Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 19, Number 2 (2023), pp. 387-404

https://dx.doi.org/10.37622/GJPAM/19.2.2023.387-404

MODA Method - A Revised Version: For Optimality Testing and Optimizing a Solution in Transportation Problems

Dr. R. MURUGESAN
Associate Professor, Department of Mathematics, St. John's College, Palayamkottai-627002, Tirunelveli,Tamil Nadu, India
(Affiliated to Manonmaniam Sundaranar University,Tirunelveli-627012, Tamil Nadu, India)

Abstract

Dr. R. Murugesan (2022) proposed a very simple innovative iterative method named MODA (Modified Allocation) for optimality testing and also optimization of a solution obtained by a method in transportation problems (TP). In this method, during the optimization process we have traced and considered all possible loops starting and ending at an identified basic cell and passing through only one non-basic cell. During our further research we have observed that it can be simplified further. Thereby, we have introduced the new idea of *Solution Improvement Loops* and considered such loops only for the solution improvement process. This makes the optimality testing process much easier to achieve the optimal solution. Thereby, we present a revised version of the MODA method. The revised version of the MODA method has been tested on a number of non-optimal solutions obtained for many balanced and unbalanced transportation problems. Testing results validate that the presented revised version is the best one for testing the optimality of an obtained solution and also optimizing that solution, if that's not optimal. Another added advantage of the presented method is that it also generates the possible numbers of alternative optimal solutions to a given transportation problem, it they exist to the problem. Further, the presented method is an alternative and simple than the existing MODI method to test the optimality of a solution and also optimizing it, if it is not optimal.

2010 Mathematics Subject Classification: 90C08

Key Words: Transportation Problem, Initial Basic Feasible Solution, Optimal Solution, SOFTMIN method, I-SOFT method, MODI method, MODA method.

1. INTRODUCTION

The readers of this research article know the fundamentals of Transportation problems (TPs) and the key methods such as NWCM, LCM, VAM, MODI and Stepping Stone [4, 10] available to solve them. In 2012, Abdul Quddoos and et al. [1] introduced a direct method (but actually not direct) called ASM, and its revised version in 2016 [2], to generate an optimal solution directly to a wide range of TPs.

Murugesan R. [7] showed through some illustrative TPs that the ASM method is the method to generate best IBFS only and not a direct method to generate the optimal solution. By identifying some difficulties in the allocation process when tie occurs among certain 0-entry cells, Murugesan R. and Esakkiammal T. [10] improved the existing ASM method and named it as IASM method and showed that the later produces better IBFS than the best IBFS produced by the ASM method.

In 2021, Esakkiammal T. and Murugesan R. [3] proposed an innovative zero allocations approach named SOFTMIN which produces optimal solutions to most of the TPs. In 2022, Murugesan R. [9] established that the SOFTMIN method performs much better than the IASM method, but not a direct method to produce optimal solution to any given TP.

We further analyzed the process of allocation due to the SOFTMIN method on the near optimal solutions obtained for some 'More Challenging' TPs, and identified that very few changes made in the allocation process have improved the solution. This resulted in the 'Improved SOFTMIN" (or briefly I-SOFT) method [5]. As for as our knowledge/search is concerned so far, no competing methods for generating best initial basic feasible solution (IBFS) on the identified some 'More Challenging' TPs are not available in the literature and thereby, the I-SOFT method may be the best one to produce the best IBFS to a given TP.

In 2022, Murugesan R. [6] proposed an innovative method named MODA (Modified Allocation) which tests the optimality of a solution and also optimizes the solution, if it is not optimal. By our further research we have identified the extra efforts made in the MODA method to trace and consider all possible loops starting and ending at an identified basic cell and passing through a non-basic cell. In this article we have simplified this difficulty by introducing the new idea of *Solution Improvement Loops* only to consider. This resulted in a revised version of the existing MODA method.

The paper is organized as follows: Section 1 – Briefs the introduction. Section 2 – Presents the algorithm of the existing MODA method. Section 3 – presents the algorithm of the revised version of the MODA method. Section 4 –Illustrates two 'more challenging' TPs by the proposed method. Section 5 – Lists a set of 14 "More Challenging" TPs. Section 6 – Discusses on the results obtained. Section 7 – Draws the conclusion.

Basic cell and Non-basic cell

A cell in a transportation table (TT) is said to be a *basic cell* if it is having some allocated quantity. The allocated quantity may be positive or zero. The other cells are called *non-basic cells*. A TT with size m × n will have at most (m+n-1) basic cells and the remaining cells are non-basic.

