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Variants of the Gamma Function and Logarithmic Spirals

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Abstract

Relationships between variants of the gamma function and logarithmic spirals are investigated. In particular, a relationship between the radii of logarithmic spirals associated with a variant of the gamma function and the imaginary components of the Riemann zeta function zeros is investigated. One of the variant gamma functions is relevant to the definition of the zeta function derived by Riemann. The zeta function zeros appear to have an invariance property.
Keywords: Riemann zeta function, Euler's totient function, gamma function, logarithmic spirals, invariance property.

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.) An expression given in section 10 of Chapter 9.6 of Edwards' [2] book is

$$\lim_{N\to\infty} \prod_{n=1}^{N}(1+\frac{x}{n\pi})N^{-x/\pi} \tag{2}$$

The limit exists for all x and defines a function $F(x)$ which satisfies $\sin x = x \cdot F(x) \cdot F(-x)$. Edwards defines a simpler limit by setting $y = x/\pi$ and considers

$$\lim_{N\to\infty} \Pi_{n=1}^{N}(1+\frac{y}{n})N^{-y}. \tag{3}$$

For $y = 1, 2, 3, \ldots$ this limit exists and is equal to $1/y!$. Thus if one defines a function $\Pi(y)$ by the equation

$$\frac{1}{\Pi(y)} = \lim_{N\to\infty} \Pi_{n=1}^{N}(1+\frac{y}{n})N^{-y} \tag{4}$$

whenever this limit exists and is not zero, the function $\Pi(y)$ is an extension of the factorial function.

2. EULER'S PHI FUNCTION AND EXISTENCE OF LIMITS

A plot of the logarithms of the values of the last expression above when $x = 6$ and $N = 10$ for $n = 1, 2, 3, \ldots, 6$ is

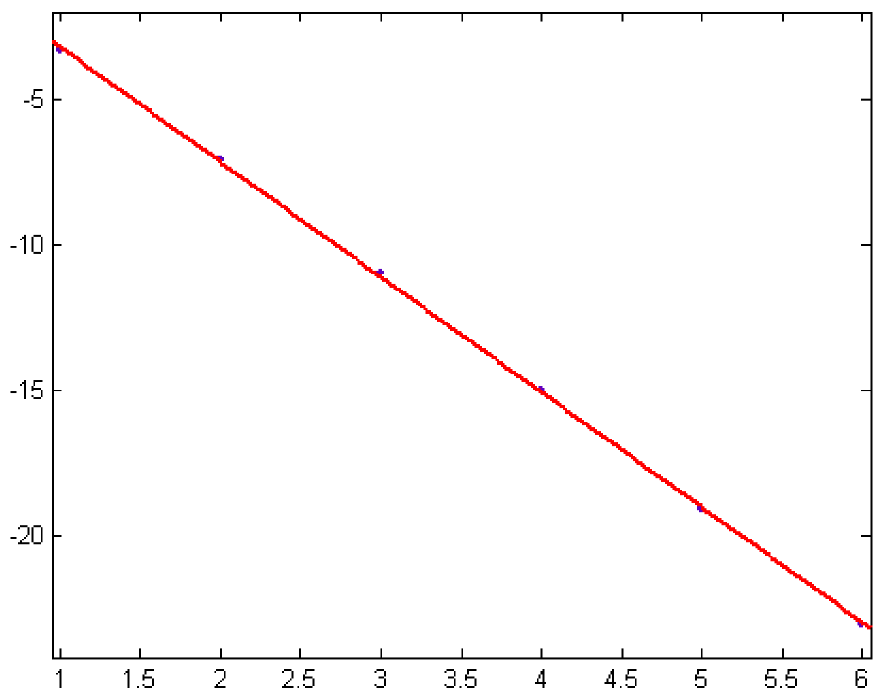

Figure 1

For a linear least-squares fit of the curve, $p_1 = -3.955$ with a 95% confidence interval of $(-4.034,\ -3.876)$, $p_2 = 0.7796$ with a 95% confidence interval of (0.4712, 1.088), SSE=0.05695, R-squared=0.9998, and RMSE=0.9993.

Let $\varphi(N)$ denote Euler's totient function. The limit also apparently exists when n is replaced with $\varphi(n)$. A plot of the logarithms of the values for the same parameters as above is

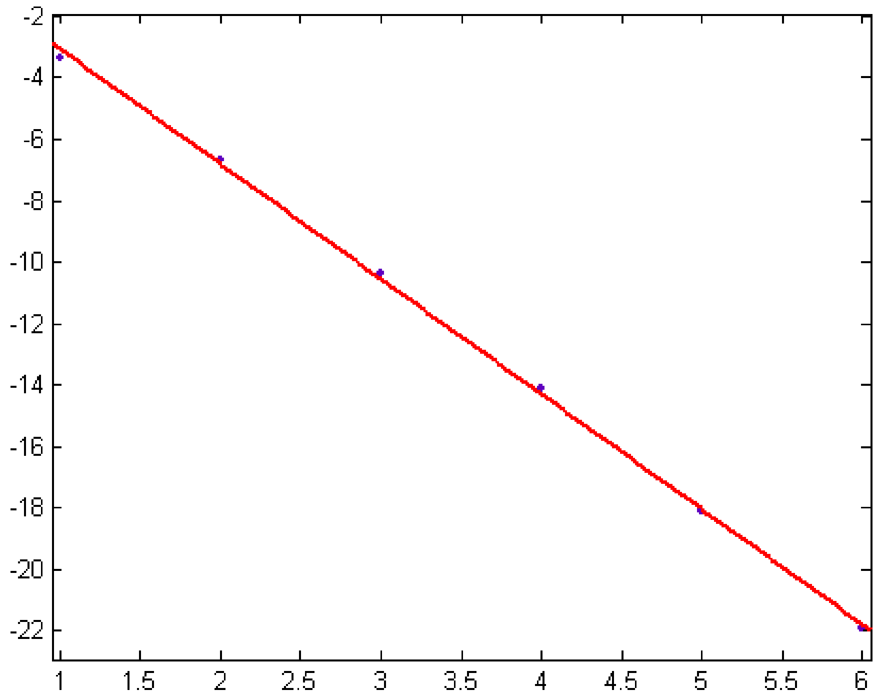

Figure 2

For a linear least-squares fit of the curve, $p_1 = -3.746$ with a 95% confidence interval of $(-3.887, -3.605)$, $p_2 = 0.6874$ with a 95% confidence interval of (0.1384, 1.236), SSE=0.1804, R-squared=0.9993, and RMSE=0.2124.

To partially explain this, it is necessary to reproduce the proof (by Edwards) that the limit exists for all values of y. Rewriting the limit as

$$[1+y][(1+\frac{y}{2})(\frac{2}{1})^{-y}]\cdots[(1+\frac{y}{n})(\frac{N}{N-1})^{-y}] \tag{5}$$

the problem is to show that the product $a_1a_2a_3\cdots$ with

$$a_n = (1+\frac{y}{n})(\frac{n-1}{n})^y = (1+\frac{y}{n})(1-\frac{1}{n})^y \tag{6}$$

converges. Now

$$a_n = (1+\frac{y}{n})(1-\frac{y}{n} + terms\ in\ \frac{1}{n^2}, \frac{1}{n^3}, \ldots) \tag{7}$$

$$= 1 + terms\ in\ \frac{1}{n^2}, \frac{1}{n^3}, \ldots \tag{8}$$

hence for large values of n the factors a_n are like $(1 + \frac{const.}{n^2})$, which indicates that their product converges. To make this rigorous, it must be proved that there is a constant K such that

$$|\log(1 + x) - x| < Kx^2 \tag{9}$$

for all sufficiently small x, hence that $|\log a_n| < K(y^2 + |y|)\frac{1}{n^2}$, hence that $\sum_{n=N}^{\infty} \log a_n$ converges, and hence that $\prod_{n=1}^{\infty} a_n$ converges. The product is zero only if a factor is zero.

The limit involving the totient function exists because $(1 + \frac{y}{\varphi(n)})(1 - \frac{y}{\varphi(n)})$ is less than or equal to $(1 + \frac{y}{n})(1 - \frac{y}{n})$. This is due to $\varphi(n)$ being less than n except when $n = 1$. The $\frac{1}{\varphi(n)^2}$, $\frac{1}{\varphi(n)^3}$,... terms are larger than the $\frac{1}{n^2}$, $\frac{1}{n^3}$,... terms, so a rigorous proof would still be difficult. Of course, there is a multitude of functions $f(n)$ where $f(n) < n$. Why this particular function is significant is unknown.

3. A VARIANT GAMMA FUNCTION

In a variant of the gamma function, $\varphi(N)$ is substituted for N in equation (3) of section 1.3 of Edwards' book. In this case the gamma function is denoted by $\Pi'(s)$.

Theorem (6) in section 1.3 of Edwards' first book is

Theorem 1. $\frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s$

Proof (Edwards): $\pi y/\Pi(y)\Pi(-y)$ is the limit of $\pi y \prod_{n=1}^{N}(1 - (y^2/n^2))$ which is $\sin \pi y$. Using $N - 1$ in the formula for $\Pi(y)$ shows that $y\Pi(y - 1)\Pi(y)$ is the limit as $N \to \infty$ of $(1 - (1/N))^{-y}$ which is 1.

