Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 19, Number 2 (2023), pp. 281-301

https://dx.doi.org/10.37622/GJPAM/19.2.2023.281-301

Multi - decomposition of Complete Bipartite Graphs into Stars and Bowties of Size l

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Abstract

Let $K_{m,n}$ denotes a complete bipartite graph with vertex partitions of cardinality m and n. A star S_l denotes a complete bipartite graph $K_{1,l}$. A Bowtie B_l is a graph formed by the union of two cycles of length $l/2$ intersecting at a common vertex whenever l is even. A decomposition of a graph G is a collection of edge disjoint subgraphs H such that every edge of G belongs to exactly one H. Given non - isomorphic subgraphs H_1 and H_2 of G, a (H_1, H_2) multi - decomposition of G is the decomposition of G into a copies of H_1 and b copies of H_2, such that $aH_1 \oplus bH_2 = G$, for some integers $a, b \geq 0$. In this paper, the multi - decomposition of $K_{m,n}$ into S_l and B_l has been investigated. It is proved that for given positive integers m, n and l, when $m \equiv 0(\text{mod } l)$, $n \geq l/2$ for $l \equiv 0(\text{mod } 8)$ or $l \equiv 4(\text{mod } 8)$, $K_{m,n}$ can be decomposed into a copies of S_l and b copies of B_l for some of the admissible pairs (a, b) if and only if $l(a + b) = mn$.

Keywords: Decomposition, Multi - decomposition, Complete Bipartite Graph, Star, Bowtie

Mathematics Subject Classification: 05C70.

1. INTRODUCTION

All graphs considered here are simple and finite. For basic terminologies and notations we refer [2]. The complete bipartite graph with vertex partitions of cardinality m and n is denoted by $K_{m,n}$. Let S_l denotes a Star with l edges. i.e., $S_l \cong K_{1,l}$.

A Bowtie consists of two cycles $l/2$ intersecting at a common vertex and is denoted by B_l, whenever l is even. If $H_1, H_2, ..., H_r$ are edge disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup ... \cup E(H_r)$ then we say that $\{H_1, H_2, ..., H_r\}$ decompose G. If each $H_i \cong H$, $1 \leq i \leq r$, then we say that H decomposes G and we denote it by $H|G$. If G can be decomposed into a copies of H_1 and b copies of H_2 where $H_1 \neq H_2$ then we say that G has a $\{aH_1, bH_2\}$ - decomposition or (H_1, H_2) multi - decomposition of G. If the necessary conditions are satisfied by a pair (a, b), then it is known as an admissible pair. If the multi - decomposition exists for all admissible pairs (a, b) of non - negative integers, then we say that G has a $(H_1, H_2)_{\{a,b\}}$ - decomposition. For any subset S of $E(G)$, we use $G \backslash S$ to denote the graph obtained by deleting all of the edges of S from G and for two subgraphs H_1 and H_2 of G, we use $H_1 \oplus H_2$ to denote the edge disjoint union of H_1 and H_2. For any simple graph G, let sG denotes s vertex disjoint copies of G and $G(s)$ is the edge disjoint sum of s copies of G. In particular, if $H_1 = H_2 = H$ say, then $H_1 \oplus H_2$ is denoted by $H(2)$.
T. W. Shyu [17] obtained the decomposition of complete bipartite graph $K_{m,n}$ into paths P_{k+1} and stars S_{k+1} with k edges. In particular, they found the necessary and sufficient conditions for accomplishing this when $m > k$ and $n \geq 3k$, and they have given a complete solution when $k = 3$. S. Jeevadoss and A. Muthusamy [9] obtained the decomposition of complete bipartite graph $K_{m,n}$ into paths and cycles with k edges. In particular, they have obtained the necessary and sufficient conditions for such decomposition in $K_{m,n}$, when $m \geq k/2, n \geq \lceil k + 1/2 \rceil$ for $k \equiv 0 (\text{mod } 4)$ and $m, n \geq 2k$ for $k \equiv 2 (\text{mod } 4)$. Chao - Chih Chou et al. [3] obtained the decomposition of complete bipartite graph $K_{m,n}$ into p copies of C_4, q copies of C_6 and r copies of C_8, if $m \geq 4, n \geq 6$ and m, n are even, and the same kind of decomposition exists in $K_{m,n} - I$ where I is a 1-factor of $K_{m,n}$, if n is odd.
Many authors discussed the multi - decomposition as combination of stars with cycles or paths in complete multi - graphs, complete bipartite graphs and product graphs [1, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18].
All these studies motivate us to study the multi - decomposition of $K_{m,n}$ into stars and bowties of size l for the given positive integers m, n and l.