Non-degenerate BFS and Degenerate BFS

A basic feasible solution (BFS) to a TP is said to be *non-degenerate* if it contains exactly (m+n-1) numbers of basic cells, which are in independent positions. A BFS is said to be *degenerate* if it contains less than (m+n-1) numbers of basic cells.

Loop in a TT

A *loop* in a transportation table is an ordered set of even numbers (≥ 4) of cells having only one non-basic cell (NBC) and the remaining basic cells. The cells in a loop are called *corner cells*. The one NBC of a loop may occur at an add position or at an even position in the order of cells of the loop.

Net Cost Change (NCC) value of a loop

Consider one traced loop. Mark with a + sign and a – sign alternatively at each of the corner cells of the considered loop, starting from the non-basic cell (i, j) in it. Compute the *effect on cost* of the considered loop by adding together the original UTC found in each corner cell containing a + sign and then subtracting the original UTC found in each corner cell containing a – sign. This effect on cost is called the *net cost change* (NCC) value for the considered loop.

Solution improvement (SI) loop

A loop with negative NCC value only will improve the objective function value. Such a loop is called a *Solution Improvement (SI) loop*. If the identified basic cell (h, k) has two or more than two SI loops, then select the one having the minimum NCC value for implementation. If tie occurs among the SI loops with the same minimum NCC value, then implement any one loop arbitrarily. Such a situation will generate an alternative solution to the given TP. If the SI loop has NCC value as zero, this situation also generates an alternative solution to the given TP. In general, a SI loop will either eliminate the IBC from the basis or reduce the allocation at the IBC. However, if the SI loop has the NBC which occurs at the odd position in the order of cells of the loop it may increase the allocation value at the IBC. But, sthe IBC will be eliminated during the successive iterations.

2. ALGORITHM FOR THE EXISTING 'MODA' METHOD

The readers are requested to refer the article [6] due to Murugesan R. for the detailed algorithm of the MODA method.

3. ALGORITHM FOR THE PROPOSED ‘REVISED VERSION OF THE MODA’ METHOD

The term MODA has been coined from the first three letters of the word ‘Modified’ and the first one letter of the word ‘Allocation’. MODA is an iterative method which can be used for testing the optimality of an initial basic feasible solution (IBFS) and also optimize the IBFS, if it is not optimal, for transportation problems. The innovative way of improving a non-optimal solution to an optimal solution by the revised version of the MODA method is based on redistributing the allocation available at a currently allocated cell (basic cell) with largest ‘unit transportation cost’ (UTC) to another un-allocated cell (non-basic cell) and its subsequent induced reallocations. The algorithm of the proposed revised version of the MODA method consists of two stages. In Stage #1, an IBFS is obtained to the given TP. In Stage #2, optimality testing of the obtained IBFS and also optimizing it, if it is not optimal, is carried out.

We use the following notations and abbreviations in the development of the algorithm of the revised version of the MODA method:

$m \times n$	– Size of the unit cost matrix of the given TP
TT	– Transportation table
BTP	– Balanced transportation problem
UTP	– Unbalanced transportation problem
UTC	– Unit transportation cost
C_{ij}	– UTC available at the cell (i, j)
IBFS	– Initial basic feasible solution
$X = [x_{ij}]$	– A solution
X^*	– An optimal solution
TTC	– Total transportation cost
Z(X)	– TTC
$Z(X^*)$	– Minimum TTC
NBC	– Non-Basic Cell
IBC	– Identified Basic Cell
NCC	– Net Cost Change
SI	– Solution Improvement

STAGE #1: OBTAIN AN IBFS TO THE GIVEN TP

For the given TP, first obtain an IBFS say $X^{(0)}$ with its associated total transportation cost $Z(X^{(0)})$ using any available method in TPs. We use the I-SOFT method [7] to obtain an IBFS because at present day it has been identified and established as the best method to find the best IBFS to TPs.

STAGE #2: TEST THE OPTIMALITY OF THE OBTAINED IBFS

Step 1: Construct the current solution table

Consider the transportation table (TT) en-squared with the obtained allocations (IBFS) $X^{(0)} = [x_{ij}]$ as the current solution table. Also, compute the corresponding TTC $Z(X^{(0)})$.

Step 2: Ensure the Non-degeneracy condition

Ensure the numbers of basic cells in the TT exactly equal to (m+n-1).

Step 3: Perform the Optimality Test on the IBFS $X^{(0)}$

(a) Determine $C(X^{(0)}) = \text{Max}\{c_{ij} : x_{ij} > 0\}$ and the corresponding basic cell as the identified basic cell (IBC). Let it be (h, k).