A plot of $\frac{\sqrt{\pi s}}{\Pi'(s)}$ for $s = (0.5, 14.1347251)$ (the first non-trivial Riemann zeta function zero) and $N = 1$ to 685 is

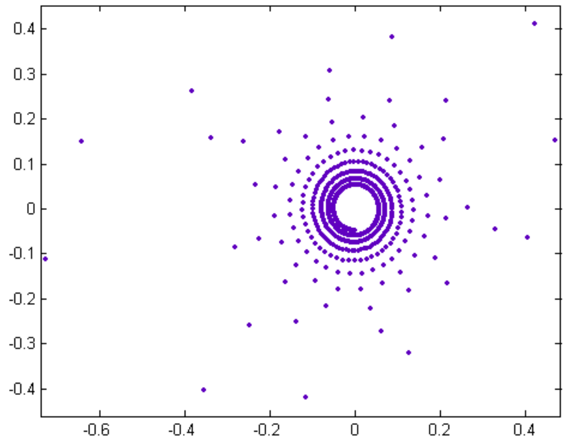

Figure 3

This is an inward logarithmic spiral. The final N value of 685 was chosen so that the real component is close to zero and at an inflection point. The N values for all such inflection points are 3, 5, 8, 12, 19, 30, 47, 74, 116, 180, 282, 439, and 685. A plot of logarithms of these values is

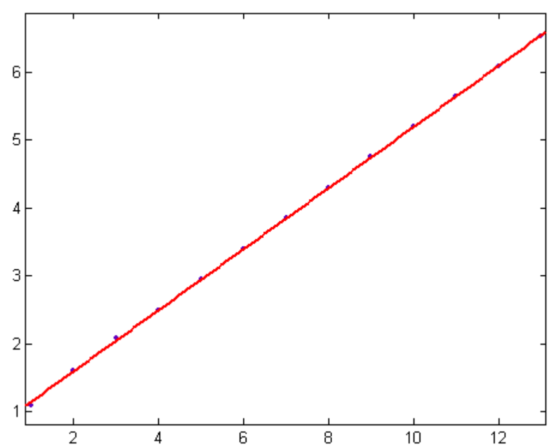

Figure 4

For a linear least-squares fit of the curve, $p_1 = 0.4498$ with a 95% confidence interval of (0.4466, 0.4529), $p_2 = 0.6959$ with a 95% confidence interval of (0.6706, 0.7213), SSE=0.004217, R-squared=0.9999, and RMSE=0.01958.

The spirals are centered on (0,0). A plot of $\frac{\sqrt{\pi s}}{\Pi'(s)}$ for $s = (0.5, 21.0220396)$ (the second non-trivial Riemann zeta function zero) and $N = 100$ to 1000 is

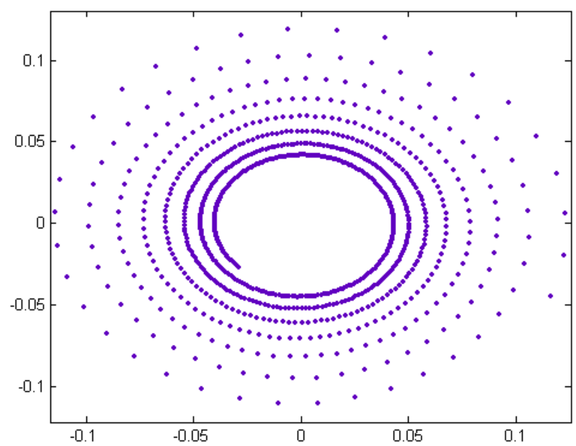

Figure 5

A plot of the real and imaginary components versus N for $N = 1$ to 1000 is

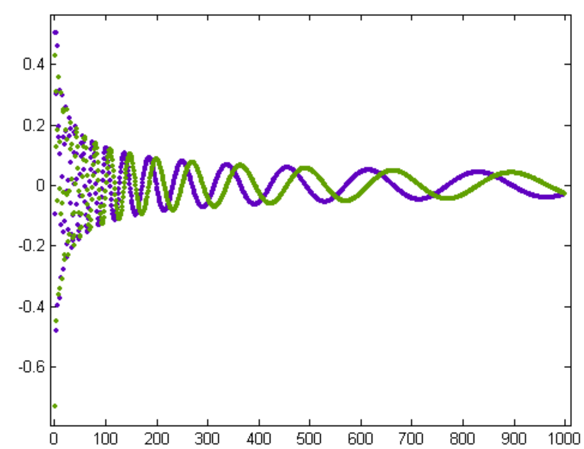

Figure 6

A plot of $\frac{\sqrt{\pi s}}{\Pi'(s)}$ for $s = (0.25, 10.0)$ and $N = 100$ to 1000 is

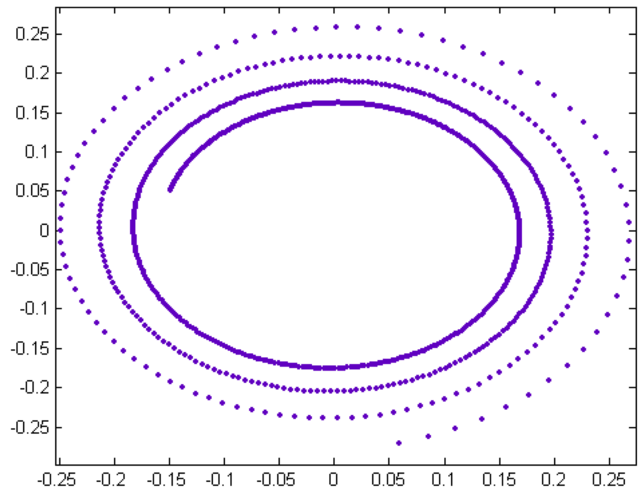

Figure 7

A plot of the real and imaginary components versus $\log N$ for $N = 1$ to 1000 is

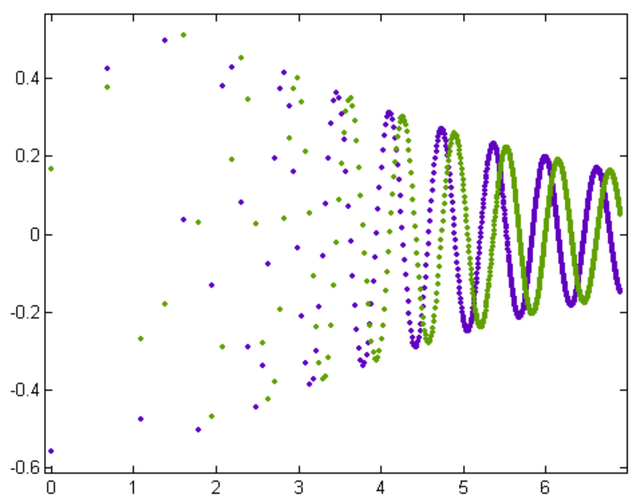

Figure 8

A plot of $\frac{\pi s}{\Pi'(s)\Pi'(-s)}$ for $s = (0.5, 14.1347251)$ and $N = 100$ to 1000

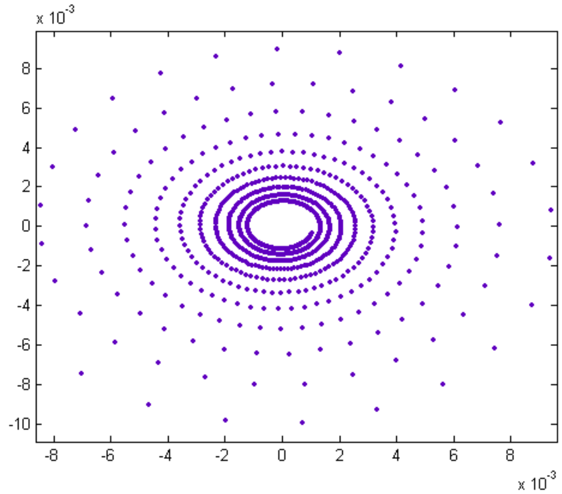

Figure 9

As $N \to \infty$, the value of $\frac{\pi s}{\Pi'(s)\Pi'(-s)}$ appears to be (0,0). (Note: The imaginary components of expressions such as $1-s$ and $-s$ are taken to be positive.)
The inflection points for the logarithmic spiral of $\frac{\sqrt{\pi s}}{\Pi(s)}$ for $s = (0.5, 14.1347251)$ and $N \leq 1000$ are at N equal to 3, 5, 8, 12, 19, 30, 48, 74, 116, 181, 282, 441, and 687. These are almost the same as for $\frac{\sqrt{\pi s}}{\Pi'(s)}$ - the values deviate by at most 2. $\frac{\sqrt{\pi s}}{\Pi(s)}$ and $\frac{\sqrt{\pi s}}{\Pi'(s)}$ are then basically the same.

