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Some Interesting Examples of Group Rings Structure *R*[*G*]

Dr. Hiteshwar Singh and Dhananjay Kumar Mishra

Retd. Senior Assistant Professor
Dept. of Mathematics, M.L.S.M. College, Darbhanga
Univ. Dept. of Mathematics, L.N.M.U. Darbhanga

Abstract

In this paper we have discussed upon the different kinds of examples of group rings algebraic structure. Some of these examples are of special types. In a group ring $R[G]$ if (x^n-1) is principal ideal then, $R[G] \cong \frac{R[x]}{(x^n-1)}$. If $C[G]$ is a complex group ring and M_i is a maximal ideal then, $C[G] \cong \frac{C[x]}{\prod_{i=1}^{n} M_i}$. Again if $R[G]$ is a group ring then $R[G] \cong \prod_{i=1}^{n} M_{n_i}(D_i)$, here $M_{n_i}(D_i)$ is *i-th* matrix division ring. If $C[Z]$ is a complex group ring with integral group then we have observed that $C[Z] \cong C[x, x^{-1}]$. We have also discussed about the group ring over a finite group as well as over an infinite group. We have tried to present group rings over a quaternion group. This paper also explains the conjecture of Keplansky over an infinite group in complex field.

Key words: Principal ideal, Matrix division rings, quaternion group, isomorphism, Keplansky conjecture, cyclic group, Maschke's theorem.

Different examples of group rings *R*[*G*]:

1. Group rings over a cyclic group in real field *R*

Let us suppose that G be a cyclic group of order n. So we can write G = $\{1_g, g, g^2, \ldots\ldots\ldots, g^{n-1}\}$. Now let there be a ring of real number R, then group ring $R[G]$, may be a ring of real number R, and group ring $R[G]$, may be written as $\{1 + r_1 g + r_2 g^2 +$

r_3g^3 + …… + $r_{n-1}g^{n-1}$} here $r_i \in R$. We have supposed that $R[x]$ be the set of all polynomials in an indeterminate x with coefficients from real number R. Let us suppose a ring homomorphism ϕ such that $\phi: R[x] \rightarrow R[G]$. This mapping is surely a surjective mapping. It is clear that if we choose $x^n - 1$ as a polynomial in $R[x]$, then these types of polynomials give identity element of $R[G]$ under surjective mapping ϕ. That is $\phi : (x^n - 1) \rightarrow 1_g.1_R$ as we know that $1_g.1_R = 1 \in R[G]$. So we can say that $(x^n \pm 1)$ are polynomial in the set of Kernel ϕ. Let us suppose that $p(x)$ be any polynomials in the kernel of ϕ then we can write polynomial $p(x) = q(x).(x^n - 1) + r(x)$. As it is obvious that degree of $r(x)$<n. But there is not any polynomial whose degree is less than n therefore $r(x) = 0$. If the ring R has a unit element 1 then we suppose a function f such that

$$f(g) = 1_R \cdot 1_G + \sum_{g \neq G} 0.g = \begin{cases} 1 & g = 1_G \\ 0 & g \neq 1_G \end{cases}$$

Thus we have observed that polynomial $p(x) = q(x).(x^n - 1)$ and Ker ϕ is a principal ideal domain (**PID**) while polynomial $(x^n - 1)$ is principal ideal. So we can say that Ker ϕ is principal ideal domain generated by polynomial $(x^n - 1)$. Finally we have found that $R[G]$ is isomorphic to the quotient ring $R[x]/(x^n - 1)$ or mathematically it will be represented as,

$$R[G] \cong R[x]/(x^n - 1).$$

2. Group rings *R*[*G*] based on the complex field *C*

Let us suppose that C be a complex field and G be a cyclic group of order n. Thus group G will be a set {1, g, g^2, ………,g^{n-1}}. Let us choose a polynomial $(x^n - 1)$ of $C[x]$. This polynomial can be factored as,