(i) If the IBC is unique, then go to Step (b) directly.

(ii) If there is two or more basic cells having the same largest UTC $C(X^{(0)})$, then select the basic cell having the maximum quantity of allocation as the IBC. Let it be (h, k) and go to Step (b).

(iii) If there is two or more basic cells having the same largest UTC $C(X^{(0)})$ and with the same maximum allocation quantity, then select any one such basic cell as the IBC. Let it be (h, k) and go to Step (b).

(b) Trace a SI loop starting and ending at the IBC (h, k) and passing through a non-basic cell. As it is a SI loop, it will have the Net Cost Change (NCC) value as non-positive (≤ 0). If there is a tie between two or more than two SI loops with the same NCC value, then select any one loop. Such a situation may generate alternative solutions to the given TP. If the NCC value of the SI loop is zero, then this will also indicate the existence of an alternative solution to the given TP.

(c) Implement this loop and obtain the better BFS, say $X^{(1)}$ with its associated TTC $Z(X^{(1)})$.

(d) If it is not possible to trace a SI loop starting and ending at the current IBC, then consider the next basic cell having UTC next to $C(X^{(0)})$ as the new IBC (h, k) and go to Step (a(i)).

Step 4: Repeat Steps 3(a) to (d) until no SI loop can be traced starting and ending at the new IBC with the current largest UTC. At this level, the solution under optimality test is the optimal one. Write the optimal solution X^* with its minimum TTC as $Z(X^*)$.

ALTERNATIVE OPTIMAL SOLUTION

At the 'optimal level', if the NCC value of the SI loop is zero, then this indicates that the given TP has an alternative optimal solution. By implementing this loop we can get the alternative optimal solution to the given TP.

IMPORTANT NOTE

1. One cannot restrict a SI loop with corner cells having UTCs less than or equal to the UTC of the identified basic cell.
2. One cannot restrict the place (even position or odd position) of the non-basic cell in a SI loop.

SIMILARITY BETWEEN 'MODI' AND 'MODA'

One can verify the following two statements regarding the similarity between the existing MODI (Modified Distribution) method and the proposed revised version of the MODA (Modified Allocation) method:

1. For a given non-optimal solution of a TP, the MODI method and the revised version of the MODA method take the same numbers of iterations to reach the 'optimal level'.
2. For a given non-optimal solution of a TP, the MODI method and the revised version of the MODA method identify the same identical 'non-basic cell' to enter as a 'basic cell' during the corresponding iteration.

DIFFERENCE BETWEEN 'MODI' AND 'MODA'

1. In a TT with size $m \times n$, there are mn cells in total. Out of these, (m+n-1) numbers are basic and the remaining mn - (m+n-1) are non-basic. Note that, $[mn-(m+n-1)] > (m+n-1)$, when $m, n > 3$. In MODI method one has to compute the net evaluation (opportunity cost) corresponding to each of the [mn - (m+n-1)] non-basic cells in order to identify the appropriate one to enter as a basic one. But, in the proposed revised version of the MODA method, it is enough to test a very few of the (m+n-1) basic cells (or at most (m+n-1) basic cells) to identify the 'leaving' basic cell (identified basic cell) into a non-basic one and an 'entering' non-basic cell into a basic one. Due to this the computational time is considerably less in the proposed revised version of the MODA method.
2. In MODI method, we trace a loop starting and ending at the identified non-basic cell with the most negative net evaluation. But in the revised version of the MODA method, we trace the Solution Improvement Loop starting and ending at the identified basic cell and through an opt NBC.

Tested optimal solution

An optimal solution to a TP is regarded as 'tested optimal' if it comes out of an algorithm after testing its optimality.

Default optimal solution

An optimal solution is regarded as 'default optimal' if it comes out of an algorithm as optimal directly. Default optimal is optimal by default.

IMPORTANT NOTE

1. In transportation problems, the optimal solutions generated through either the MODI method or Stepping Stone method [4, 11] are 'tested optimal'. So far, no direct method has been developed to produce the 'default optimal' solutions directly to transportation problems. This is the fact from the article titled Neeya? Naana? due to Murugesan R. [7]
2. For the assignment problems, the optimal solutions generated by the NILA technique [8] are 'tested optimal' and that of generated by the 'Hungarian' method and the 'Mantra' technique [8] are 'default optimal' by nature. The latter two are the 'direct' methods which produce 'default optimal' solutions to assignment problems. It is the design (divine!) specialty of the two methods.