2. SOME USEFUL RESULTS

The following results will be used in proving our main results.

Theorem 2.1. *[18] A complete bipartite graph $K_{m,n}$ can be decomposed into union of S_k if and only if m and n satisfy one the following three conditions:*

1. *$n \equiv 0$(mod k) when $m < k$*

2. $m \equiv 0$*(mod k) when* $n < k$

3. $mn \equiv 0$*(mod k) when* $m \geq k$ *and* $n \geq k$. □

Lemma 2.2. *[9] If* k, m *and* n *be positive even integers such that* $k \equiv 0$*(mod 4)*, $k/2 \leq m \leq n < k$ *and* $n \neq k/2$*, then the graph* $K_{m,n}$ *has a* (pP_{k+1}, qC_k) *- decomposition.* □

Lemma 2.3. *[9] If* k *be positive even integers such that* $k \equiv 0$*(mod 4), then the graph* $K_{k/2,k}$ *has a* (pP_{k+1}, qC_k) *- decomposition.* □

Lemma 2.4. *[9] If* k *be positive even integers such that* $k \geq 4$*, then the graph* $K_{k,k}$ *has a* (pP_{k+1}, qC_k) *- decomposition.* □

Theorem 2.5. *[2] A complete bipartite graph has no odd cycle.* □

Theorem 2.6. *If* G *has a* (H_1, H_2) *multi- decomposition then so are* sG *and* $G(s)$*.* □

3. NECESSARY CONDITION

In this section the necessary condition for the existence of multi - decomposition of $K_{m,n}$ into stars and bowties of size l is obtained.

Theorem 3.1. *Let* m *and* n *be positive integers. If* $K_{m,n}$ *can be decomposed into* a *copies of* S_l *and* b *copies of* B_l *for non - negative integers* a *and* b*, then* $l(a+b) = mn$.□

Remark 3.2. *The decomposition of* $K_{m,n}$ *into* B_l *exist only when* m *and* $n \geq l/2$*. And the multi - decomposition* $\{aS_l, bB_l\}$ *in* $K_{m,n}$ *exists only when*
(i) $m \geq l$ *and* $n \geq l/2$
(ii) $n \geq l$ *and* $m \geq l/2$*, for* $l \equiv 0$*(mod 8) or* $l \equiv 4$*(mod 8).*

Proof. Assume that $K_{m,n}$ has a $\{aS_l, bB_l\}$ - decomposition. Further, if m (or n) $< l$, then the star decomposition does not exist. For the bowtie decomposition the necessary condition is satisfied only when m and $n \geq l/2$. Since, by Theorem 2.5 the complete bipartite graph can have only even cycles and a bowtie needs two cycles, we have $l \equiv 0$(mod 8) or $l \equiv 4$(mod 8). Hence the multi - decomposition $\{aS_l, bB_l\}$ in $K_{m,n}$ exists only when
(i) $m \geq l$ and $n \geq l/2$
(ii) $n \geq l$ and $m \geq l/2$. □

Notation: Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be the vertices of $K_{m,n}$. A star S_l with centre at x_1 and end vertices at $y_1, y_2, ..., y_l$ is denoted by $S(x_1 : y_1, y_2, ..., y_l)$ and a bowtie B_l having common vertex as x_1 and the two cycles $(x_1, y_i, ..., x_j, y_k, x_1)$ and $(x_1, y_p, ..., x_q, y_r, x_1)$ of length $l/2$ is denoted by $B[(x_1, y_i, ..., x_j, y_k, x_1) \cup (x_1, y_p, ..., x_q, y_r, x_1)]$ where $\{i \neq k \neq p \neq r\} \in \{1, 2, ..., n\}$ and $\{j \neq q\} \in \{2, 3, ..., m\}$.

4. SUFFICIENT CONDITIONS

In this section some sufficient conditions for the existence of $\{aS_l, bB_l\}$ - decomposition in $K_{m,n}$ have been obtained, for the admissible pair of non - negative integers (a, b).