$$x^n - 1 = (x - y_1)(x - y_2)(x - y_3) \ldots\ldots\ldots\ldots (x - y_n)$$

Here we have taken n as a prime positive integer. Then this factorization of $(x^n - 1)$ will be as *n-th* root of unity and all *n-th* roots of unity will be distinct. This means all y_1, y_2, y_3, …..,y_n are distinct.If M_i be the maximal ideal of $C[x]$ which is generated by the polynomial $(x - y_i)$. As we know that if $n_1, n_2, n_3, \ldots n_k$, are integers greater than 1 and are divisor of N also $N = n_1. n_2. n_3. \ldots n_k$ then the ring of integers modulo N is isomorphic to the product of the rings of integers modulo the n_i. Thus we have as follow,

$$C[G] \cong \frac{C[x]}{\prod_{i=1}^{n} M_i}.$$

3. Group rings $R[G]$ based on the theorem of Wedderburn–Artin

Let us suppose that $R[G]$ be a group ring which is semi-simple. Then group ring $R[G]$ can be written as direct sum of n_iV_i, here n_i is *i-th* positive integer and V_i is *i-th* mutually non-isomorphic simple right $R[G]$ – sub module. So we have,

$$R[G] = n_1V_1 \oplus n_2V_2 \oplus \dots\dots\dots\dots \oplus n_tV_t$$

$$\Rightarrow \text{End}_{R[G]}(R[G]) = \text{End}_{R[G]}(n_1V_1 \oplus n_2V_2 \oplus \dots\dots\dots\dots \oplus n_tV_t)$$

$$= \text{End}_{R[G]}(n_1V_1) \text{ x } \text{End}_{R[G]}(n_2V_2) \text{ x } \dots\dots\dots\dots \text{ x } \text{End}_{R[G]}(n_tV_t)$$

$$\Rightarrow \text{End}_{R[G]}(R[G]) = Mn_1(D_1) \text{ x } Mn_2(D_2) \text{ x } \dots\dots\dots\dots \text{ x } Mn_t(D_t)$$

$$\Rightarrow R[G] \cong Mn_1(D_1) \text{ x } Mn_2(D_2) \text{ x } \dots\dots\dots\dots \text{ x } Mn_t(D_t)$$

$$\Rightarrow R[G] \cong \prod_{i=1}^{n} M_{n_i}(D_i).$$

4. Group rings $R[G]$ when there is a complex field C and integral group Z

If we have a complex field C and group of integers Z then,

$$C[Z] \cong C[x^0\ x^{-1}].$$

5. Group rings over cyclic group G

Let us suppose that G be a cyclic group of three elements, with generator α and identity element 1. Thus we have $G = \{1_G, \alpha, \alpha^2\}$ and an element r of $C[G]$ will be written as $r = Z_0.1_G + Z_1.\alpha + Z_2.\alpha^2$ here Z_0, Z_1 and Z_2 are elements of complex field C. Similarly let us choose other element of $C[G]$ as $s = W_0.1_G + W_1.\alpha + W_2.\alpha^2$ here W_0, W_1 and W_2 are elements of complex field C. Then the sum of these two elements r and s are $r + s = \{(Z_0 + W_0).\ 1_G + (Z_1 + W_1).\ \alpha + (Z_2 + W_2)\ \alpha^2\}$ and product will be

$$r.s = \{ (Z_0\ W_0 + Z_1\ W_2 + Z_2\ W_1).\ 1_G + (Z_0\ W_1 + Z_1\ W_0 + Z_2\ W_2).\ \alpha$$

$$+ (Z_0\ W_2 + Z_2\ W_0 + Z_1\ W_1)\ \alpha^2\}$$

6. Group rings over quaternion group Q_8

Let us suppose that Q_8 be the quaternion group, whose elements are written in a set $\{e, \bar{e}, i, \bar{i}, j, \bar{j}, k, \bar{k}\}$. Now we consider the group ring $R[Q_8]$ where R is the set of real numbers. Any arbitrary element of this group ring will be written in the form, $x_1 \cdot e + x_2 \cdot \bar{e} + x_3 \cdot i + x_4 \cdot \bar{i} + x_5 \cdot j + x_6 \cdot \bar{j} + x_7 \cdot k + x_8 \cdot \bar{k}$, here x_i is a real number.