3. NUMERICAL ILLUSTRATION

Suitable illustrative explanation helps the readers to understand the algorithm of the proposed revised version of the MODA method in a better way. Keeping in mind, two 'more challenging' TPs from the literature are illustrated.

Example-1: Consider the following cost minimization type BTP with five sources and five destinations, as given in Table 1.

Table 1: The given BTP

Sources	Destinations					Supply
	D1	**D2**	**D3**	**D4**	**D5**	
S1	8	8	2	10	2	**40**
S2	11	4	10	9	4	**70**
S3	5	2	2	11	10	**35**
S4	10	6	6	5	2	**90**
S5	8	11	8	6	4	**85**
Demand	**80**	**55**	**60**	**80**	**45**	**320**

SOLUTION BY THE PROPOSED 'REVISED VERSION OF THE MODA METHOD'

Stage #1: Obtain an IBFS

In Stage #1, we solve the given BTP by using the I-SOFT method and obtain the IBFS ($X^{(0)}$) table. This is shown in Table 2

Stage #2: Optimizing the obtained solution by the proposed revised version of the MODA method

Construct the current solution table

Consider the transportation table (TT) en-squared with the obtained allocations (solution) as the current solution table. This is the IBFS and is shown in Table 2.

Table 2: The IBFS table obtained by the I-SOFT method

Sources	Destinations D1	D2	D3	D4	D5	Supply
S1	8	8	[40] 2	10	2	**40**
S2	[15] 11	[55] 4	10	9	4	**70**
S3	[15] 5	2	[20] 2	11	10	**35**
S4	[45] 10	6	6	5	[45] 2	**90**
S5	[05] 8	11	8	[80] 6	4	**85**
Demand	**80**	**55**	**60**	**80**	**45**	**320**

Writing the IBFS $X^{(0)}$

As of Table 2, the IBFS is $X^{(0)} = \{x_{13} = 40, x_{21} = 15, x_{22} = 55, x_{31} = 15, x_{33} = 20, x_{41} = 45, x_{45} = 45, x_{51} = 05, x_{54} = 80\}$and the associated TTC is $Z(X^{(0)}) = \$1640$.

Optimality Testing for the IBFS $X^{(0)}$ [First Iteration]

(a) Determine $C(X^{(0)}) = \text{Max}\{c_{ij} ; x_{ij} > 0\} = \text{Max}\ \{2, \mathbf{11}, 4, 5, 2, 10, 2, 8, 6\} = 11$ at the unique cell (h, k) = (S2, D1). Therefore, the IBC is (S2, D1).

(b) Trace a SI loop starting and ending at the IBC (S2, D1) and passing through one opt NBC.

(d) As it is not possible to trace a SI loop, we consider the next IBC as (h, k) = (S4, D1) having $c_{41} = 10$, which is less than $C(X^{(0)}) = 11$.

(b) Trace a SI loop starting and ending at the next IBC (S4, D1). Actually two SI loops can be traced as

Loop 1 = {(S4, D1), (S4, D3), (S3, D3), (S3, D1)} passing through the NBC (S4, D3) with c_{43} = 6 and NCC value as -1.

Loop 2 = {(S4, D1), (S4, D4), (S5, D4), (S5, D1)} passing through the NBC (S4, D4) with $\mathbf{c_{44} = 5}$ and NCC value as **-3**.

Since Loop 2 is having a NBC with a minimum UTC and having minimum NCC value, we choose the Loop 2 for implementation.

(c) By implementing this loop we obtain the following better BFS shown in Table 3.

Table 3: A better BFS

	Destinations					
Sources	D1	D2	D3	D4	D5	Supply
S1	8	8	**40** 2	10	2	**40**
S2	**15** 11	**55** 4	10	9	4	**70**
S3	**15** 5	2	**20** 2	11	10	**35**
S4	10	6	6	**45** 5	**45** 2	**90**
S5	**50** 8	11	8	**35** 6	4	**85**
Demand	**80**	**55**	**60**	**80**	**45**	**320**

Writing the better BFS $X^{(1)}$

From Table 3, we see that the BFS is $X^{(1)}$ = {x_{13} = 40, x_{21} = 15, x_{22} = 55, x_{31} = 15, x_{33} = 20, x_{44} = 45, x_{45} = 45, x_{51} = 50, x_{54} = 35} and the computed TTC is $Z(X^{(1)})$ = $1505. Note that $X^{(1)}$ is a better BFS than $X^{(0)}$ as $Z(X^{(1)}) < Z(X^{(0)})$.