Lemma 4.1. *The graph $K_{4,4}$ has a B_8 - decomposition.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ are the vertices of $K_{4,4}$. Here, we have 2 copies of B_8 in the decomposition of $K_{4,4}$ as follows:
$B[(x_1, y_1, x_2, y_2, x_1) \cup (x_1, y_3, x_3, y_4, x_1)]$, $B[(x_4, y_1, x_3, y_2, x_4) \cup (x_4, y_3, x_2, y_4, x_4)]$.
Hence the proof. □

Lemma 4.2. *The graph $K_{4,6}$ has a B_8 - decomposition.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ are the vertices of $K_{4,6}$. Here, we have 3 copies of B_8 in the decomposition of $K_{4,6}$ as follows:
$B[(y_1, x_1, y_2, x_2, y_1) \cup (y_1, x_3, y_3, x_4, y_1)]$, $B[(y_4, x_1, y_3, x_2, y_4) \cup (y_4, x_3, y_5, x_4, y_4)]$, $B[(y_6, x_1, y_5, x_2, y_6) \cup (y_6, x_3, y_2, x_4, y_6)]$.
Hence the proof. □

Lemma 4.3. *The graph $K_{8,12}$ has $\{aS_8, bB_8\}$ - decomposition for some admissible pairs (a, b) such that $a + b = 12$.*

Proof. Assume that there exists (a, b) such that $a + b = 12$. Let $X = \{x_1, x_2, ..., x_8\}$ and $Y = \{y_1, y_2, ..., y_{12}\}$ are the vertices of $K_{8,12}$. The admissible pairs (a, b) are $a = 0$, $b = 12$; $a = 1$, $b = 11$; $a = 2$, $b = 10$; $a = 3$, $b = 9$; $a = 4$, $b = 8$; $a = 5$, $b = 7$; $a = 6$, $b = 6$; $a = 7$, $b = 5$; $a = 8$, $b = 4$; $a = 9$, $b = 3$; $a = 10$, $b = 2$; $a = 11$, $b = 1$ and $a = 12$, $b = 0$.
The existence or non - existence of the multi - decomposition $\{aS_8, bB_8\}$ in $K_{8,12}$ for these values of a and b is discussed in the following cases:
Case (i) $a = 0$, $b = 12$.
Note that, $K_{8,12}$ can be written as 4 edge disjoint copies of $K_{4,6}$. By Lemma 4.2, $K_{4,6}$ has a bowtie (B_8) - decomposition. Hence, $(0, 12)$ is a suitable pair for the required decomposition in $K_{8,12}$.
Case (ii) $a = 1$, $b = 11$.
In this case, if we remove the 8 edges of one S_8 in $K_{8,12}$ then the resultant graph $K_{8,12} \backslash S_8$ has odd degree and using that we cannot form 11 bowties. Hence $(1, 11)$ is not a suitable pair for the required decomposition in $K_{8,12}$.
Case (iii) $a = 2$, $b = 10$.

The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,2}$, $K_{4,6}(2)$ and $K_{4,4}(2)$ and by Theorem 2.1, Lemmas 4.1 and 4.2, the multi - decomposition $\{2S_8, 10B_8\}$ exists. Hence, $(2, 10)$ is a suitable pair for the required decomposition in $K_{8,12}$.

Case (iv) $a = 3$, $b = 9$.

In this case, we have a set of 3 copies of S_8 and 9 copies of B_8 in the multi - decomposition of $K_{8,12}$ as follows:

$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$, $S(x_2 : y_1, y_2, y_3, y_4, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_3 : y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$.

If we remove the 3 stars from the graph $K_{8,12}$ then the resultant graph $K_{8,12}\backslash S_8(3)$ has the adjacency matrix as given below:

$$A(K_{8,12}\backslash S_8(3)) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{1}$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{6,4}$ where

i) $X = \{x_3, x_4, x_5, x_6, x_7, x_8\}$; $Y = \{y_1, y_2, y_3, y_4\}$,

ii) $X = \{x_2, x_4, x_5, x_6, x_7, x_8\}$; $Y = \{y_5, y_6, y_7, y_8\}$ and

iii) $X = \{x_1, x_4, x_5, x_6, x_7, x_8\}$; $Y = \{y_9, y_{10}, y_{11}, y_{12}\}$. By Lemma 4.2, $K_{8,12}$ has a B_8 - decomposition. Hence, $(3, 9)$ is a suitable pair for the required decomposition in $K_{8,12}$.

Case (v) $a = 4$, $b = 8$.

The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,4}$ and $K_{4,4}(4)$. The graph $K_{4,8}$ has 4 copies of S_8 by Theorem 2.1, and by Lemma 4.1, $K_{4,4}$ has 2 copies of B_8. Hence, $(4, 8)$ is a suitable pair for the required decomposition in $K_{8,12}$.