The multiplication of the elements of group rings $R[Q_8]$ in which group is quaternion will be written as, if we choose $\left(3\cdot e+\sqrt{2}\cdot i\right)$ and $\left(\frac{1}{2}\right)\cdot j$ as any two arbitrary elements of $R[Q_8]$, then

$$\left(3\cdot e+\sqrt{2}\cdot i\right)\left(1/2\cdot \bar{j}\right) = 3/2\left(e\cdot \bar{j}\right) + \sqrt{2}/2\left(i\cdot j\right)$$

$$=\left(3/2\right)\bar{j} + \left(\sqrt{2}/2\right)k$$

The quaternion group Q_8 is a non-abelian group of order eight, which is isomorphic to the eight element subset {1, *i. j, k,* –1, –*i*. –*j*, –*k*} of the quaternion under multiplication. It can be written by the group presentation $Q_8 = \{ \bar{e}, i. j, k \}$ $\bar{e}^2 = e$, $i^2 = j^2 = k^2 = i. j. k = \bar{e}$, here e is the identity element and $\bar{e}$ commutes with other element of the group. In another way we can present $Q_8 = \{ a, b | \ a^4 = e.\ a^2 = b^2,\ ba = a^{-1}b \}$

7. Group rings over symmetric group S_3

We can create another example of group rings structure by choosing symmetric group S_3 of order 3 and ring Z as an integral ring. This is a non-abelian group ring $Z[S_3]$. If we have [1 – (12)]. [1 + (12)] = 1 – (12) (12) + (12) – (12) = 1 – 1 = 0, here the element (12) $\in S_3$ as it is a transposition and permutation which only takes 1 and 2. Therefore we can say that the given group rings need not be an integral domain even if the underlying ring is an integral domain.

8. Group rings *R*[*G*] over an infinite group *G*

Let Group rings *R*[*G*] be over an infinite group. We have very less information about group *G* which is countably infinite or uncountable. Under such condition if we have chosen field of complex number *C*, then it was observed by Irving Kaplansky that if one choose *a* and *b* as elements of *C*[*G*] with *ab* = 1 which implies *ba* = 1. But this is true if field *C* has positive characteristic, while field *C* under such condition is not known till now. We have a Kaplansky conjecture which says that if *G* is a torsion free group and *K* is a field then the group ring *K*[*G*] has no non-trivial zero divisor. This conjecture is equivalent to *K*[*G*] which has no non-trivial nilpotent under the similar hypotheses for *K* and *G*. Thus *K* is a field can be relaxed to any ring. This ring can be embedded into an integral domain. But in some cases it is seen that the torsion free group also satisfy the zero divisor conjecture. These special cases are,

(i) Unique product groups
(ii) Elementary amenable groups
(iii) Diffuse groups

9. Group rings *R*[*G*] over a finite group *G*

Group algebra is seen naturally in the theory of group representations of finite groups. It is the free vector space on *G* over field *K*. For *x* in *K*[*G*], it is clear that

$$x = \sum_{g \in G} \alpha_g \cdot g \,.$$

Interpretation of the free vector space as *K* valued functions on group *G* as the algebra of multiplication forms the convolution of functions. Group ring *R*[*G*] over finite group *G* forms discrete topology which corresponds with compact support. The group algebra *C*[*G*] of a finite group over the complex numbers is a semi-simple ring. According to the Maschke's theorem, *C*[*G*] as a finite product of matrix rings with entries in complex field *C*. Thus we can write it as algebra of homomorphism.

$$\rho : C[G] \to \overset{m}{\underset{K=1}{\oplus}} \text{End}\,(V_K) \cong \overset{m}{\underset{K=1}{\oplus}} M_{d_K}(C)$$

10. Laurent polynomials

Laurent polynomials over a ring *R*, is group ring of the infinite cyclic group *Z* over *R* as *R*[*Z*].

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