A plot of $\sin(s)(e^{Nk})^s$ for $s = (0.5, 6.0)$, $k = -0.01$ and $N = 1$ to 1000 is

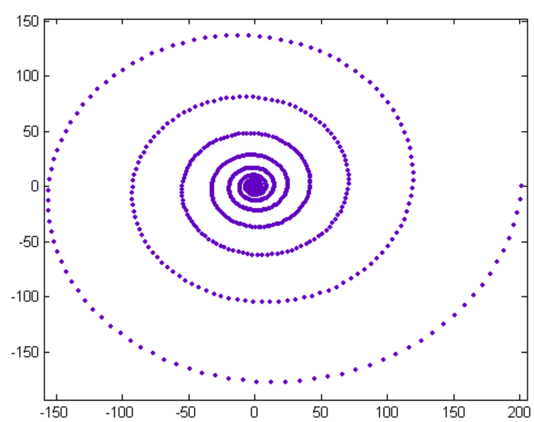

Figure 10

The inflections points are at N=79, 184, 288, 393, 498, 603, 707, 812, and 917. A plot of these values versus N is

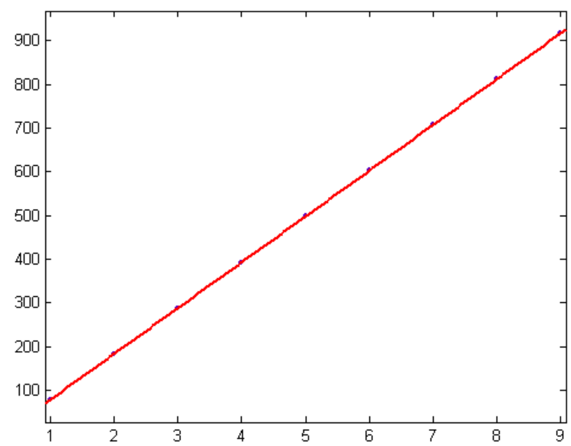

Figure 11

For a linear least-squares fit of the curve, $p_1 = 104.7$ with a 95% confidence interval of (104.6, 104.8), $p_2 = -25.78$ with a 95% confidence interval of $(-26.29, -25.27)$, SSE=.6222, R-squared=1, and RMSE=0.2981.

A plot of the real and imaginary components versus N is

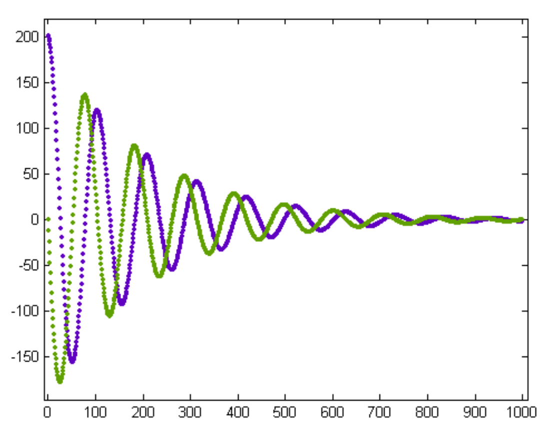

Figure 12

The inflection points of $\frac{\sqrt{\pi s}}{\Pi(s)}$ for $s = (0.5, 6.0)$ and $N = 1$ to 1000 are at N=2, 7, 20, 57, 163, and 463. The logarithms of these values increase linearly.

A plot of the real and imaginary components versus N is

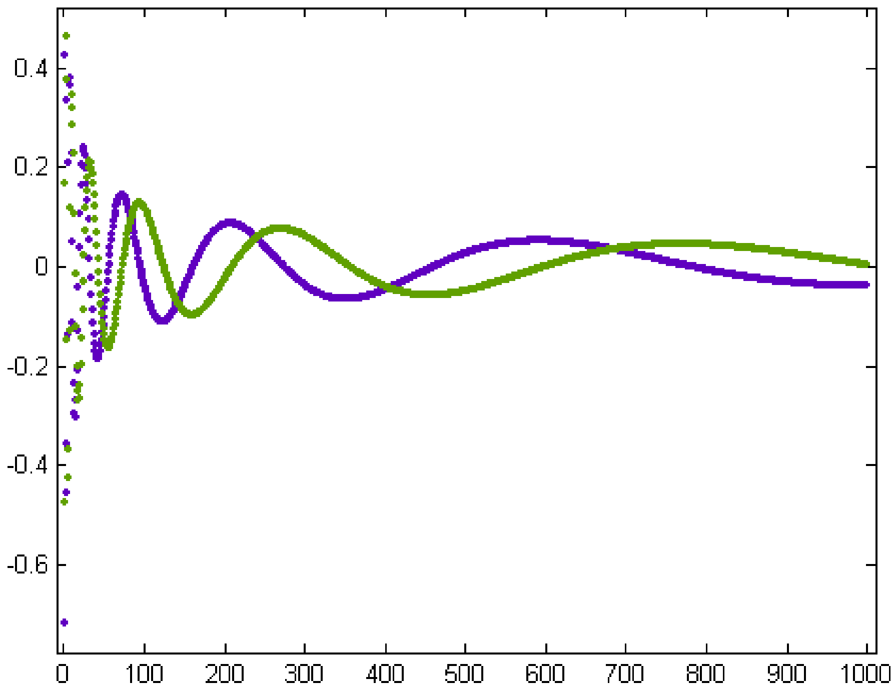

Figure 13

Other than the values converging more slowly, $\sin(s)(e^{Nk})^s$ is useful for modeling $\frac{\sqrt{\pi s}}{\Pi(s)}$.

4. ANOTHER GAMMA FUNCTION VARIANT

Another variant gamma function with more interesting limits is

$$\beta(s) = \Pi(s)\Pi(-s), 1 > \Re s > -1 \tag{10}$$

$\Pi'(s)\Pi'(-s)$ is denoted by $\beta'(s)$. A plot of $\frac{\pi s}{\beta'(s)}$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

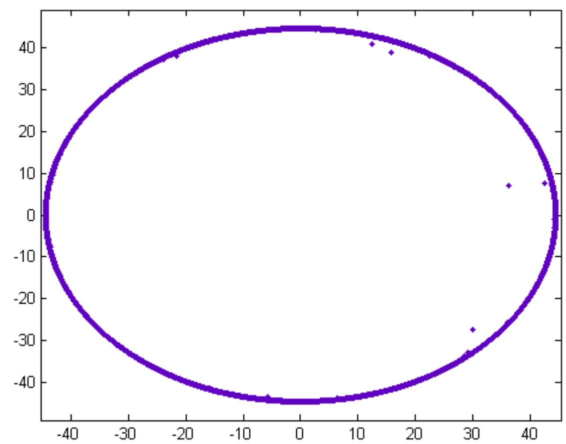

Figure 14

The interpretation of this circle is that the values are a degenerate logarithmic spiral (where the points keep circling without spiraling inward or outward) with an amplitude of $\pi\Im s$.

A plot of the real and imaginary components versus N is

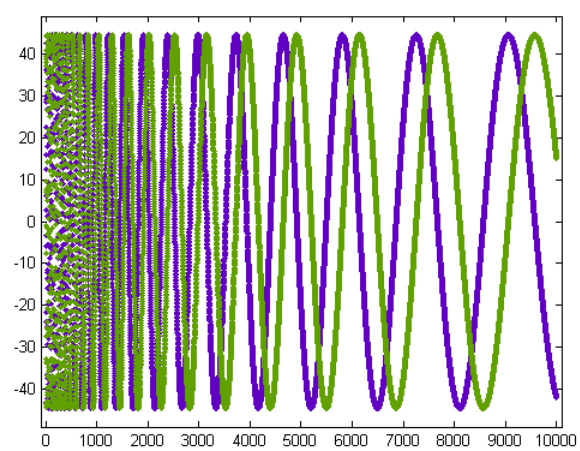

Figure 15

As with $\frac{\sqrt{\pi s}}{\Pi'(s)}$, the wavelengths are constant when plotted against $\log(N)$.

A plot of the real components of $\frac{\pi s}{\beta(s)}$ versus the real components of $\frac{\pi s}{\beta'(s)}$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

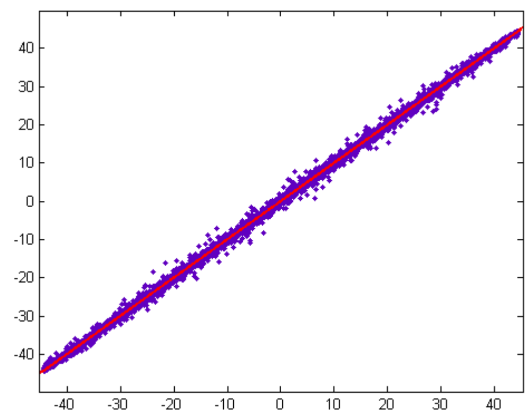

Figure 16

For a linear least-squares fit of the curve, $p_1 = 0.9994$ with a 95% confidence interval of (0.999, 0.9998), $p_2 = 0.01318$ with a 95% confidence interval of (0.0007935, 0.02557), SSE=3993, R-squared=0.9996, and RMSE=0.632.