Case (vi) $a = 5$, $b = 7$.

In this case, we have a set of 5 copies of S_8 and 7 copies of B_8 in the multi - decomposition of $K_{8,12}$ as follows:

$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$, $S(x_2 : y_1, y_2, y_3, y_4, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_3 : y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$, $S(x_4 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$,
$S(x_5 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$.

If we remove the 5 stars from the graph $K_{8,12}$ then the resultant graph $K_{8,12}\backslash S_8(5)$ has the adjacency matrix as given below:

$$A(K_{8,12}\backslash S_8(5)) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{2}$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{4,4}$ or $K_{6,4}$ where
i) $X = \{x_3, x_6, x_7, x_8\}$; $Y = \{y_1, y_2, y_3, y_4\}$,
ii) $X = \{x_2, x_6, x_7, x_8\}$; $Y = \{y_5, y_6, y_7, y_8\}$ and
iii) $X = \{x_1, x_4, x_5, x_6, x_7, x_8\}$; $Y = \{y_9, y_{10}, y_{11}, y_{12}\}$. By Lemma 4.1 and 4.2, $K_{8,12}$ has a B_8 - decomposition. Hence, $(5, 7)$ is a suitable pair for the required decomposition in $K_{8,12}$.
Case (vii) $a = 6$, $b = 6$.
The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,6}$ and $K_{4,6}(2)$. By Theorem 2.1, $K_{8,6}$ has 6 copies of S_8 and by Lemma 4.2, $K_{4,6}$ has a B_8 - decomposition. Hence, $(6, 6)$ is a suitable pair for the required decomposition in $K_{8,12}$.
Case (viii) $a = 7$, $b = 5$.
In this case, we have a set of 7 copies of S_8 and 5 copies of B_8 in the multi - decomposition of $K_{8,12}$ as follows:
$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$, $S(x_2 : y_1, y_2, y_3, y_4, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_3 : y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$, $S(x_4 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$,
$S(x_5 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$, $S(x_6 : y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_7 : y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$.
If we remove the 7 stars from the graph $K_{8,12}$ then we get the resultant graph $K_{8,12}\backslash S_8(7)$. Using the edges of $K_{8,12}\backslash S_8(7)$ we can form 10 cycles of length 4 and taking two cycles having a common vertex as one bowtie (B_8), we can form 5 bowties (B_8) as shown below: $B[(x_8, y_5, x_2, y_6, x_8) \cup (x_8, y_9, x_1, y_{10}, x_8)]$,
$B[(x_8, y_7, x_2, y_8, x_8) \cup (x_8, y_{11}, x_4, y_{12}, x_8)]$,
$B[(x_5, y_9, x_4, y_{10}, x_5) \cup (x_5, y_{11}, x_1, y_{12}, x_5)]$, $B[(x_3, y_1, x_6, y_2, x_3) \cup (x_3, y_3, x_7, y_4, x_3)]$,
$B[(x_8, y_1, x_7, y_2, x_8) \cup (x_8, y_3, x_6, y_4, x_8)]$.

Hence, $(7, 5)$ is a suitable pair for the required decomposition in $K_{8,12}$.
Case (ix) $a = 8$, $b = 4$.
The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,8}$ and $K_{4,4}(2)$. By Theorem 2.1 and Lemma 4.1, $(8, 4)$ is a suitable pair for the required decomposition in $K_{8,12}$.
Case (x) $a = 9$, $b = 3$.
The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,9}$ and $K_{8,3}$. By Theorem 2.1, $K_{8,9}$ is decomposable into 9 copies of S_8. But, $K_{8,3}$ is not bowtie decomposable. Hence, $(9, 3)$ is not a suitable pair for the required decomposition in $K_{8,12}$.
Case (xi) $a = 10$, $b = 2$.
The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,10}$ and $K_{8,2}$. By Theorem 2.1, $K_{8,10}$ is decomposable into 10 copies of S_8. But $K_{8,2}$ is not bowtie decomposable. Hence, $(10, 2)$ is not a suitable pair for the required decomposition in $K_{8,12}$.
Case (xii) $a = 11$, $b = 1$.
The graph $K_{8,12}$ can be written as an edge disjoint union of $K_{8,11}$ and $K_{8,1}$. By Theorem 2.1, $K_{8,11}$ is decomposable into 11 copies of S_8. But, $K_{8,1}$ is not bowtie decomposable. Hence, $(11, 1)$ is not a suitable pair for the required decomposition in $K_{8,12}$.
Case (xiii) $a = 12$, $b = 0$.
By Theorem 2.1, $K_{8,12}$ is decomposable into 12 copies of S_8.
Thus, $K_{8,12}$ has a $(S_8, B_8)_{\{a,b\}}$-decomposition where $(a, b) \notin \{(1, 11), (9, 3), (10, 2), (11, 1)\}$. □

