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Approximating Fixed Points for Suzuzki's Generalized Nonexpansive Mapping in CAT(0) Space via New Iteration Process

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Abstract

In this paper, we establish strong and Δ-converges theorem of a new iteration process for Suzuki's generalized nonexpansive mapping in the setting of CAT(0) space. A numerical example is provided to demonstrate the fastness of the new iteration process. Our results are improved and generalized form of the earlier results.

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1. INTRODUCTION

Let C be a nonempty subset of a metric space (X, d) and $T\colon C \to C$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C\colon x = Tx\}$.

Remember that a selfmap T on a metric space subset C is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \;\; \forall x, y \in C. \tag{1.1}$$

Kirk [10, 11] was the first to introduce fixed point theory of nonexpansive operators in the context of nonlinear CAT(0) spaces. Suzuki's [15] made a significant breakthrough

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in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a selfmap T of a metric space subset C is said to satisfy Condition (C) (also known as Suzuki's map) if for any $x, y \in C$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y). \tag{1.2}$$

Remark 1.1. It is clear that every nonexpansive map is Suzuki's nonexpansive. However, an example in [15] shows that there exists maps which are Suzuki's nonexpansive but not nonexpansive.

2. PRELIMINARIES

Study of fixed point theory in CAT(0) spaces was initiated by Kirk [10, 11]. He proved that one can always find a fixed point for every nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping C from a closed interval $[0, r] \subset \mathbb{R}$ to X such that

$$c(0) = x, \ c(r) = y, \ d(c(t), c(s)) = |t - s|$$

for all $s, t \in [0, r]$. In particular, C is an isometry and $d(x, y) = r$. The image of C is call a geodesic segment (or metric segment) joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We denote the point $w \in [x, y]$ such that $d(x, w) = \alpha d(x, y)$ by $w = (1 - \alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$.

The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $D \subseteq X$ is said to be convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consist of three points (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle (or $\Delta(x_1, x_2, x_3)$) in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a CAT(0) space if all geodesic triangle of appropriate size satisfy the following CAT(0) comparison axiom:

Let Δ be a geodesic triangle in C and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality impels

$$d^2(x, \frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

this inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings ([2]), Pre-Hilbert spaces, $\mathbb{R}$-trees ([1]), the complex Hilbert ball with a hyperbolic metric ([8]) is a CAT(0) space. Further, Complete CAT(0) spaces are called Hadamard spaces.

Now, we give some elementary properties about CAT(0) spaces as follows:

Lemma 2.1. *[7] Let X be a CAT(0) space, $x, y, z \in X$ and $t \in [0,1]$. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z).$$

Let $\{x_n\}$ be a bounded sequence in X, complete CAT(0) spaces. For $x \in X$ set:

$$r(x, \{x_n\}) = \lim_{n\to\infty} sup\, d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = inf\{r(x, x_n) \colon x \in C\}.$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in C \colon r(x, x_n) = r(\{x_n\})\}.$$

$A(\{x_n\})$ consists of exactly one point in CAT(0) spaces see ([4], Proposition 7).

In 2008, Kirk and Panyanak [11] gave the following definition of Δ-convergence.

Definition 2.2. [11] A sequence $\{x_n\}$ in a given complete CAT(0) space X is said to Δ-converges to $x \in X$ if x is the unique asymptotic center for every subsequence $\{a_n\}$ of $\{x_n\}$. In this case we write $\Delta - lim_n x_n = x$ and read as x is the $\Delta - limit$ of $\{x_n\}$.

Lemma 2.3. *[1] Let X be a CAT(0) space, $x, y, z \in X$ and $t \in [0,1]$. Then*

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2.$$

Definition 2.4. [14] Let T be a selfmap on a subset C of a given CAT(0) space and f be a selfmap of $[0,\infty)$. We say that T has condition (I) if the following holds:

1. $f(g) = 0$ if and only if $g = 0$.

2. $f(g) > 0$ for every $g > 0$.

3. $d(x, Tx) \geq f(d(x, f(T)))$.

Lemma 2.5. *[11] Let $T\colon C \to C$ be a Suzuki's generalized nonexpansive mapping defined on a nonempty closed and Convex subset of a complete CAT(0) space such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x$. Then $x \in F(T)$.*

Lemma 2.6. *Let X be a complete CAT(0) space, C be closed convex subset of X.*
(1) If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C (see [[6], Proposition 2.1]).
(2) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence (see [[11], Proposition 3.7]).