A plot of the imaginary components of $\frac{\pi s}{\beta(s)}$ versus the imaginary components of $\frac{\pi s}{\beta'(s)}$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

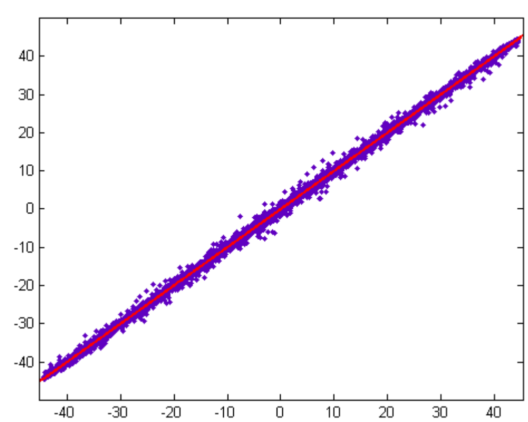

Figure 17

For a linear least-squares fit of the curve, $p_1 = 0.9997$ with a 95% confidence interval of (0.9993, 1.0), $p_2 = -0.0004807$ with a 95% confidence interval of (-0.01703, 0.007419), SSE=3881, R-squared=0.9996, and RMSE=0.623.

$\frac{\pi s}{\beta(s)}$ and $\frac{\pi s}{\beta'(s)}$ are then basically the same.

5. THEOREM (5)

Theorem (5) in section 1.3 of Edwards' first book is

Theorem 2. $\Pi(s) = s\Pi(s-1)$

A plot of $\Pi'(s)$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

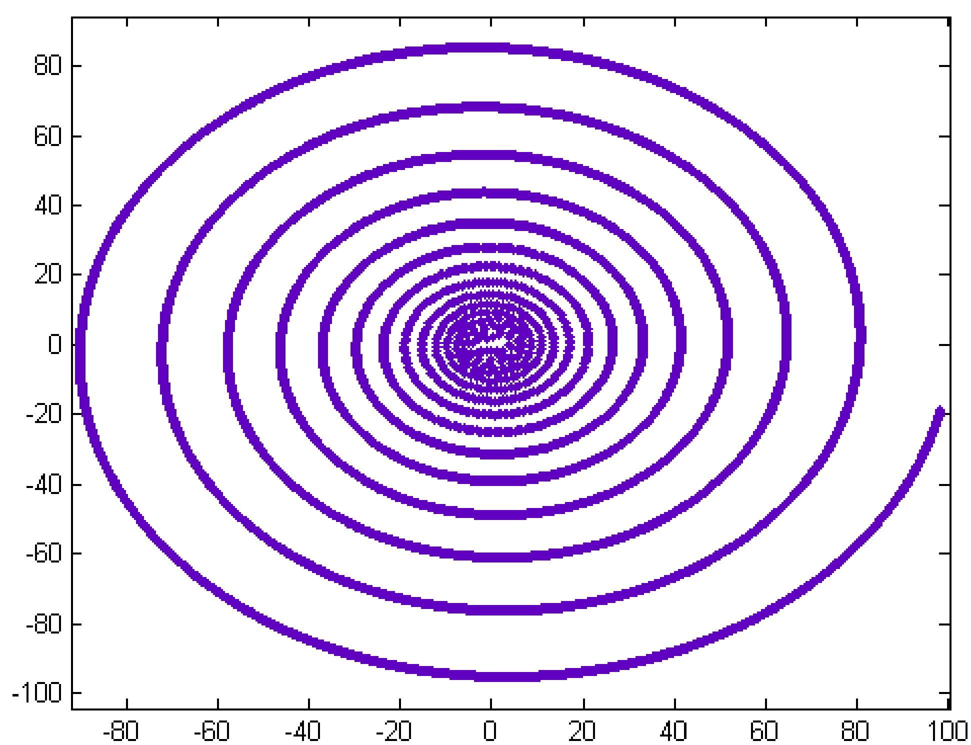

Figure 18

The N values of the inflection points are 4, 7, 11, 18, 28, 43, 68, 106, 166, 259, 403, 630, 982, 1532, 2390, 3738, 5814, and 9069. A plot of the logarithms of these values is

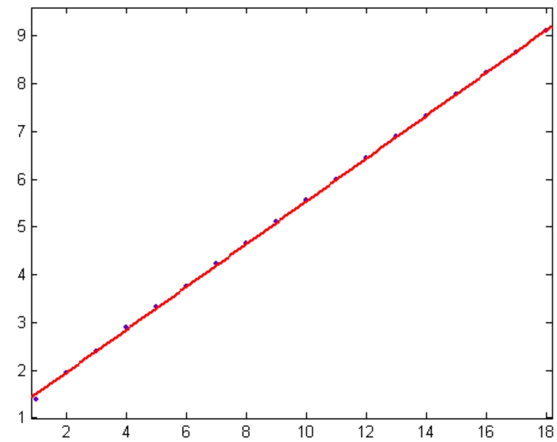

Figure 19

For a linear least-squares fit of the curve, $p_1 = 0.4491$ with a 95% confidence interval of (0.4458, 0.4524), $p_2 = 1.051$ with a 95% confidence interval of (1.015, 1.087), SSE=0.01906, R-squared=0.9998, and RMSE=0.03452.
The slopes for the first 10 zeta function zeros are 0.4491, 0.2999, 0.2527, 0.2075, 0.1916, 0.1679, 0.154, 0.1457, 0.1313, and 0.1265. A plot of the reciprocals of the slopes versus the imaginary components of the zeta function zeros is

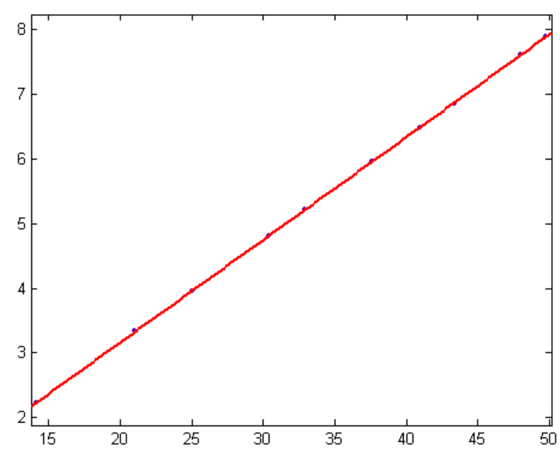

Figure 20

For a linear least-squares fit of the curve, $p_1 = 0.159$ with a 95% confidence interval of (0.1585, 0.1594), $p_2 = -0.01655$ with a 95% confidence interval of $(-0.03411, 0.001014)$, SSE=0.0004455, R-squared=1, and RMSE=0.007463.
A plot of $\beta(s)$ for $s = (0.25, 14.1747251)$ and $\beta(s)$ for $s = (0.75, 14.1747251)$ and $N = 1$ to 10000 is

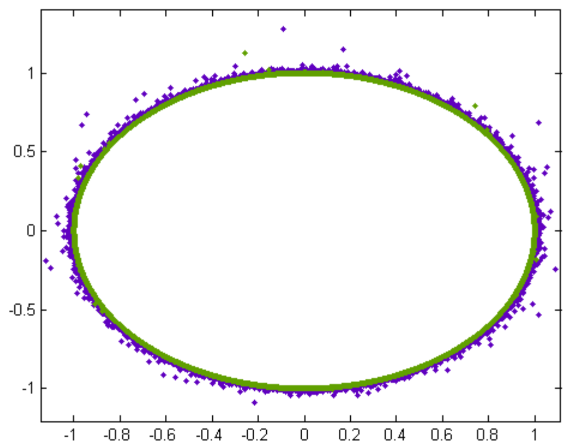

Figure 21

A plot of real component of $\beta(s)$ for $s = (0.25, 14.1747251)$ and the real component of $\beta(s)$ for $s = (0.75, 14.1747251)$ is

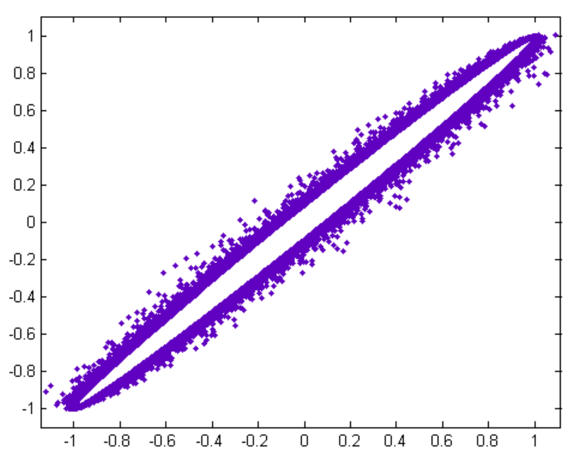

Figure 22

A plot of imaginary component of $\beta(s)$ for $s = (0.25, 14.1747251)$ and the imaginary component of $\beta(s)$ for $s = (0.75, 14.1747251)$ is