Lemma 4.4. *The graph $K_{8,n}$ where $1 \leq n \leq 11$, has $\{aS_8, bB_8\}$-decomposition.*

Proof. Table 1 gives the set of admissible pairs (a, b) and the corresponding forms of edge disjoint union of $K_{8,n}$ into subgraphs belong to the set $\{K_{4,4}, K_{4,6}\}$. Then the proof follows from Theorem 2.1, Lemma 4.1 and Lemma 4.2. □

Lemma 4.5. *The graph $K_{8s,n}$ where $s \geq 1$ and $n \geq 12$, has $\{aS_8, bB_8\}$ - decomposition.*

Proof. Clearly,

$$K_{8s,n} = K_{8,n}(s). \tag{3}$$

So, we have to prove $K_{8,n}$ admits $\{aS_8, bB_8\}$ - decomposition, when $n \geq 12$.
If $n = 12$, then by Lemma 4.3, the graph $K_{8,12}$ admits a multi - decomposition $\{aS_8, bB_8\}$.
If $n = 13$, then the graph $K_{8,13}$ can be written as an edge disjoint union of $K_{8,12}$ and $K_{8,1}$.

$$\text{i.e., } K_{8,13} = K_{8,12} \oplus K_{8,1}.$$

$G = K_{m,n}$	Admissible pairs (a, b)	Edge disjoint union of subgraphs of G
$K_{8,1}$	$(1, 0)$	$K_{8,1}$
$K_{8,2}$	$(2, 0)$	$K_{8,2}$
$K_{8,3}$	$(3, 0)$	$K_{8,3}$
$K_{8,4}$	$(0, 4)$	$K_{4,4}$ (2)
	$(4, 0)$	$K_{8,4}$
$K_{8,5}$	$(1, 4)$	$K_{8,1} \oplus K_{4,4}(2)$
	$(5, 0)$	$K_{8,5}$
$K_{8,6}$	$(0, 6)$	$K_{4,6}(2)$
	$(2, 4)$	$K_{8,2} \oplus K_{4,4}$ (2)
	$(6, 0)$	$K_{8,6}$
$K_{8,7}$	$(1, 6)$	$K_{8,1} \oplus K_{4,6}(2)$
	$(3, 4)$	$K_{8,3} \oplus K_{4,4}(2)$
	$(7, 0)$	$K_{8,7}$
$K_{8,8}$	$(0, 8)$	$K_{4,4}$ (4)
	$(2, 6)$	$K_{8,2} \oplus K_{4,6}(2)$
	$(4, 4)$	$K_{8,4} \oplus K_{4,4}(2)$
	$(8, 0)$	$K_{8,8}$
$K_{8,9}$	$(1, 8)$	$K_{8,1} \oplus K_{4,4}(4)$
	$(3, 6)$	$K_{8,3} \oplus K_{4,6}(2)$
	$(5, 4)$	$K_{8,5} \oplus K_{4,4}(2)$
	$(9, 0)$	$K_{8,9}$
$K_{8,10}$	$(0, 10)$	$K_{4,4}(2) \oplus K_{4,6}(2)$
	$(2, 8)$	$K_{8,2} \oplus K_{4,4}(4)$
	$(4, 6)$	$K_{8,4} \oplus K_{4,6}$ (2)
	$(6, 4)$	$K_{8,6} \oplus K_{4,4}(2)$
	$(10, 0)$	$K_{8,10}$
$K_{8,11}$	$(1, 10)$	$K_{8,1} \oplus K_{4,4}(2) \oplus K_{4,6}(2)$
	$(3, 8)$	$K_{8,3} \oplus K_{4,4}(4)$
	$(5, 6)$	$K_{8,5} \oplus K_{4,6}(2)$
	$(7, 4)$	$K_{8,7} \oplus K_{4,4}(2)$
	$(11, 0)$	$K_{8,11}$

Table 1: Decomposition of $K_{8,n}(1 \leq n \leq 11)$ into $K_{4,4}$ and $K_{4,6}$

