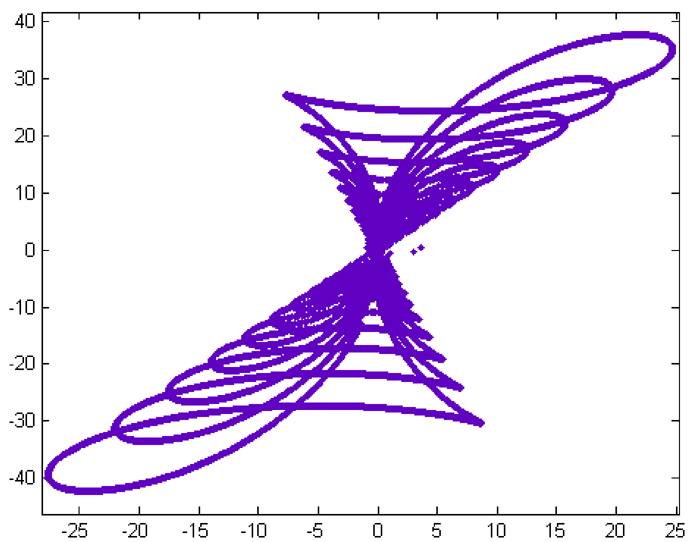

Figure 30

A plot of the logarithms of the n values of the inflection points is

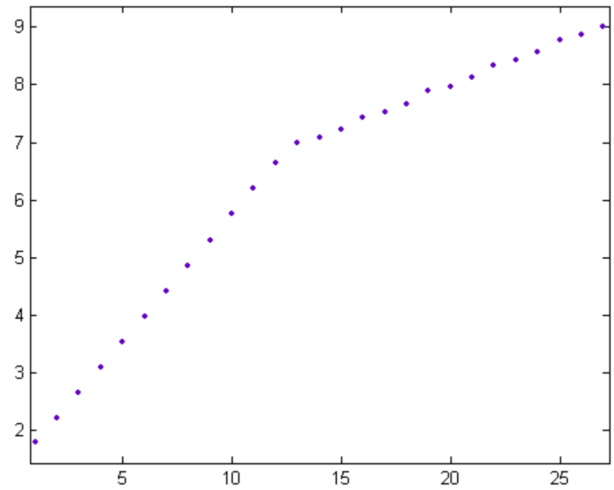

Figure 31

The slope and intercept of the first line segment is (0.4401, 1.333), about the same as that for the usual zeta function. The slope and intercept of the second line segment is (0.1482, 5.025).

A plot of $\frac{\Pi(\frac{1}{2}s)\zeta'(s)}{\pi^{s/2}}$ for the tenth zeta function zero is

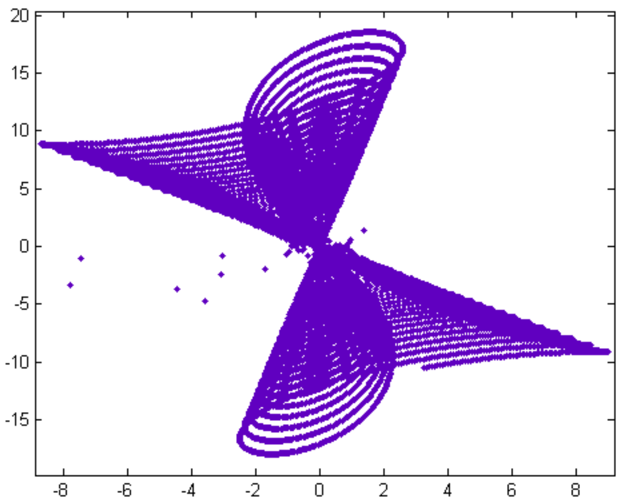

Figure 32

A plot of the logarithms of the n values of the inflection points is

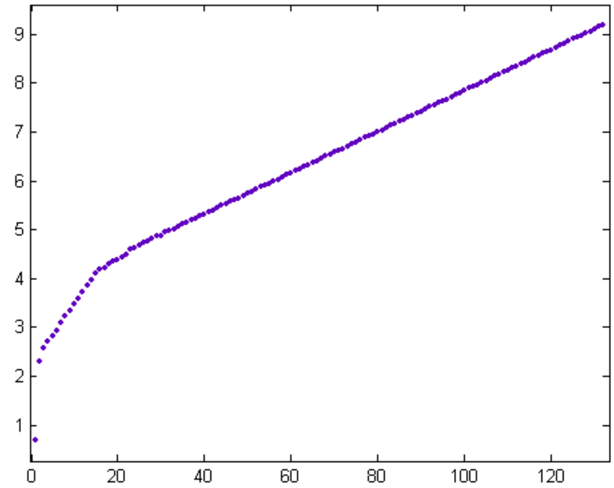

Figure 33

The slope and intercept of the first line segment is (0.1269, 2.194), about the same as that for the tenth zeta function. The slope and intercept of the second line segment is (0.04235, 3.605).

5. CONCLUSION

A Selberg zeta function is identical to Riemann's ϵ function.

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