Optimality Testing for the better BFS $X^{(1)}$ [Second Iteration]

(a) Determine $C(X^{(1)})$ = Max{c_{ij} ; $x_{ij} > 0$} = Max {2, **11**, 4, 5, 2, 5, 2, 8, 6} = 11 at the cell (h, k) = (S2, D1).

(b) Trace a SI loop starting and ending at the IBC (S2, D1). There is only one SI loop passing through the NBC (S2, D5) with NCC value of the loop as -2.

(c) By implementing this loop we obtain a further better BFS shown in Table 4.

Table 4: A further better BFS

Sources	Destinations					Supply
	D1	D2	D3	D4	D5	
S1	8	8	**40** 2	10	2	**40**
S2	11	**55** 4	10	9	**15** 4	**70**
S3	**15** 5	2	**20** 2	11	10	**35**
S4	10	6	6	**60** 5	**30** 2	**90**
S5	**65** 8	11	8	**20** 6	4	**85**
Demand	**80**	**55**	**60**	**80**	**45**	**320**

Writing the further better BFS $X^{(2)}$

As of Table 3, we see that the further better BFS is $X^{(2)} = \{x_{13} = 40, x_{22} = 55, x_{25} = 15, x_{31} = 15, x_{33} = 20, x_{44} = 60, x_{45} = 30, x_{51} = 65, x_{54} = 20\}$ and the corresponding computed TTC is $Z(X^{(2)}) = \$1475$. Note that the BFS $X^{(2)}$ is a further better than the BFS $X^{(1)}$ as $Z(X^{(2)}) < Z(X^{(1)})$.

Optimality Testing for the further better BFS $X^{(2)}$ [Third Iteration]

(a) Determine $C(X^{(2)}) = \text{Max}\{c_{ij} ; x_{ij} > 0\} = \text{Max}\ \{2, 4, 4, 5, 2, 5, 2, \mathbf{8}, 6\} = 8$ at the cell (h, k) = (S5, D1). Therefore, the new IBC is (S5, D1).

(b) Trace a SI loop starting and ending at the IBC (S5, D1) and passing through one opt NBC.

(d) It is not possible to trace a SI loop starting and ending at the IBC (S1, D5). Similarly, it is not possible to trace any SI loop from other subsequent IBCs (S5, D4), (S3, D5) and so on. This indicates that the current solution $X^{(2)}$ under optimality test is the optimal solution.

Writing the Optimal Solution

The optimal solution (X^*) to the given TP is $X^* = \{x_{13} = 40, x_{22} = 55, x_{25} = 15, x_{31} = 15, x_{33} = 20, x_{44} = 60, x_{45} = 30, x_{51} = 65, x_{54} = 20\}$ with the minimum TTC of $Z(X^*) = \$1475$.

Example-2: Consider the following cost minimization type UTP with three sources and four destinations, as given in Table 5.

Table 5: The given UTP

Sources	Destinations				Supply
	D1	**D2**	**D3**	**D4**	
S1	3	48	14	2	**24**
S2	4	2	30	10	**24**
S3	36	8	12	12	**02**
Demand	**6**	**12**	**3**	**44**	**65 \ 50**

SOLUTION BY THE PROPOSED 'REVISED VERSION OF THE MODA METHOD'

Stage #1: Obtain an IBFS

In Stage #1, we solve the given UTP by using the I-SOFT method and obtain the near optimal solution ($X^{(0)}$) table as shown in Table 6. Note that, the dummy source S4 has been added to satisfy the extra demand of 15 units. The UTC for each of the dummy cells is assumed to be zero.

Stage #2: Optimizing the obtained solution by the proposed revised version of the MODA method

Construct the current solution table

Consider the TT en-squared with the obtained allocations (solution) as the current solution table. This is shown in Table 6.

Table 6: The IBFS table by the I-SOFT method

	Destinations				
Sources	D1	D2	D3	D4	**Supply**
S1	3	48	14	[**24**] 2	**24**
S2	[**6**] 4	[**10**] 2	30	[**8**] 10	**24**
S3	36	[**2**] 8	12	12	**02**
S4	0	0	[**3**] 0	[**12**] 0	**15**
Demand	**6**	**12**	**3**	**44**	**320**

Step 2: Ensure the Non-degeneracy condition

The obtained solution is non-degenerate as there are exactly 7, that is, (m+n-1) basic cells.