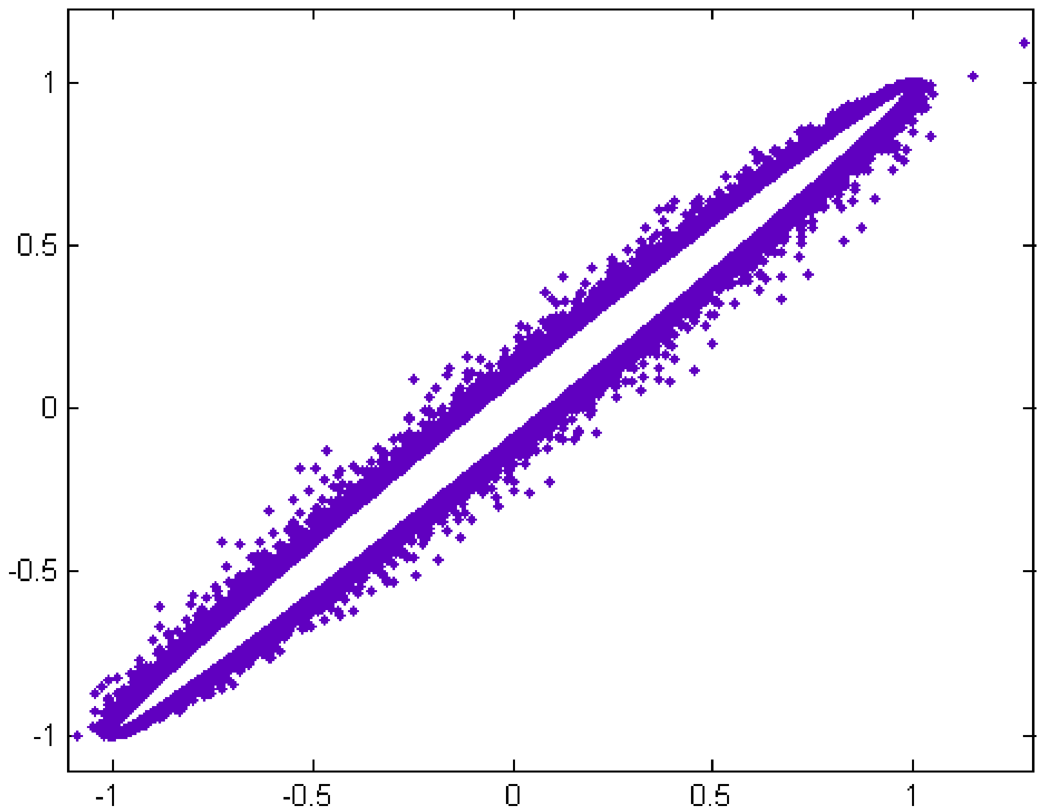

Figure 23

There is then a quantifiable difference between $\beta(s)$ and $\beta(1-s)$.

6. THEOREM (7)

Theorem (7) in section 1.3 of Edwards' book is

Theorem 3. $\Pi(s) = 2^s\Pi(\frac{s}{2})\Pi(\frac{s-1}{2})\pi^{-1/2}$

A plot of $2^s\beta'(\frac{s}{2})\beta'(\frac{s-1}{2})\pi^{-1/2}$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

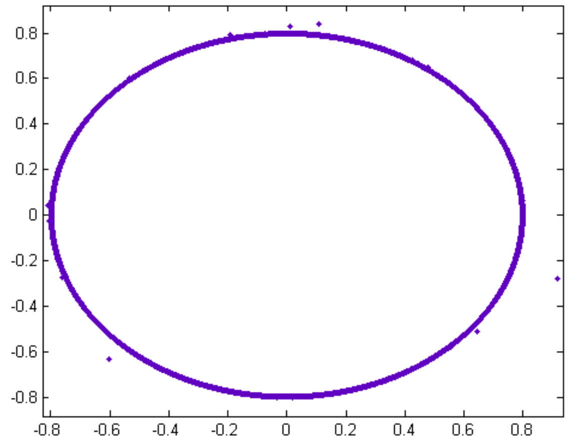

Figure 24

A plot of $2^s\Pi'(\frac{s}{2})\Pi'(\frac{s-1}{2})\pi^{-1/2}$ for $s = (0.5, 14.1347251)$ and $N = 1$ to 10000 is

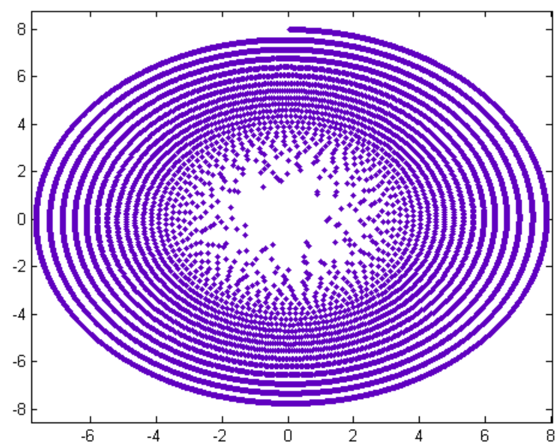

Figure 25

The N values of the inflection points are 3, 7, 9, 11, 14, 17, 22, 27, 34, 43, 53, 67, 84, 105, 131, 163, 204, 255, 319, 398, 497, 621, 776, 969, 1211, 1512, 1888, 2358, 2946, 3679, 4595, 5738, 7167, and 8950. A plot of the logarithms of these values (excluding 3) is

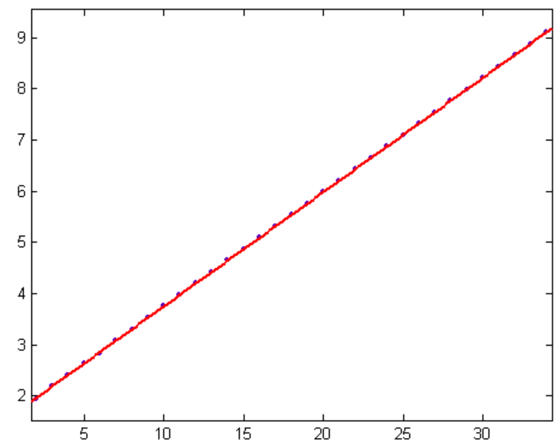

Figure 26

For a linear least-squares fit of the curve, $p_1 = 0.2232$ with a 95% confidence interval of (0.2228, 0.2235), $p_2 = 1.519$ with a 95% confidence interval of (1.513, 1.526), SSE=0.002248, R-squared=1, and RMSE=0.008516.

For the first 10 zeta function zeros, the slopes are 0.2232, 0.1499, 0.1261, 0.1036, 0.09564, 0.08401, 0.07701, 0.07279, 0.06564, and 0.06328. A plot of the reciprocals of these values versus the imaginary components of the zeta function zeros is

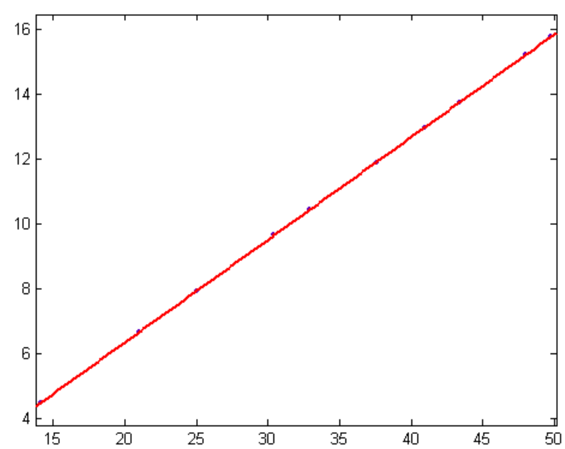

Figure 27

For a linear least-squares fit of the curve, $p_1 = 0.3173$ with a 95% confidence interval of (0.3165, 0.3181), $p_2 = -0.003419$ with a 95% confidence interval of (-0.03166, 0.02482), SSE=0.001152, R-squared=1, and RMSE=1.012.

7. THEOREM (4)

Theorem (4) in section 1.3 of Edwards' book is

Theorem 4. $\Pi(s) = \prod_{n=1}^{\infty} \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^{\infty}(1+\frac{s}{n})^{-1}(1+\frac{1}{n})^s$

This is a reformulation of formula (3). Using it one can prove that $\Pi(s)$ is an analytic function of the complex variable s which has simple poles at $s = -1, -2, -3, \ldots$ and that it has no zeros. A plot of $\frac{n^{1-s}n^{-1+s}(n+1)^s(n+1)^{-s}}{s+n}$ where $s = (0.25, 2.0)$ for $n = 1, 2, 3, \ldots, 1000$ is