From Lemma 4.3 and Theorem 2.1, the graph $K_{8,13}$ admits a multi - decomposition $\{aS_8, bB_8\}$ in which the admissible pairs are $(1, 12), (3, 12), (4, 12), ..., (9, 4), (13, 0)$. If $n = 14$, then the graph $K_{8,14}$ can be written as an edge disjoint union of $K_{8,12}$ and $K_{8,2}$.

$$\text{i.e., } K_{8,14} = K_{8,12} \oplus K_{8,2}.$$

From Lemma 4.3 and Theorem 2.1, the graph $K_{8,14}$ admits a multi-decomposition $\{aS_8, bB_8\}$ in which the admissible pairs are $(0, 14), (2, 12), (3, 11), (4, 10), ..., (10, 4), (14, 0)$.
In general,

$$K_{8,n} \cong \begin{cases} K_{8,n-1} \oplus K_{8,1}, \text{if n is odd} \\ K_{8,n-2} \oplus K_{8,2}, \text{if n is even.} \end{cases} \tag{4}$$

Here, the admissible pairs (a, b) for which the multi - decomposition of $K_{8,n}$ into $\{aS_8, bB_8\}$ are given below:

$$(a, b) \in \begin{cases} (1, n-1), (3, n-3), (4, n-4), ..., (n-4, 4), (n, 0), \text{if } n \text{ is odd} \\ (0, n), (2, n-2), (3, n-3), (4, n-4), ..., (n-4, 4), (n, 0), \text{if } n \text{ is even.} \end{cases}$$

The multi - decomposition of $K_{8,n}$ (for $n \geq 12$) into $\{aS_8, bB_8\}$ can be proved using induction hypothesis and Equation 4 for the admissible pairs (a, b) stated above. Hence, the graph $K_{8s,n}$ has $\{aS_8, bB_8\}$ - decomposition by Equation 3 and Theorem 2.6. □

Lemma 4.6. *The graph $K_{6,6}$ has a B_{12} - decomposition.*

Proof. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ are the vertices of $K_{6,6}$. Here, we have 3 copies of B_{12} in $K_{6,6}$ as follows:
$B[(x_1, y_1, x_2, y_5, x_3, y_3, x_1) \cup (x_1, y_4, x_4, y_2, x_5, y_6, x_1)]$,
$B[(x_6, y_1, x_4, y_5, x_1, y_2, x_6) \cup (x_6, y_3, x_2, y_6, x_3, y_4, x_6)]$,
$B[(x_5, y_1, x_3, y_2, x_2, y_4, x_5) \cup (x_5, y_3, x_4, y_6, x_6, y_5, x_5)]$
Hence, the graph $K_{6,6}$ has a B_{12} - decomposition. □

Lemma 4.7. *The graph $K_{6,8}$ has a B_{12} - decomposition.*

Proof. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$ are the vertices of $K_{6,8}$. Here, we have 4 copies of B_{12} in $K_{6,8}$ as follows:
$B[(x_1, y_1, x_2, y_5, x_3, y_3, x_1) \cup (x_1, y_4, x_4, y_2, x_5, y_6, x_1)]$,
$B[(x_5, y_1, x_7, y_2, x_8, y_4, x_5) \cup (x_5, y_3, x_4, y_6, x_6, y_5, x_5)]$,

$B[(x_2, y_2, x_1, y_5, x_8, y_3, x_2) \cup (x_2, y_4, x_6, y_1, x_3, y_6, x_2)]$,
$B[(x_7, y_5, x_4, y_1, x_8, y_6, x_7) \cup (x_7, y_4, x_3, y_2, x_6, y_3, x_7)]$.
Hence, the graph $K_{6,8}$ has a B_{12} - decomposition. □

Lemma 4.8. *The graph $K_{6,10}$ has a B_{12}-decomposition.*

Proof. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$ are the vertices of $K_{6,10}$. Here, we have 5 copies of B_{12} in $K_{6,10}$ as follows:
$B[(y_1, x_1, y_2, x_5, y_3, x_2, y_1) \cup (y_1, x_3, y_4, x_6, y_5, x_4, y_1)]$,
$B[(y_6, x_3, y_7, x_5, y_8, x_1, y_6) \cup (y_6, x_2, y_9, x_6, y_{10}, x_4, y_6)]$,
$B[(y_2, x_6, y_1, x_5, y_5, x_3, y_2) \cup (y_2, x_2, y_{10}, x_1, y_9, x_4, y_2)]$,
$B[(y_3, x_1, y_4, x_2, y_7, x_4, y_3) \cup (y_3, x_3, y_{10}, x_5, y_6, x_6, y_3)]$,
$B[(y_8, x_4, y_4, x_5, y_9, x_3, y_8) \cup (y_8, x_2, y_5, x_1, y_7, x_6, y_8)]$.
Hence, the graph $K_{6,10}$ has a B_{12} - decomposition. □