Definition 2.7. Suppose C is a nonempty subset of a given CAT(0) space. If $T\colon C \to C$ has Suzuki's generalized nonexpansive mapping. Then for every fixed point p of T, one has

$$d(p, Tx) \leq d(p, x), \tag{2.1}$$

for each $x \in C$.

Lemma 2.8. *[11] Let C be a nonempty subset of a CAT(0) space X and $T\colon C \to C$ a Suzuki's generalized nonexpansive mapping. Then $F(T)$ is closed. Moreover, if X is strictly convex and C is convex, then $F(T)$ is also convex.*

Lemma 2.9. *[13] Let C be a CAT(0) space and $\{t_n\}$ be any real sequence such that $0 < a \leq a_n \leq b < 1$ for $n \geq 1$. Let $\{y_n\}$ and $\{z_n\}$ be any two sequences of C such that $\lim_{n\to\infty} sup\ d(y_n, x) \leq q$, $\lim_{n\to\infty} sup\ d(z_n, x) \leq q$ and $\lim_{n\to\infty} d(a_n y_n \oplus (1 - a_n)z_n, x) = p$ hold for some $q \geq 0$. Then $\lim_{n\to\infty} d(y_n, z_n) = 0$.*

3. NEW ITERATION PROCESS AND ITS CONVERGENCE ANALYSIS

Throughout this section, we have $n \geq 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in [0,1]. Thakur et al.[16] introduced an iteration process named as "Thakur new iteration process" as follows:

$$\begin{aligned} &x_0 \in C, \\ &z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ &y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ &x_{n+1} = T y_n, \;\; n \geq 1. \end{aligned} \tag{3.1}$$

They have proven that their new iteration process is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process using a numerical example of some assignment classes.

Following this Hussain et al.[9] introduced a three-step iteration process known as the K-iteration process and proved that it is converging faster than Picard-S iteration process.

K-iteration proces:

$$\begin{aligned} &x_0 \in C, \\ &z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ &y_n = T((1-\alpha_n)T x_n + \alpha_n T z_n), \\ &x_{n+1} = T y_n, \;\; n \geq 1. \end{aligned} \tag{3.2}$$

Recently, In 2018, Ullah et al.[17] proposed a three-step iteration process known as "M Iteration Process", described as:

$$\begin{aligned} &x_0 \in C, \\ &z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ &y_n = T z_n \\ &x_{n+1} = T y_n, \;\; n \geq 1. \end{aligned} \tag{3.3}$$

And they demonstrated the efficiency of this iteration process with the help of a numerical example.

Question: Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

To answer this, we introduce the new iteration process as follows:

Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $T\colon C \to C$ be a mapping. Let $x_0 \in C$ be arbitrary and the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} &x_0 \in C, \\ &z_n = T((1-\alpha_n)x_n \oplus \alpha_n T x_n), \\ &y_n = T((1-\beta_n)z_n \oplus \beta_n T z_n), \\ &x_{n+1} = T((1-\gamma_n)y_n \oplus \gamma_n T y_n), \;\; n \geq 1, \end{aligned} \tag{3.4}$$

where $\{\alpha_m\}$,$\{\beta_m\}$ and $\{\gamma_m\}$ are appropriate sequences in the interval (0,1). Motivated by what has been said above, in this research work, we establish convergence result of a new iteration process for Suzuki's generalized nonexpansive mapping in the setting of CAT(0). A numerical example is provided to demonstrate the fastness of the new iteration process. Our results are improved and generalized form of the earlier results.

4. MAIN RESULT

In this section, we prove strong and Δ-convergence theorem for Suzuki's generalized nonexpansive mapping for our results will generalize the results of Thakur et al. [16] and Ullah et al. [17] in CAT(0) space.

Theorem 4.1. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined as (3.4) then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof: For any $p \in F(T)$, By Definition 2.7, so we have

$$\begin{aligned} d(z_n, p) &= d(T((1-\alpha_n)x_n \oplus \alpha_n T x_n), p) \\ &\leq ((1-\alpha_n))d(x_n, p) + \alpha_n d(Tx_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{4.1}$$

Using (3.4),(4.1) and Definition 2.7, we have

$$\begin{aligned} d(y_n, p) &= d(T((1-\beta_n)z_n \oplus \beta_n T z_n), p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{4.2}$$

Using (4.1), (4.2) and Definition 2.7, we get

$$\begin{aligned} d(x_{n+1}, p) &= d(T((1-\gamma_n)y_n \oplus \gamma_n T y_n), p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{4.3}$$