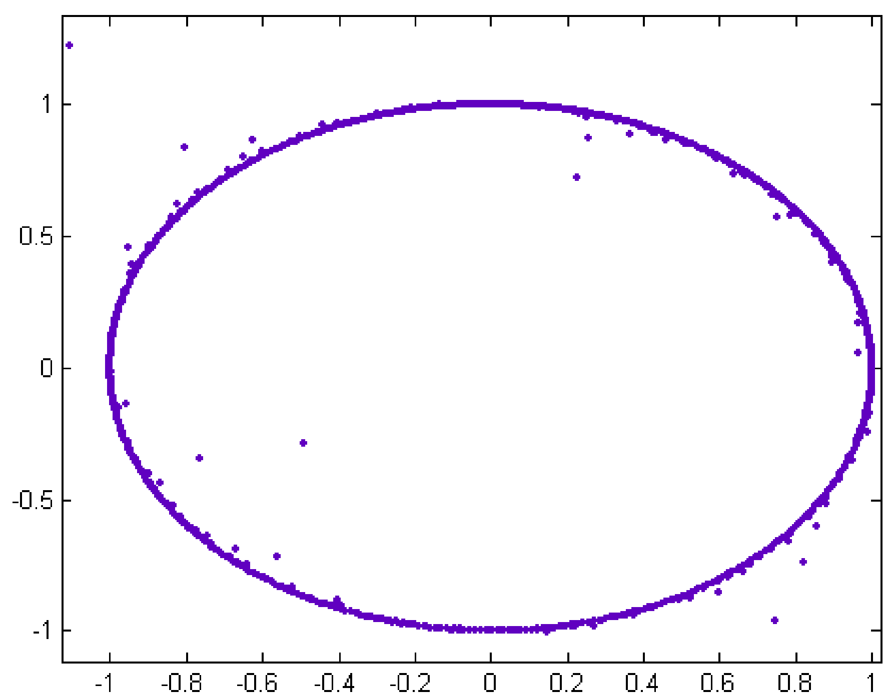

Figure 28

The imaginary component of $\frac{1}{s+n}$ is taken to be 1.0. A plot of the real and imaginary components versus $\log n$ is

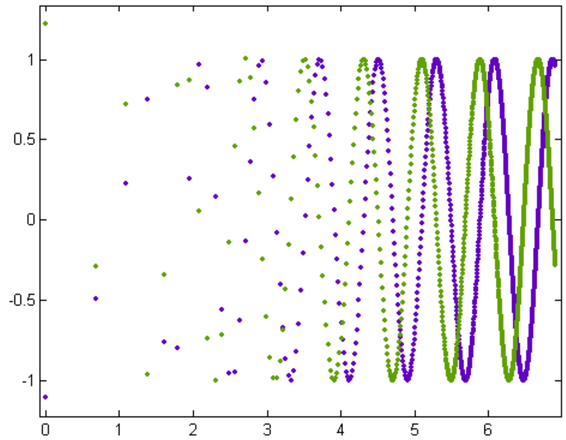

Figure 29

A plot of $\frac{\varphi(n)^{1-s}\varphi(n)^{-1+s}(\varphi(n)+1)^{s}(\varphi(n)+1)^{-s}}{s+\varphi(n)}$ where $s = (0.25, 2.0)$ for $n = 1, 2, 3, \ldots, 1000$ is

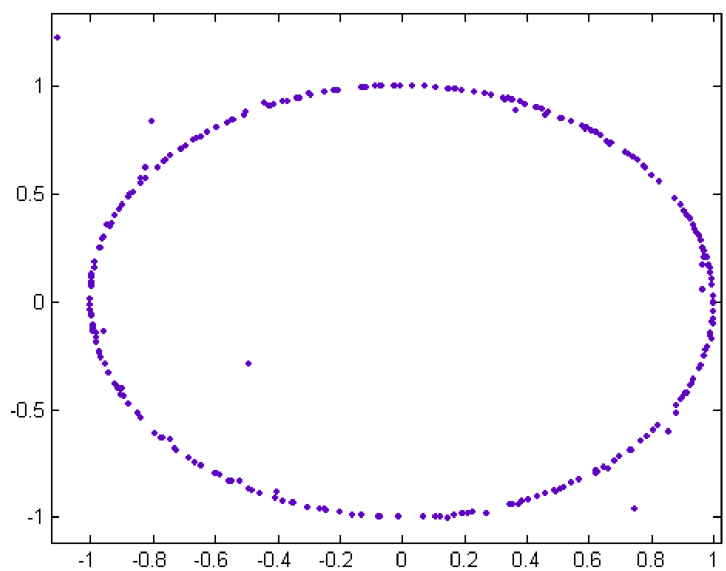

Figure 30

A plot of $\frac{n^{1-s}(n+1)^{s}}{s+n}$ where $s = (0.5, 14.134725)$ for $n = 1, 2, 3, \ldots, 10000$ is

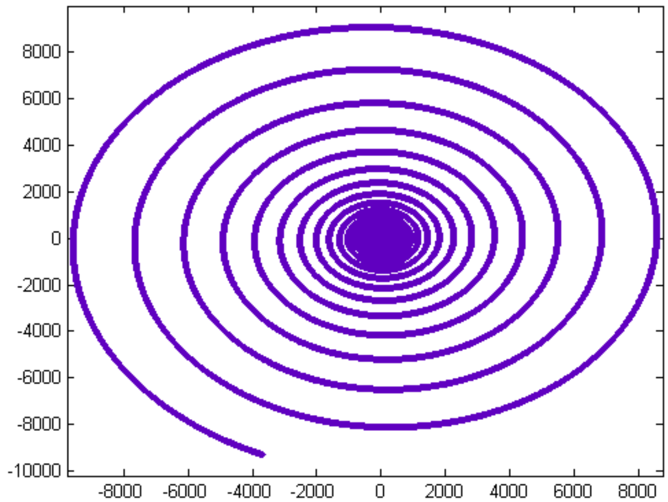

Figure 31

The n values of the inflection points are 2, 8, 10, 13, 16, 20, 25, 31, 39, 49, 61, 76, 95, 119, 149, 186, 232, 289, 361, 451, 564, 704, 879, 1098, 1371, 1713, 2139, 2671, 3336, 4166, 5203, 6498, and 8116. A plot of the logarithms of these values (excluding 2) is

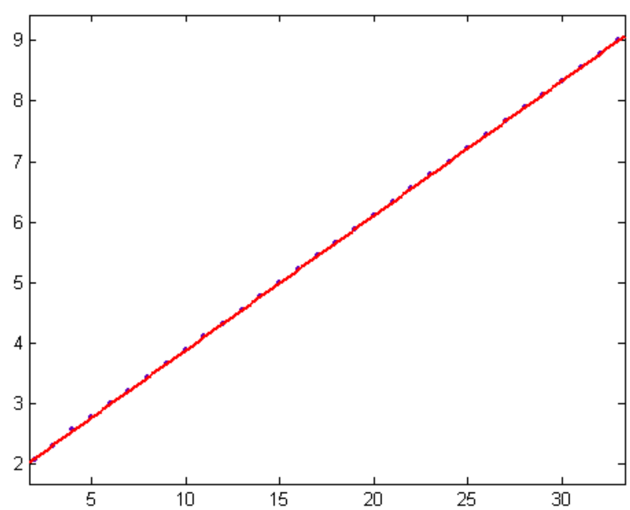

Figure 32

For a linear least-squares fit of the curve, $p_1 = 0.2227$ with a 95% confidence interval of (0.2224, 0.223), $p_2 = 1.657$ with a 95% confidence interval of (1.651, 1.663), SSE=0.001644, R-squared=1, and RMSE=0.007403.

The slopes for the first 10 zeta function zeros are 0.2227, 0.1494, 0.1255, 0.1033, 0.0955, 0.08365, 0.07681, 0.06258, 0.06551, and 0.0632. A plot of the reciprocals of these values versus the imaginary components of the zeta function zeros is

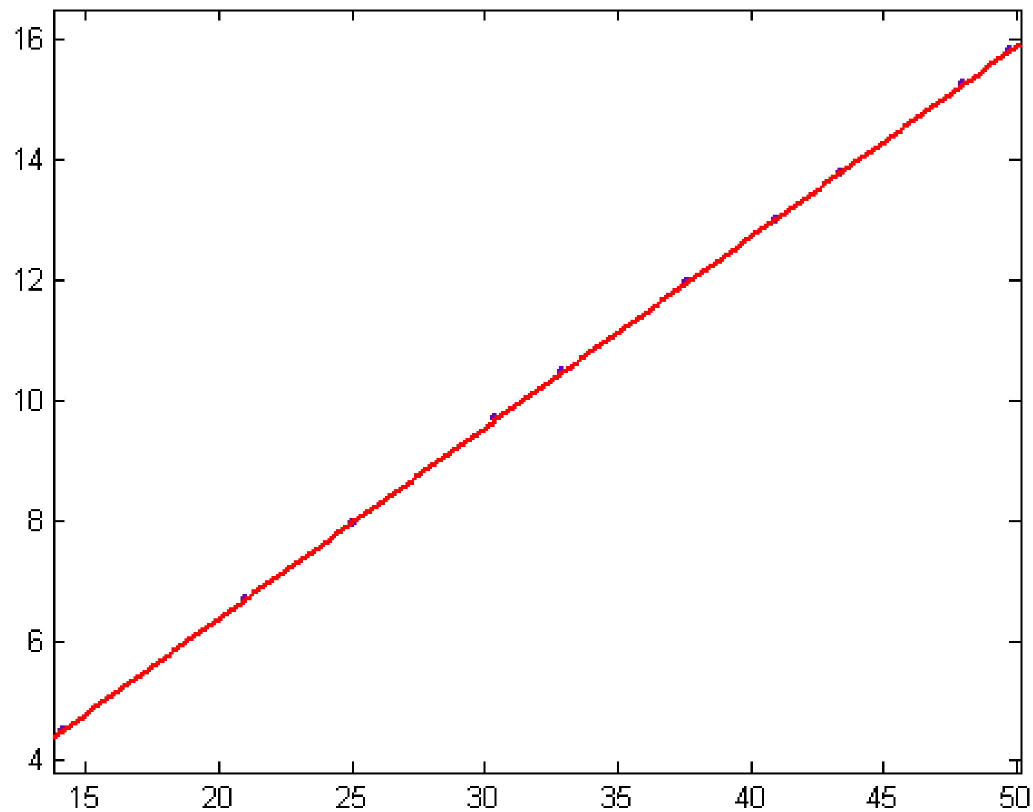

Figure 33

For a linear least-squares fit of the curve, $p_1 = 0.3177$ with a 95% confidence interval of (0.3172, 0.3182), $p_2 = 0.01199$ with a 95% confidence interval of (-0.006522, 0.0305), SSE=0.0004949, R-squared=1, and RMSE=0.007866.