Lemma 4.9. *The graph $K_{12,18}$ has $\{aS_{12}, bB_{12}\}$ - decomposition for some admissible pairs (a, b) such that $a + b = 18$.*

Proof. Assume that there exists (a, b) such that $a+b = 18$. Since $K_{12,18}$ has 216 edges, the necessary condition is satisfied. Let $X = \{x_1, x_2, ..., x_{12}\}$ and $Y = \{y_1, y_2, ..., y_{18}\}$ are the vertices of $K_{12,18}$. The admissible pairs (a, b) are $a = 0$, $b = 18$; $a = 1$, $b = 17$; $a = 2$, $b = 16$; $a = 3$, $b = 15$; $a = 4$, $b = 14$; $a = 5$, $b = 13$; $a = 6$, $b = 12$; $a = 7$, $b = 11$; $a = 8$, $b = 10$; $a = 9$, $b = 9$; $a = 10$, $b = 8$; $a = 11$, $b = 7$; $a = 12$, $b = 6$; $a = 13$, $b = 5$; $a = 14$, $b = 4$; $a = 15$, $b = 3$; $a = 16$, $b = 2$; $a = 17$, $b = 1$ and $a = 18$, $b = 0$.
The existence or non - existence of the multi - decomposition $\{aS_{12}, bB_{12}\}$ in $K_{12,18}$ for the admissible pairs (a, b) is discussed in the following cases:
Case (i) $a = 0$, $b = 18$.
Note that, $K_{12,18}$ can be written as 6 edge disjoint copies of $K_{6,6}$. By Lemma 4.6, $K_{6,6}$ has a B_{12} - decomposition. Hence, $(0, 18)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (ii) $a = 1$, $b = 17$.
In this case, if we remove the 12 edges of one S_{12} in $K_{12,18}$ then the resultant graph $K_{12,18} \backslash S_{12}$ has odd degree and using that we cannot form 17 bowties. Hence $(1, 17)$ is not a suitable pair for the required decomposition in $K_{12,18}$.
Case (iii) $a = 2$, $b = 16$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,2}$ and $K_{6,8}(4)$. Hence, by Theorem 2.1 and Lemma 4.7, the multi - decomposition $\{2S_{12}, 16B_{12}\}$ exists in $K_{12,18}$. Hence, $(2, 16)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (iv) $a = 3$, $b = 15$.
In this case, we have a set of 3 copies of S_{12} and 15 copies of B_{12} in the multi -

decomposition of $K_{12,18}$ as follows:
$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_2 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_3 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$.
If we remove the 3 stars from the graph $K_{12,18}$ then the resultant graph $K_{12,18} \backslash S_{12}(3)$ has the adjacency matrix as given below:

$$A(K_{12,18} \backslash S_8(3)) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad (5)$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{10,6}$ where
i) $X = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$,
ii) $X = \{x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}$ and
iii) $X = \{x_1, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}\}$. By Lemma 4.8, $K_{10,6}$ has a B_{12} - decomposition. Hence, $(3, 15)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (v) $a = 4$, $b = 14$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,4}$, $K_{6,6}(2)$ and $K_{6,8}(2)$. By Theorem 2.1, $K_{12,4}$ has 4 copies of S_{12}, by Lemma 4.6, $K_{6,6}$ has 3 copies of B_{12} and by Lemma 4.7, $K_{6,8}$ has 4 copies of B_{12}. Hence, $(4, 14)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (vi) $a = 5$, $b = 13$.

In this case, we have a set of 5 copies of S_{12} and 13 copies of B_{12} in the multi - decomposition of $K_{12,18}$ as follows:
$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_2 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_3 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_4 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_5 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$.
If we remove the 5 stars from the graph $K_{12,18}$ then the resultant graph $K_{12,18} \backslash S_{12}(5)$ has the adjacency matrix as given below:

$$A(K_{12,18} \backslash S_8(5)) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array} \\ \left(\begin{array}{cccccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array} \right) \end{array} \quad (6)$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{8,6}$ or $K_{10,6}$ where
i) $X = \{x_3, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$,
ii) $X = \{x_2, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}$ and
iii) $X = \{x_1, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}\}$. By Lemma 4.7, $K_{8,6}$ has 4 copies of B_{12} and by Lemma 4.8, $K_{10,6}$ has 5 copies of B_{12}. Hence, $(5, 13)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (vii) $a = 6$, $b = 12$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,6}$ and $K_{6,6}(4)$. By Theorem 2.1, $K_{12,6}$ has 6 copies of S_{12} and Lemma 4.6, $K_{6,6}$ has 3 copies of B_{12}.