Thus, $\{d(x_n, p)\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} d(x_n, p)$ exists $\forall\, p \in F(T)$. $\square$

Theorem 4.2. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined as in (3.4), then* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof: Suppose that $F(T) \neq \emptyset$ and $p \in F(T)$. Then by Theorem 4.1, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is bounded. Put for $c \geq 0$

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{4.4}$$

If $c \neq 0$, then by (3.4)

$$\lim_{n\to\infty} sup\, d(z_n, p) \leq \lim_{n\to\infty} sup\, d(x_n, p) = c. \tag{4.5}$$

By using Definition 2.7, we have

$$\lim_{n\to\infty} sup\, d(Tx_n, p) \leq \lim_{n\to\infty} sup\, d(x_n, p) = c. \tag{4.6}$$

Again by the proof of Theorem 4.1, we have $d(y_n, p) \leq d(x_n, p)$

Therefore,

$$\begin{aligned} d(x_{n+1},p) &= d(T((1-\gamma_n)y_n \oplus \gamma_n T y_n), p) \\ &\leq (1-\gamma_n)d(y_n,p) + \gamma_n d(Ty_n,p) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(y_n,p). \end{aligned}$$

It follows that

$$\begin{aligned} d(x_{n+1},p) - d(x_n,p) &\leq \frac{d(x_{n+1},p) - d(x_n,p)}{\gamma_n} \\ &\leq d(y_n,p) - d(x_n,p) \\ &\leq (1-\beta_n)d(z_n,p) + \beta_n d(z_n,p) - d(x_n,p) \\ &\leq d(z_n,p) - d(x_n,p). \end{aligned}$$

So, we can get $d(x_{n+1},p) \leq d(z_n,p)$ and form (4.4), we have

$$c \leq \lim_{n\to\infty} inf\ d(z_n,p). \tag{4.7}$$

Hence, from (4.5) and (4.7), we obtain

$$c = \lim_{n\to\infty} d(z_n,p). \tag{4.8}$$

Therefore, from (4.8), we have

$$\begin{aligned} c = \lim_{n\to\infty} d(z_n,p) &= \lim_{n\to\infty} d(T((1-\alpha_n)x_n \oplus \alpha_n T x_n), p) \\ &\leq \lim_{n\to\infty} (1-\alpha_n)d(x_n,p) + \alpha_n d(Tx_n,p) \\ &\leq \lim_{n\to\infty} (1-\alpha_n)d(x_n,p) + \lim_{n\to\infty} \alpha_n d(Tx_n,p) \\ &\leq c. \end{aligned} \tag{4.9}$$

Hence,

$$\lim_{n\to\infty} (1-\alpha_n)d(x_n,p) + \alpha_n d(Tx_n,p) = c. \tag{4.10}$$

Now, from (4.5),(4.6),(4.10) and Lamma 2.9, we conclude that,

$$\lim_{n\to\infty} d(Tx_n,x_n) = 0. \quad \square$$

Theorem 4.3. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (3.4). Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof: By Theorem 4.2, the sequence $\{x_n\}$ is bounded. Hence one can take $A(\{x_n\}) = \{c\}$ for some $c \in X$. We are going to prove $A(\{x_n\}) = \{c\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_n\}) = \{c\}$ Since $\{x_{n_k}\}$ is bounded, one can find a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_j}\}$ Δ-converges to p for some $p \in C$. By Theorem 4.2, Lemma 2.5 one has $p \in F(T)$ and hence $\lim_{n\to\infty} sup\, d(x_n, p)$ exists. If $p \neq x$, then the singletoness of the cardinality of the asymptotic centers allows us the following

$$\begin{aligned}\lim_{n\to\infty} sup\, d(x_n, p) &= \lim_{j\to\infty} sup\, d(x_{n_j}, p) < \lim_{j\to\infty} sup\, d(x_{n_j}, x) \\ &\leq \lim_{k\to\infty} sup\, d(x_{n_k}, x) < \lim_{k\to\infty} sup\, d(x_{n_k}, p) \\ &= \lim_{n\to\infty} sup\, d(x_n, p),\end{aligned} \tag{4.11}$$

which is contradiction. Therefore, $x = p \in F(T)$. Suppose that $x \neq c$. Then

$$\begin{aligned}\lim_{n\to\infty} sup\, d(x_n, x) &= \lim_{k\to\infty} sup\, d(x_{n_k}, x) \leq \lim_{k\to\infty} sup\, d(x_{n_k}, c) \\ &\leq \lim_{n\to\infty} sup\, d(x_m, c) < \lim_{n\to\infty} sup\, d(x_m, x) \\ &= \lim_{n\to\infty} sup\, d(x_n, x).\end{aligned} \tag{4.12}$$

$\{x_n\}$ Δ-converges to an element $c \in F(T)$.