Writing the IBFS $X^{(0)}$

As of Table 2, the IBFS is $X^{(0)} = \{x_{14} = 24, x_{21} = 06, x_{22} = 10, x_{24} = 08, x_{32} = 02, x_{43} = 03, x_{44} = 12\}$and the associated TTC is $Z(X^{(0)}) = \$188$.

Optimality Testing for the IBFS $X^{(0)}$ [First Iteration]

(a) Determine $C(X^{(0)}) = \text{Max}\{c_{ij}\ ;\ x_{ij} > 0\}$ = Max {2, 4, 2, **10**, 8} = 10 at the unique cell (h, k) = (S2, D4). Therefore, the IBC is (S2, D4).

(b) Trace a SI loop starting and ending at the IBC (S2, D4) and passing through one NBC. There is only one such loop.

Loop = {(S2, D4), **(S3, D4)**, (S3, D2), (S2, D2), }, NCC = 12-10+2-8 = **-4**

(c) By implementing this loop we obtain the following better BFS shown in Table 7.

Table 7: The better BFS (Optimal Solution) by the MODA method

Sources	Destinations				
	D1	D2	D3	D4	**Supply**
S1	3	48	14	[**24**] 2	**24**
S2	[**6**] 4	[**12**] 2	30	[**6**] 10	**24**
S3	36	8	12	[**2**] 12	**02**
S4	0	0	[**3**] 0	[**12**] 0	**15**
Demand	**6**	**12**	**3**	**44**	**320**

Writing the better BFS $X^{(1)}$

As of Table 6, the better BFS is $X^{(1)} = \{x_{14} = 24, x_{21} = 06, x_{22} = 12, x_{24} = 06, x_{34} = 02, x_{43} = 03, x_{44} = 12\}$and the associated TTC is $Z(X^{(1)}) = \$180$. Note that the BFS $X^{(1)}$ is a better BFS than $X^{(0)}$ as $Z(X^{(1)}) < Z(X^{(0)})$.

Optimality Testing for the IBFS $X^{(1)}$ [Second Iteration]

(a) Determine $C(X^{(1)}) = \text{Max}\{c_{ij}\ ;\ x_{ij} > 0\}$ = Max {2, 4, 2, 10, **12**} = 12 at the unique cell (h, k) = (S3, D4). Therefore, the IBC is (S3, D4).

(b) Trace a SI loop starting and ending at the IBC (S3, D4) and passing through one

NBC. There is only one such loop.

Loop = {(S3, D4), **(S3, D2)**, (S4, D3), (S4, D4), }, NCC = 12-12+0-0 = **0**

(d) Further, it is not possible to trace a SI loop starting and ending at the new IBC (S2, D4). Similarly, it is not possible to trace any SI loop from other subsequent new IBCs (S2, D2), (S2, D2) and so on. This indicates that the current solution $X^{(1)}$ under optimality test is the optimal solution.

Writing the Optimal Solution

The optimal solution (X^*) to the given TP is $X^* = \{x_{14} = 24, x_{21} = 06, x_{22} = 12, x_{24} = 06, x_{34} = 02, x_{43} = 03, x_{44} = 12\}$ and the associated TTC is $Z(X^*) = \$180$.

Alternative Optimal Solution

As the NCC value of the SI loop starting and ending at the IBC (S3, D4) and passing through one NBC is zero, the given UTP has an alternative optimal solution. By implementing this loop we get an alternative optimal solution as shown in Table 8.

Table 7: The better BFS (Alternative Optimal Solution) table by the MODA method

	Destinations				
Sources	D1	D2	D3	D4	**Supply**
S1	3	48	14	[24] 2	**24**
S2	[6] 4	[12] 2	30	[6] 10	**24**
S3	36	8	[2] 12	12	**02**
S4	0	0	[1] 0	[14] 0	**15**
Demand	**6**	**12**	**3**	**44**	**320**

Writing the Alternative Optimal Solution

The alternative optimal solution (X^*) to the given TP is $X^* = \{x_{14} = 24, x_{21} = 06, x_{22} = 12, x_{24} = 06, x_{33} = 02, x_{43} = 01, x_{44} = 14\}$ and the associated TTC is $Z(X^*) = \$180$.

4. NUMERICAL EXAMPLES

To validate the efficiency of the proposed revised version of the MODA method, we have solved a set of 14 numbers of "more challenging" TPs of balanced and unbalanced categories in different small sizes, from various literatures and textbooks, which are listed in Table 9.