8. LOGARITHMIC SPIRALS AND RIEMANN'S ZETA FUNCTION ZEROS

In this section, zeta function zeros are substituted into the formula $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)^s}$. The first few zeta function zeros are 14.134725, 21.022039, 25.010857, 30.424876, 32.935061, 37.586178,.... A plot of $\sum_{n=1}^{66872} \frac{1}{\varphi(n)^s}$ for $s = (\frac{1}{2}, 14.134725)$ is

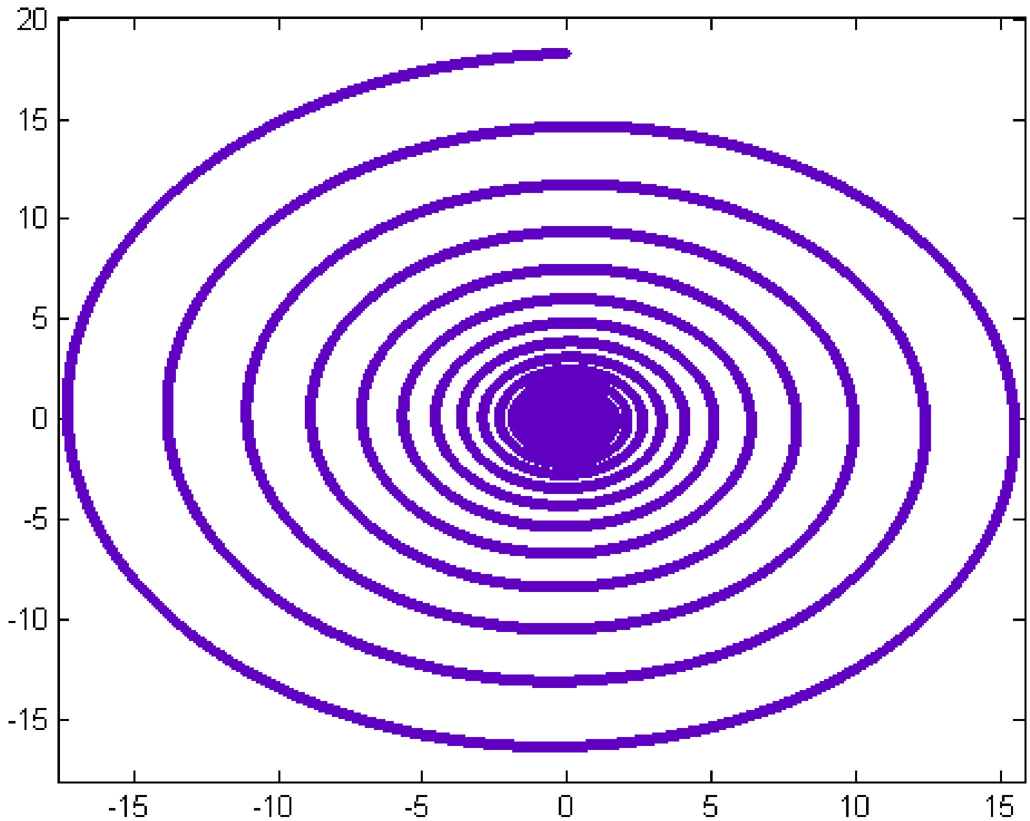

Figure 34

The n value of 66872 was chosen so that the spiral ends at a real component close to zero (and at an inflection point). The n values ($n \leq 100000$) of the inflection points are 4, 6, 9, 14, 22, 35, 54, 85, 133, 207, 323, 503, 785, 1224, 1909, 2978, 4644, 7244, 11299, 17623, 27487, 42873, and 66872. A plot of the logarithms of these n values is

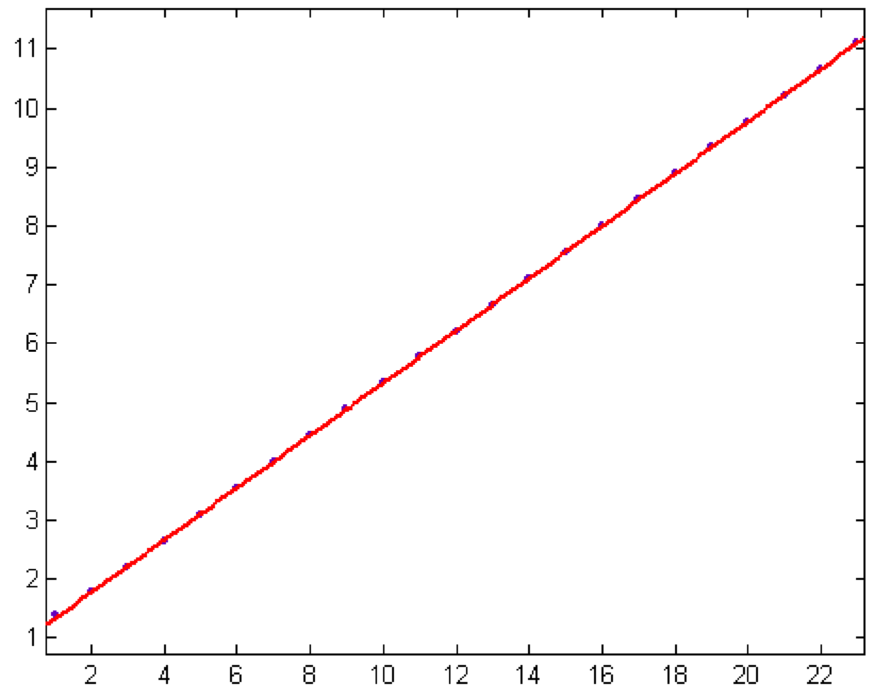

Figure 35

For a linear least-squares fit of the curve, $p_1 = 0.4443$ with a 95% confidence interval of (0.4433, 0.4453), $p_2 = 0.8891$ with a 95% confidence interval of (0.8755, 0.9028), SSE=0.004874, R-squared=1, and RMSE=0.01524. For the usual Riemann zeta function, the inflection points are at the same n values.

9. RIEMANN'S ZETA FUNCTION

Equation (1) in section 1.4 of Edwards' first book is

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Pi(s-1) \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{11}$$

A contour integral (developed by Riemann) is

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = (e^{i\pi s} - e^{-i\pi s}) \int_0^\infty \frac{x^{s-1} dx}{e^x - 1} \tag{12}$$

A plot of this integral for $\Pi'(s)$, $s = (0.5, 10.0)$ and $x = 1$ to 10000 is

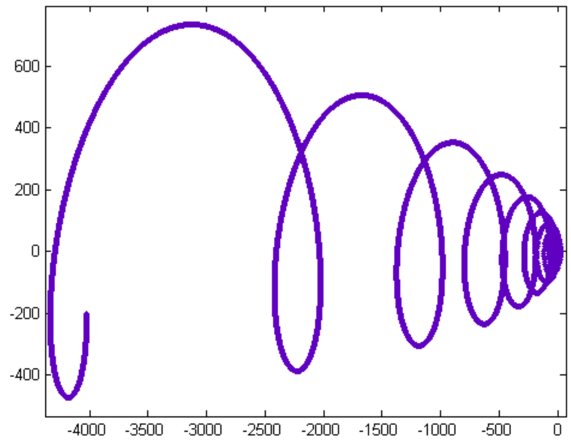

Figure 36

The x values of the inflection points are 2, 5, 9, 17, 33, 63, and 121. A plot of the logarithms of these values (excluding 2) is

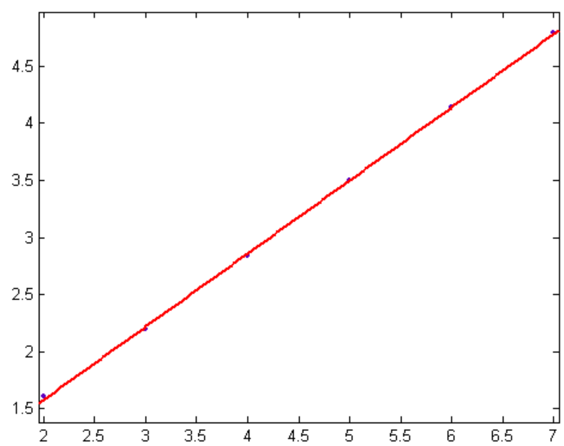

Figure 37

For a linear least-squares fit of the curve, $p_1 = 0.6409$ with a 95% confidence interval of (0.6248, 0.657), $p_2 = 0.295$ with a 95% confidence interval of (0.2175, 0.3725), SSE=0.002355, R-squared=0.9997, and RMSE=0.02427.