Note that the strong convergence of our scheme on a non-compact domain is valid by following result. $\square$

Theorem 4.4. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (3.4), then $\{x_n\}$ strongly converges to a fixed point of T if and only if* $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Proof: If the sequence $\{x_n\}$ converges to a point $p \in F(T)$, then

$$\lim_{n\to\infty} inf d(x_n, p) = 0.$$

so,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, assume that $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$. From Theorem 4.1, we have

$$d(x_{n+1}, p) \leq d(x_n, p) \; for \; any \; p \in F(T),$$

so we have,

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)). \tag{4.13}$$

Thus, $d(x_n, F(T))$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} d(x_n, F(T))$ exists. As, $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$ so $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $F(T)$ such that $d(x_{n_j}, x_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 4.1, we have

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{n_j}, x_j) \\ &\leq \frac{1}{2^j}. \end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{j+1}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \\ &\to 0 \; as \; j \to \infty. \end{aligned}$$

So, $\{x_j\}$ is a Cauchy sequence in $F(T)$. Lemma 2.8 $F(T)$ is closed, so $\{x_j\}$ converges to some $x \in F(T)$.

Again, owing to triangle inequality, we have

$$d(x_{n_j}, x) \leq d(x_{n_j}, x_j) + d(x_j, x).$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $x \in F(T)$.

Since $\lim_{n\to\infty} inf d(x_n, x)$ exists by Theorem 4.1, therefore $\{x_n\}$ converges to $x \in F(T)$. $\square$

Eventually, we discuss the strong convergence for our scheme (3.4) by using the condition(I) given by definition 2.4.

Theorem 4.5. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (3.4) and T satisfies the Condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof: From (4.13), $\lim_{n\to\infty} d(x_n, F(T))$ exists.

Also, by Theorem 4.2 we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

It follows from the Condition (I) that

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n, F(T))) &\leq \lim_{n\to\infty} d(x_n, Tx_n)\\ &= 0.\end{aligned}$$

So $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

By Theorem 4.4, the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. □

5. NUMERICAL EXAMPLE

In this part, we'll look at a Suzuki's generalized nonexpansive mapping as an example.

Example:[17] Define a mapping $T\colon [0,1] \to [0,1]$ by

$$Tx = \begin{cases} 1 - x \ \ if \ \ x \in [0, \frac{1}{8}), \\ \frac{x+7}{8} \ \ if \ \ x \in [\frac{1}{8}, 1]. \end{cases}$$

Now, we prove that T is Suzuki generalized nonexpansive mapping but not nonexpansive.

If $x = \frac{3}{25}$, $y = \frac{1}{8}$ we see that

$$\begin{aligned} d(Tx, Ty) &= \mid Tx - Ty \mid \\ &= \left|1 - \frac{3}{25} - \frac{57}{64}\right| \\ &= \left|\frac{1600 - 192 - 1425}{1600}\right| \\ &= \frac{17}{1600} \\ &> \frac{1}{200} \\ &= d(x, y). \end{aligned}$$

Hence T is not nonexpansive mapping.

To verify that T is Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in [0, \frac{1}{8})$, then $\frac{1}{2}d(x, Tx) = \frac{1-2x.}{2} \in (\frac{3}{8}, \frac{1}{2}]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$ we must have $\frac{1-2x}{2} \leq |y - x|$, i.e., $\frac{1}{2} \leq y$, hence $y \in [\frac{1}{2}, 1]$. We have

$$d(Tx, Ty) = \left|\frac{y-7}{8} - (1-x)\right| = \left|\frac{y + 8x - 1}{8}\right| < \frac{1}{8},$$

and

$$d(x, y) =| x - y |> \left|\frac{1}{8} - \frac{1}{2}\right| = \frac{3}{8}.$$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y)$.

Case II: Let $x \in [\frac{1}{8}, 1]$, then $\frac{1}{2}d(x, Tx) = \frac{1}{2}\left|\frac{x+7}{8} - x\right| \in [0, \frac{49}{128}]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$ we must have $\frac{7-7x}{16} \leq| y - x |$, which gives two possibilities:

(a) Let $x < y$, then $\frac{7-7x}{16} \leq |y - x| \implies y \geq \frac{7+9x}{16} \implies y \in [\frac{65}{128}, 1] \subset [\frac{1}{8}, 1]$. so

$d(Tx, Ty) = \left|\frac{x+7}{8} - \frac{y+7}{8}\right| = \frac{1}{8}d(x, y) \leq d(x, y)$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y)$.