Table 9: A set of some “More Challenging” balanced and unbalanced TPs

BTP Problem No.	UTP Problem No.
Problem1 $[C_{ij}]$ 3×5= [1 9 13 36 51; 24 1216 20 1; 14 33 1 23 26] $[S_i]$ 3×1= [50, 100, 150] $[D_j]$ 1×5= [100, 70, 50, 40,40]	**Problem 1** $[C_{ij}]$ 3×3= [6 10 14; 12 19 21; 15 14 17] $[S_i]$ 3×1= [50, 50, 50] $[D_j]$ 1×3= [30, 40, 55]
Problem 2 $[C_{ij}]$ 4×5= 4 9 810 12; 6 10 3 2 3; 3 2 7 10 3; 3 5 5 4 8] $[S_i]$ 4×1= [24, 18, 20, 16]$[D_j]$ 1×5= [10, 20, 10, 18, 20]	**Problem 2** $[C_{ij}]$ 3×4= [19 30 50 10; 70 30 40 60; 40 8 70 20] $[S_i]$ 3×1= [7, 9, 18]$[D_j]$ 1×4= [40, 8, 7, 14]
Problem 3 $[C_{ij}]$ 4×6= [1 2 1 4 5 2;3 3 2 1 4 3;4 2 5 9 6 2;3 1 7 3 4 6] $[S_i]$ 4×1= [30, 50, 75, 20] $[D_j]$ 1×6= [20, 40, 30, 10, 50, 25]	**Problem 3** $[C_{ij}]$ 3×4= [10 15 12 12; 8 10 11 9; 11 12 13 10] $[S_i]$ 3×1= [20, 15, 12] $[D_j]$ 1×4= [14, 12, 8, 22]
Problem 4 $[C_{ij}]$ 5×5= [73 40 9 79 20; 62 93 96 8 13; 96 65 80 50 65; 57 58 29 12 87; 56 23 87 18 12] $[S_i]$ 5×1= [8, 7, 9, 3, 5] $[D_j]$ 1×5= [6, 8, 10, 4, 4]	**Problem 4** $[C_{ij}]$ 3×4= [42 48 38 37; 40 49 52 51; 39 38 40 43] $[S_i]$ 3×1= [160, 150, 190] $[D_j]$ 1×4= [80, 90, 110, 160]
Problem 5 $[C_{ij}]$ 5×5= [8 8 2 10 2; 11 4 10 9 4; 5 2 2 11 10; 10 6 6 5 2; 8 11 8 6 4] $[S_i]$ 5×1= [40, 70, 35, 90, 85] $[D_j]$ 1×5= [80, 55, 60, 80, 45]	**Problem 5** $[C_{ij}]$ 4×3= [2 7 14; 3 3 1; 5 4 7; 1 6 2] $[S_i]$ 4×1= [5, 8, 7, 15] $[D_j]$ 1×3= [7, 9, 18]
Problem 6 $[C_{ij}]$ 3×5= [5 7 10 5 3; 8 69 12 14; 10 9 8 10 15] $[S_i]$ 3×1= [5, 10, 10] $[D_j]$ 1×5= [3, 3, 10, 5, 4]	**Problem 6** $[C_{ij}]$ 3×4= [3 48 14 2; 4 230 10; 36 8 12 12] $[S_i]$ 3×1= [24, 24, 2] $[D_j]$ 1×4= [6, 12, 3, 44]
Problem 7 $[C_{ij}]$ 3×4= [6 1 9 3; 11 5 2 8; 10 12 4 7] $[S_i]$ 3×1= [70, 55, 90] $[D_j]$ 1×4= [85, 35, 50, 45]	**Problem 7** $[C_{ij}]$ 3×3= [4 8 8; 16 24 16; 8 16 24] $[S_i]$ 3×1= [76, 82, 77] $[D_j]$ 1×3= [72, 102, 41]

5. RESULT ANALYSIS

For the identified 14 “More Challenging” TPs (7 BTPs and 7 UTPs), as listed in Table 9, the IBFS generated by the I-SOFT method, and the optimal solution derived through the proposed revised version of the MODA method are shown in Table 10 and Table 11 respectively for the BTPs and UTPs .

Table 10: Derivation of optimal solutions to some "More Challenging" BTPs

BTP #	I-SOFT	Optimal Solution by MODA method	Minimum No. of iterations required to reach the optimal
1.	2810	2700	1
2.	332	316	2
3.	440	430	1
4.	1103	1102	1
5.	1640	1475	2
6.	187	183	1
7. *	1165	1160	1

Table 11: Derivation of optimal solutions to some "More Challenging" UTPs

UTP #	I-SOFT	Optimal Solution by MODA method	Minimum No. of iterations required to reach the optimal
1.	1695	1650	2
2.	883	743	3
3.	4780	4720	1
4.	17060	17050	1
5.	93	75	1
6. *	188	180	1
7. *	2752	2424	1

NOTE: The problems marked with * have alternative optimal solutions.

