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# Estimate of Number of Zeros of Random Polynomials.

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**Abstract**

Let us consider the random trigonometric polynomial

$T = T_n(\theta,\omega) = \sum_{k=1}^{n} a_k(\omega)\cos k\theta$ in the interval (Φ', Φ''). We have to prove that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions ($T_n$(θ,w) have $\frac{2n}{\sqrt{6}} + 0(\log n)$ zeros when n is large.

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**Theorem 1.** Let EN (T, Φ', Φ'') denote the average number of real roots of the random trigonometric polynomial

$$T = T_n(\theta,\omega) = \sum_{k=1}^{n} a_k(\omega)\cos k\theta$$

in the interval (Φ', Φ''). Clearly, T can have at most 2n zeros in the interval (0, 2π). Assuming $a_k$(w)s to be mutually independent and identically distributed normal random variables. Dunnage has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions ($T_n$(θw) have $\left(2n/\sqrt{3}\right) + O\left\{n^{11/13}(\log n)^{3/13}\right\}$ zeros when n is large. We consider the same family of trigonometric polynomials and use the Kac-Rice formula for the exception of the number of real roots and obtain that

$$EN(T;0,2\pi) \sim \frac{2n}{\sqrt{6}} + 0(\log n). \qquad (1)$$

This result is better than that of Dunnage since our constants is (1/√2) times his constant and our error term is smaller. The proof is base on the convergence of an integral of which an asymptotic estimation is obtained.

## 1. INTRODUCTION

Let N (T, Φ', Φ'') be the number of real roots of the trigonometric polynomial

$$T = T_n(\theta, \omega) = \sum_{k=1}^{n} a_k(\omega) \cos k\theta$$

in the interval (Φ', Φ'') where the coefficients $a_k(w)$ are mutually independent random variables identically distributed according to the normal law, and when multiple roots are counted only once. Let EN (T, Φ', Φ'') denote the exception of N (T, Φ', Φ''). Obviously, $T_n(\theta w)$ can have at most 2n zeros in the interval (0, 2π). Dunnage [1] has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions ($T_n(\theta w)$ have $\frac{2n}{\sqrt{3}} + O(n^{11/13}(\log n)^{3/13}$

zeros when n is large. The measure of the exceptional set does not exceed $(\log n)^{-1}$. Subsequently, Das[2] and Qualls[3] have obtained similar results. In this note our purpose is to show that it is possible to obtain a still lower estimate for the exception of the number of real roots of (1) by using the method of Logan and Sheep [4]. We show that

$$EN(T; 0, 2\pi) \sim \frac{2n}{\sqrt{6}} + 0(\log n)$$

The result is better than that of Dunnage since our constants is (1/√2) times his constant and our error term is smaller.

## 2. The approximation for $EN(T; 0, 2\pi)$

Let L(n) be a positive-valued function of n such that L(n) and n/L(n) both approach infinity with n. We take $\varepsilon = L(n)/n$ throughout.

Outside a small exceptional set of values of w, ($T_n(\theta w)$ has a negligible number of zeros in each of the intervals $(0, \varepsilon), (\pi - \varepsilon, \pi + \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$ . By periodicity, the number of zeros in $(0, \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$ is the same as the number in $(-\varepsilon, \varepsilon)$ . We shall use the following lemma.

***LEMMA1 :-*** *The probability that $T_n(\theta w)$ has more than*

$$1 + (2/\log 2)(\log n + 2n\varepsilon)$$

zeros in $\omega - \varepsilon \le \theta \le \omega + \varepsilon$ does not exceed 2 exp (-nε).

This lemma is due to Das[2], in the special case $D_n = \sum b_n = n.$

The expected number of zeros of T in the interval (Φ', Φ'') is given by the Kac-Rice formula.

$$EN(T; \Phi', \Phi'') = \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} |\eta| \rho(0,\eta) d\eta \qquad (2)$$

where the probability density ρ(ζ, n) for T= ζ and T=n is given by the Fourier inversion formula

$$\rho(\xi,\eta) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\xi\xi - i\eta\eta z)\varphi(yz)dydz$$

$\varphi(y,z) = E\{\exp(iTy + iT'z)$ being the characteristic function of the combined variable (T,T'). In our case, we have

$$T = \sum_{k=1}^{n} a_k(w)\cos k\theta, T' = \sum_{k=1}^{n} ka_k(w)\sin k\theta$$

$$\varphi(y,z) = \exp\left\{-\sum_{k=1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}$$

$$\rho(0,\eta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \exp(1 - i\eta z)\exp\left\{-\sum_{k=1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}dy$$

For ε>0,

$$\int_{-\infty}^{\infty} |\eta| \exp(-\in|\eta|)\,\rho\,(0,\eta)\,dn$$

$$= \operatorname{Re}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)\,dn\int_{-\infty}^{\infty} dz$$

$$\int_{-\infty}^{\infty}\exp(-i\eta z)\exp\left\{-\sum_{1}^{n}(y\cos k\theta - zk\sin k\theta\right\}dy$$

$$= \operatorname{Re}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\varepsilon - iz)^2} + \frac{1}{(\varepsilon - iz)^2}\right\}$$

$$x\exp\left\{-\sum_{1}^{n}(y\cos k\theta - zk\sin k\theta\right\}dy(3)$$

where Re stands for the real part.