(b) Let $x > y$, then $\frac{7-7x}{16} \leq x - y \implies y \leq x - \frac{7-7x}{16} = \frac{23x-7}{16} \implies y \in [\frac{-33}{128}, 1]$. Since $y \in [0, 1]$, so $y \leq \frac{23x-7}{16} \implies x \geq \frac{16y+7}{23} \implies x \in [\frac{7}{23}, 1]$. So in this case $x \in [\frac{7}{23}, 1]$ and $y \in [0, 1]$.

Now $x \in [\frac{7}{23}, 1]$ and $y \in [\frac{1}{8}, 1]$ is already included in case (a). So let $x \in [\frac{7}{23}, 1]$ and $y \in [0, \frac{1}{8})$, then

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x+7}{8} - (1-y)\right| \\ &= \left|\frac{x + 8y - 1}{8}\right|. \end{aligned}$$

For convenience, first we consider $x \in [\frac{7}{23}, \frac{1}{2}]$ and $y \in [0, \frac{1}{8})$, then $d(Tx, Ty) \leq \frac{1}{16}$ and $d(x, y) > \frac{33}{184}$. Hence $d(Tx, Ty) \leq d(x, y)$.

Next consider $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{8})$, then $d(Tx, Ty) \leq \frac{1}{8}$ and $d(x, y) > \frac{3}{8}$. Hence $d(Tx, Ty) \leq d(x, y)$. so

$\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y)$.

Hence, T is not a nonexpansive mapping but it satisfies condition(C). Using $\alpha_n = \frac{1}{\sqrt{n^3+4}}$ $\beta_n = \frac{2}{\sqrt{n^3+5}}$ and $\gamma_n = \frac{3}{\sqrt{n^3+200}}$ in the given example with $x_0 = 0.5$ we get table 1, comparison of the convergence of our iteration process with M iteration, K iteration and Thakur New iteration processes are given, where $x_0 = 0.5$.

We can easily see that the new iteration was the first converging one than the M iteration, the K iteration and the Thakur New iteration processes.

Graphical representation is given in Fig.1. Also, We can easily see the efficiency of the New iteration process.

Table 1: Sequence generated by Thakur new, K, M and New iteration

S.No.	Thakur	K	M	New iteration
x0	0.500000000000000	0.500000000000000	0.500000000000000	0.500000000000000
x1	0.992188374608183	0.992242281082824	0.992242281082824	0.999063797455401
x2	0.999877957017476	0.999879635594404	0.999879635594404	0.999998247049591
x3	0.999998093291878	0.999998132493548	0.999998132493548	0.999999996717767
x4	0.999999970211021	0.999999971024820	0.999999971024820	0.999999999993854
x5	0.999999998534599	0.999999999550437	0.999999999550437	0.999999999999988
x6	0.999999999334588	0.999999999993025	0.999999999993025	1
x7	0.999999999134598	0.999999999999892	0.999999999999892	1
x8	0.999999999992729	0.999999999999998	1	1
x9	0.999999999999886	1	1	1
x10	0.999999999999998	1	1	1
x11	1	1	1	1

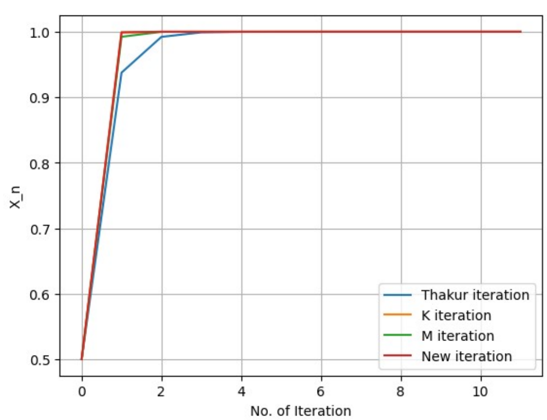

Figure 1: Convergence of New iteration, M, K and Thakur New iterations to the fixed point 1.

6. CONCLUSION

In this paper, we have presented a new type of iteration procedure for Suzuki's generalized nonexpansive mapping in CAT(0) spaces. Our result generalizes results of Thakur et al. [16] and Ullah et al. [17] in the sense of faster iteration process.

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