DECISION

One can easily verify that the existing MODI method also takes the same minimum numbers of iterations, as shown in Column-4 of Table 10 and Table 11, to reach the optimal solutions. Thereby, the revised version of the MODA method is an alternative to the MODI method to test the optimality of a solution and also optimizing a solution, if it's not optimal.

ADVANTAGE OF THE PROPOSED 'REVISED VERSION OF THE MODA METHOD' OVER THE EXISTING 'MODI' METHOD

In MODI method, one has to compute the net evaluation (opportunity cost) corresponding to each of the non-basic cells in order to identify the appropriate non-basic cell to enter into a basic one and thereby identify a basic cell to leave as a non-basic cell. But in the proposed revised version of the MODA method, it is enough to test a very few of the basic cells only to identify the appropriate basic cell to leave as a non-basic one and thereby identify a non-basic cell to enter as a basic cell. Due to this the computational time is considerably less in the proposed revised version of the MODA method.

6. CONCLUSION

A revised version of the existing MODA method proposed by R. Murugesan (2022) is presented for optimality testing and optimizing a solution found by any available method in transportation problems. Actually, the revised version reduces the efforts of tracing and considering all possible loops starting and ending at an identified basic cell in order to improve a solution. The revised version has been tested on a number of near optimal solutions obtained by the I-SOFT method. Testing outcomes authenticate that the presented method is the ideal one for testing the optimality of an obtained solution and also optimizing of that solution, if that's not optimal. Also, the presented method generates the alternative optimal solution(s) of a given transportation problem, it they exist to the problem. It is the added benefit of this presented method. Also, the presented method is much easier and an alternative to the existing MODI method with considerably less computation time than the MODI method.

REFERENCES

[1] Abdul Quddoos, Shakeel, and Khalid M.M., 2012, *A New Method for Finding an Optimal Solution for Transportation Problems,* International Journal on Computer Science and Engineering (IJCSE), 4(7), pp. 1271-1274.

[2] Abdul Quddoos, Shakeel, and Khalid M.M., 2016, *A Revised Version of ASM method for Solving Transportation Problem,* International Journal of Agricult. Stat. Sci., 12,(Supplement 1), pp. 267-272.

[3] Esakkiammal T., Murugesan R., 2021, *SOFTMIN – An Innovative and Efficient Zeros Allocation Method to Solve Transportation Problems,* 1st International Conference on Mathematical Modeling and its Soft Computing Applications (ICMMSCA-2021, November 29-30, 2021, Puttaparthi, Andhra Pradesh, India, Paper ID C-11/ISBN 978-81-952716-2-7.

[4] Kanti Swarup, Gupta P.K. and Man Mohan, 2017, *Operations Research,* 19th Edition, Sultan Chand & Sons, Educational Publishers, New Delhi.

[5] Murugesan R., 2022, *I-SOFT – The Best Method for Finding the Best IBFS to Transportation Problems,* Global Journal of Pure and Applied Mathematics, Volume 18, Number 01, pp. 377-391.

[6] Murugesan R., 2022, *MODA – An Innovative Method for Optimality Testing and Optimizing a Solution in Transportation Problems,* Global Journal of Pure and Applied Mathematics, Volume 18, Number 02, pp. 479-498.

[7] Murugesan R., 2022, *NEEYA? NAANA? – Some Recent Direct Methods versus Some Challenging Problems in Transportation Problems,* International Journal of Advanced Research, Vol.10, No.05, pp. 526-534.

[8] Murugesan R., 2022, *NILA – An Innovative Technique for Optimality Testing and Optimizing a Solution in Assignment Problems*, Global Journal of Pure and Applied Mathematics, Volume 18, Number 2, pp. 433-452.

[9] Murugesan R., 2022, *SOFTMIN Method versus IASM Method on Some Challenging Transportation Problems*, Global Journal of Pure and Applied Mathematics, Volume 18, Number 1, pp. 341-351.

[10] Murugesan R., Esakkiammal T., 2020, *An Improved ASM Method for the Transportation Problem*, Advances in Mathematics: Scientific Journal, 9(10**),** pp.8259–8271.

[11] Sharma J.K., 2017, *Operations Research-Theory and applications*, Trinity Press, New Delhi.