Here, if we allow cos kθ, k sin kθ to be arbitrary, that is we take each of them to be constant in k, then the probability density $\rho(\xi,\eta)$of $\xi = T(\theta) - AX$ and $\eta = T'(\theta) = BX$, say, degenerates and we get from (3) the following identity, valid for non-zero A and B which can be chosen suitably.

$$0 = \operatorname{Re}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\varepsilon - iz)^2} + \frac{1}{(\varepsilon - iz)^2}\right\}\exp\left\{-(Ay - Bz)^2\right\}dy. \qquad (4).$$

Subtracting (4) from (3) we get

$$\int_{-\infty}^{\infty} |\eta| \exp(-\in|\eta|)\, \rho\,(0,\eta)\, dn$$

$$= \mathrm{Re}\frac{1}{2\pi^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \left\{ \frac{1}{(\varepsilon - iz)^2} + \frac{1}{(\varepsilon - iz)^2} \right\}$$

$$\text{x} \exp\left\{ -\sum_{1}^{n} (y \cos k\theta - zk \sin k\theta)^2 \right\} - \exp\,(-(Ay - Bz)^2)dy$$

$$= \mathrm{Re}\frac{1}{\pi^2}\int_{0}^{\infty} dz \int_{-\infty}^{\infty} \left\{ \frac{1}{(\varepsilon - iz)^2} + \frac{1}{(\varepsilon - iz)^2} \right\}$$

$$\text{x} \{\exp\,(-Gz^2) - \exp\,(-Hz^2)\} du. \qquad (5).$$

by transforming the integrals putting y=-uz or y=uz and denoting

$$G = \sum_{k=1}^{n} (u \cos k\theta + k \sin k\theta)^2$$

and $H=(Au+B)^2$

Now using the identity (see Logan and Shepp [4], for α=2),

$$\int_{0}^{\infty} \{\exp(-Hz^2) - \exp\,(Gz^2)\} \frac{dz}{z} = \frac{1}{2} \log\,(G/H)$$

in the limit as $\varepsilon \to 0$ we obtain from (5) that

$$\int_{-\infty}^{\infty} |\eta| \rho(0,\eta) d\eta = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \log \left\{ \frac{\sum_{k=1}^{n} (u \,\mathrm{Cos}\, k\theta + k \,\mathrm{Sin}\, k\theta)^2}{(Au + B)^2} \right\} du \qquad (6).$$

which has been shown in ξ3 to be a convergent integral.

The double integral appearing in (5) is dominated by a decreasing exponential function. So the involved integrals are uniformly convergent on any interval. Since the integral on the right side of (6) converges, we conclude that both the passage to the limit by letting ε→0 and the subsequent change of the order of integration to produce the equation (6) are justified.

**3. Estimation of the integral of equation :-** In this section we obtain an asymptotic estimation for the integral

$$I = \int_{-\infty}^{\infty} \log \left\{ \frac{\sum_{k=1}^{n} (u \,\mathrm{Cos}\, k\theta + k \,\mathrm{Sin}\, k\theta)^2}{(Au + B)^2} \right\} du$$

where A and B are fixed non-zero real numbers. This integral exists in general as a principal value i.e.,

$$\lim_{R\to\infty}\int_{-R}^{R}......,\text{if } A^2=\sum_{k=1}^{n}\cos^2 k\theta$$

Let $B^2=\sum_{1}^{n}k^2\sin^2 k\theta$ and $C^2=\sum_{1}^{n}k\cos k\theta\sin k\theta.$

As in Das[2,pp27] we have for

$$A^2=\frac{1}{2}\{1+0(1/\log n)\}n=\frac{1}{2}Sn,$$

say

$$B^2=\frac{1}{6}\{1+0(1/\log n)\}n^3=\frac{1}{6}Sn^3,$$

and

$$C^2=O(n^2/\log n)=\beta n^2/\log n,(\beta=\text{constant}),$$

taking

L(n)=log n,

We have always by Cauchy's inequality, $AB\geq C^2$. In what follows we will assume that $AB>C^2$. This happens if θ does not take values from the set {0±π, ±2 π,…….}.