A plot of the curve when $s = (0.5, 14.1347251)$ is

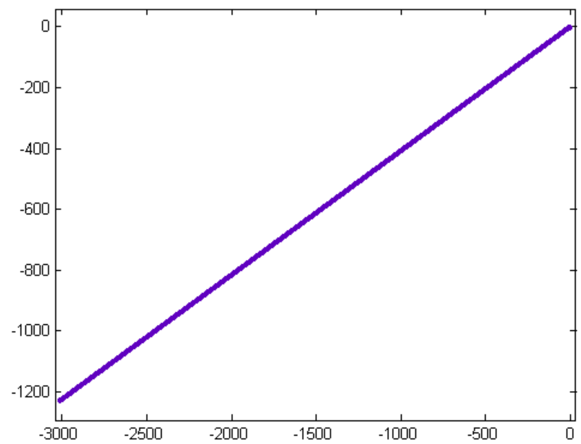

Figure 38

For a linear least-squares fit of the curve, $p_1 = 0.448$ with a 95% confidence interval of (0.448, 0.448), $p_2 = -1.928$ with a 95% confidence interval of $(-1.931, -1.925)$, SSE=51.85, R-squared=1, and RMSE=0.07202. The slopes and intercepts for the first 10 zeta function zeros are (0.448, -1.928), (0.05172, 1.923), (0.01718, 1.985), (4.204, -9.567), (-0.2416, 2.131), (-4.159, -9.701), -0.2979, 2.215), (1.757, 4.591), (0.007205, -2.138), (-0.9573, -3.088), (-0.1115, 2.177), (6.042, -14.46), and (2.047, 5.31). The slopes and intercepts of the lines are apparently random.
Equation (3) (Riemann's definition of $\zeta(s)$) in section 1.4 of Edwards' book is

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \tag{13}$$

The slopes and intercepts of the x values of the inflection points of the logarithmic spirals of $\frac{\Pi'(-s)}{2\pi i}$ for the first 10 zeta function zeros are (0.4491, 1.051), (0.3000, 0.9287), (0.2527, 0.2624), (0.2078, 1.63), (0.1916, 1.556), (0.1679, 1.562), (0.1542, 1.754), (0.1457, 1.883), (0.1313, 1.985), and (0.1265, 0.1299). (Note: In this instance, the imaginary components of the zeta function zeros are negated. This reverses the direction of the logarithmic spiral but gives slightly better results.) A plot of the reciprocals of these values versus the imaginary components of the zeta function zeros is

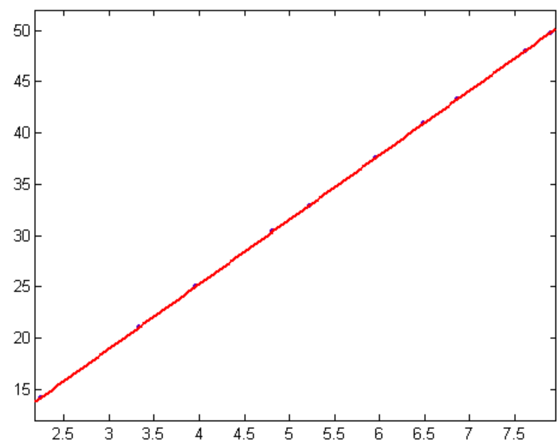

Figure 39

For a linear least-squares fit of the curve, $p_1 = 6.296$ with a 95% confidence interval of (6.279, 6.313), $p_2 = 0.7251$ with a 95% confidence interval of $(-0.02429$, 0.1693), SSE=0.01357, R-squared=1, and RMSE=0.04119.

Multiplying one of these logarithmic spirals with the corresponding straight line gives another logarithmic spiral. The resulting logarithmic spiral for the first zeta function zero is

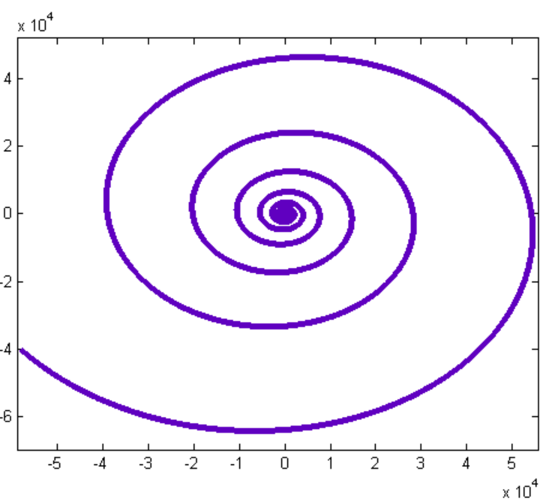

Figure 40

The slopes and intercepts of the x values of the inflection points for the first 10 zeta function zeros are (0.4451, 0.911), (0.2994, 1.336), (0.2516, 1.626), (0.207, 1.891), (0.1914, 1.788), (0.1677, 1.954), (0.1542, 2.197), (0.1451, 2.0275), (0.1314, 2.205), and (0.1265, 2.373). These slopes and intercepts are almost the same as the slopes and intercepts of the x values of the inflection points of the logarithmic spirals of $\frac{\Pi'(-s)}{2\pi i}$ given above. A zeta function zero can be thought of as having a straight line associated with it which when multiplied by the contour integral gives $\frac{\Pi'(-s)}{2\pi i}$.

10. CONCLUSION

A logarithmic spiral is defined by the equation $r = ae^{\theta \cot b}$ where r is the radius of each turn in the spiral and θ is the angle of rotation as the curve spirals. The constants a and b depend on the particular spiral. For logarithmic spirals associated with a variant of the gamma function, there is a linear relationship between the reciprocals of r and the imaginary components of the Riemann zeta function zeros. The zeta function zeros apparently have an invariance property.

11. METHODS

A C program for computing Riemann's contour integral is as follows.
A variant gamma function is used.

```
//
// compute Mobius function
//
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table, primes up to 1500000
int mobius(unsigned int a, unsigned int *table, unsigned int tsize) {
unsigned int i,count,p;
count=0;
for (i=0; i<tsize; i++) {
    p=table[i];
    if (p>a)
        break;
    if (a==(a/p)*p) {
        a=a/p;
        if (a==(a/p)*p)
```

```
            return(0);
        count=count+1;
        if (a==1)
            break;
        }
    }
if ((count&1)==0)
    return(1);
else
    return(-1);
}
//
// compute Euler's phi function
//
int mobius(unsigned int a, unsigned int *t, unsigned int tsize);
unsigned int nueuler(unsigned int n, unsigned int *table, unsigned int tsize) {
unsigned int d;
int sum;
if (n==1)
    return(1);
sum=0;
for (d=1; d<=n; d++) {
    if (n==(n/d)*d)
        sum=sum+(n/d)*mobius(d, table, tsize);
    }
return((unsigned int)sum);
}
//
// Gamma function
//
unsigned int nueuler(unsigned int a, unsigned int *table, unsigned int tsize);
unsigned int max=10000;
double s=0.25;
double t=10.0;
//double t=14.1347251;
//double t=21.0220396;
```

```
//double t=25.0108576;
//double t=30.4248761;
double pi=3.14159265;
unsigned int n=0; // select n
unsigned int xmin=0;
unsigned int out=4; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int tsize=114155; // size of prime look-up table
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
FILE *Outfp;
Outfp = fopen("cox6b.dat","w");
if (max>1500000) {
    printf("max too large \n");
    return;
    }
b=pi*s;
a=pi*t;
if (b>=0.0)
    temp1=pow((double)(exp(1.0)),b);
else {
    temp1=pow((double)(exp(1.0)),-b);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(a*log(exp(1.0))));
tempt=temp1*(sin(a*log(exp(1.0))));
temp1=temps*temps+tempt*tempt;
e=temps/temp1;
f=tempt/temp1;
e=temps-e;
f=tempt+f;
printf(" %.10llf %.10llf \n",e,f);
prods=1.0;
sums=0.0;
sumt=0.0;
```

```
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    prods=prods*(double)temp/((double)temp+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s);
    else {
        temp1=pow((double)(x+1),-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x)));
    tempt=temp1*(sin(t*log(x)));
    temp1=temps*temps+tempt*tempt;
    c=temps/temp1;
    d=tempt/temp1;
    sums=sums+c;
    sumt=sumt-d;
    c=a*sums-b*sumt;
    d=a*sumt+b*sums;
    temps=c*e-d*f;
    tempt=c*f+d*e;
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
```

```
        if ((out==2)&&((olds>0.0)&&(temps<0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %d %.10llf %.10llf backslashn",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)

[2] H. M. Edwards, *Advanced Calculus*, Houghton Mifflin Company, Boston 1969