In fact,

$$A^2B^2-2C^4=\frac{S^2n^4}{12}\left\{1-\frac{24\beta^2}{S^2(\log n)}\right\}\cong\frac{S^2n^4}{12}=A^2B^2 \qquad (7)$$

So that

$$I=\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{k=1}^{n}(u\cos k\theta+k\sin k\theta)^2}{(Au+B)^2}\right\}du$$

$$= \int_{-\infty}^{\infty} \log\left\{\frac{(A^2u^2+B^2)^2-4u^2C^4}{A^4u^4+B^4-2u^2A^2B^2}\right\}du$$

$$\cong \int_{-\infty}^{\infty} \log\left\{\frac{A^4u^4+B^4+2u^2A^2B^2}{A^4u^4+B^4-2u^2A^2B^2}\right\}du, \text{ by (7)}$$

$$= \text{I}', \text{say}$$

$$= \int_0^{\infty} \log\left\{\frac{1+x}{1-x}\right\}du, \text{ writing x} = (2u^2A^2B^2)/(A^4u^4+B^4)$$

$$= \int_0^{\infty} \log\left\{1-\frac{4x}{(1+x)^2}\right\}^{-1/2} du,$$

$$= \frac{1}{2}\int_0^{\infty} \{-\log(1-\text{z})\}du, \textit{ putting } \text{z} = 4\text{x}/(1+\text{x})^2. (8)$$

Now $x\to 0^+$ as $u\to 0$ or $\infty$. But $\text{x} > \varepsilon > 0$, if $\varepsilon\text{A}^4\text{u}^4 + \varepsilon\text{B}^4 - 2\text{u}^2\text{A}^2\text{B}^2 < 0$, which occurs for all u in the interval $(\text{d}_1\{0(\text{n}^2)/\sqrt{\text{e}}\}\text{-d}_2)$, where $d_1$,$d_2$ are functions of ε tending to zeros as ε→0. Thus for all u in the interval (0, ∞) we can safely assume that ε=1/n, and x=0{1/L(n)}, where n is tending to infinity.

Thus

$$\text{I}' > 2\int_0^{\infty} \left\{\frac{\text{x}}{(1+\text{x})^2}\right\}\text{du}$$

$$= 2\int_0^{\infty} \left\{1-\frac{1}{\text{L(n)}+1}\right\}^2 x\,du$$

$$= 4\left\{1-\frac{1}{\text{L(n)}+1}\right\}^2 \int_0^{\infty} \left\{\frac{\text{u}^2\text{A}^2\text{B}^2}{\text{A}^4\text{u}^4+\text{B}^4}\right\}du$$

$$= \frac{4\text{B}}{\text{A}}\left\{1-\frac{1}{\text{L(n)}+1}\right\}^2 \int_0^{\infty} \left\{\frac{\text{v}^2}{\text{v}^4+1}\right\}dv$$

$$= \left\{1-\frac{1}{\text{L(n)}+1}\right\}^2 . \frac{2\pi\text{n}}{\sqrt{6}} \qquad \textit{(9)}$$

Again

$$I' < \frac{1}{2}\int_0^\infty \left\{\frac{z}{(1-z)}\right\}du = \frac{1}{2}\int_0^\infty \left\{\frac{4x}{(1-x)^2}\right\}\mathrm{du}$$

$$= 4\left\{1-\frac{1}{L(n)+1}\right\}^2 x du = \left\{1+\frac{1}{L(n)-1}\right\}^2 . \frac{2\pi n}{\sqrt{6}} \qquad (10)$$

Now from (9) and (10) $\quad \mathrm{I}' \sim \dfrac{2\pi \mathrm{n}}{\sqrt{6}} \qquad (11)$

Now from (8) and (11) $\quad \mathrm{I}' \sim \dfrac{2\pi \mathrm{n}}{\sqrt{6}} \qquad (12)$

**4. EN (T; Φ', Φ''):-** From (2), (6) and (12), we obtain

$$\mathrm{EN}(\mathrm{T}; \Phi', \Phi'') = \frac{(\Phi'' - \Phi')\mathrm{n}}{\sqrt{6}}$$

In view of our choice of A, B and C

$$\mathrm{EN}(\mathrm{T}; \pi+\varepsilon, 2\pi-\varepsilon) = \mathrm{EN}(\mathrm{T}; \varepsilon, \pi-\varepsilon)$$

Again, by lemma, we have

$$\mathrm{EN}(\mathrm{T}; 0, \varepsilon) + \mathrm{EN}(\mathrm{T}; \pi-\varepsilon, \pi+\varepsilon) + \mathrm{EN}(\mathrm{T}; 2\pi-\varepsilon, 2\pi)$$

$$= 2\mathrm{EN}(\mathrm{T}; -\varepsilon, \varepsilon) \le 2\{1 + (2/\log 2)(\log \mathrm{n} + 2\mathrm{n}\varepsilon)\}.$$

Now choosing $\varepsilon = (\log \mathrm{n})/\mathrm{n}$, the desired results follows.

## CONCLUSION

By considering the random trigonometric polynomial $\mathrm{T} = \mathrm{T}_\mathrm{n}(\theta, \omega) = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a}_\mathrm{k}(\omega)\cos \mathrm{k}\theta$

We have shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions ($\mathrm{T_n}(\theta w)$) have $\left(2\mathrm{n}/\sqrt{3}\right) + \mathrm{O}\left\{\mathrm{n}^{11/13}(\log \mathrm{n})^{3/13}\right\}$ zeros when n is large. We consider the same family of trigonometric polynomials and use the Kac-Rice formula for the exception of the number of real roots and obtain that

$$\mathrm{EN}(\mathrm{T}; 0, 2\pi) \sim \frac{2\mathrm{n}}{\sqrt{6}} + 0(\log \mathrm{n}).$$

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