Hence, $(6, 12)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (viii) $a = 7$, $b = 11$.
In this case, we have a set of 7 copies of S_{12} and 11 copies of B_{12} in the multi-decomposition of $K_{12,18}$ as follows:
$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_2 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_3 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_4 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_5 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_6 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_7 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$.
If we remove the 7 stars from the graph $K_{12,18}$ then the resultant graph $K_{12,18} \backslash S_{12}(7)$ has the adjacency matrix as given below:

$$A(K_{12,18} \backslash S_8(7)) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{7}$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{6,6}$ or $K_{8,6}$ where
i) $X = \{x_2, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}$
ii) $X = \{x_3, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, and
iii) $X = \{x_1, x_4, x_5, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}\}$.
By Lemma 4.7, $K_{6,6}$ has 3 copies of B_{12} and 4.8, $K_{8,6}$ has 4 copies of B_{12}. Hence,

$(7, 11)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (ix) $a = 8$, $b = 10$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,8}$ and $K_{6,10}(2)$. By Theorem 2.1, $K_{12,8}$ has 8 copies of S_{12} and by Lemma 4.9, $K_{6,10}$ has 5 copies of B_{12}. Hence, $(8, 10)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (x) $a = 9$, $b = 9$.
In this case, we have a set of 9 copies of S_{12} and 9 copies of B_{12} in the multi - decomposition of $K_{12,18}$ as follows:
$S(x_1 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_2 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_3 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_4 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_5 : y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$,
$S(x_6 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_7 : y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_8 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$,
$S(x_9 : y_1, y_2, y_3, y_4, y_5, y_6, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18})$.
If we remove the 9 stars from the graph $K_{12,18}$ then the resultant graph $K_{12,18}\backslash S_{12}(9)$ has the adjacency matrix as given below:

$$A(K_{12,18}\backslash S_8(9)) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \quad (8)$$

From the above matrix, we can obtain 3 blocks of X and Y such that $E(X \cup Y) = K_{6,6}$ where
i) $X = \{x_3, x_6, x_7, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$,
ii) $X = \{x_2, x_8, x_9, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}$ and
iii) $X = \{x_1, x_4, x_5, x_{10}, x_{11}, x_{12}\}$; $Y = \{y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}\}$. By Lemma 4.7, $K_{6,6}$ has 3 copies of B_{12}. Hence, $(9, 9)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (xi) $a = 10$, $b = 8$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,10}$ and $K_{6,8}(2)$. By Theorem 2.1, $K_{12,10}$ has a 10 copies of S_{12}. And by Lemma 4.8, $K_{6,8}$ has 4 copies of B_{12}. Hence, $(10, 8)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (xii) $a = 11$, $b = 7$.
In this case, if we remove 11 copies of S_{12} in $K_{12,18}$ then the resultant graph $K_{12,18} \backslash S_{12}(11)$ has the edge disjoint sum of $K_{4,6}(2)$ and $K_{6,6}$. From Lemma 4.7, $K_{6,6}$ is B_{12} decomposable but $K_{4,6}$ is not B_{12} decomposable. So we cannot make bowties from the remaining edges. Hence, $(11, 7)$ is not a suitable pair for the required decomposition in $K_{12,18}$.
Case (xiii) $a = 12$, $b = 6$.
The graph $K_{12,18}$ can be written as an edge disjoint union of $K_{12,12}$ and $K_{6,6}(2)$. By Theorem 2.1, $K_{12,12}$ has 12 copies of S_{12} and Lemma 4.8, $K_{6,6}$ has 3 copies of B_{12}. Hence, $(10, 6)$ is a suitable pair for the required decomposition in $K_{12,18}$.
Case (xiv) $a = 13$, $b = 5$.
Case (xv) $a = 14$, $b = 4$.
Case (xvi) $a = 15$, $b = 3$.
Case (xvii) $a = 16$, $b = 2$.
Case (xviii) $a = 17$, $b = 1$.
As in the context of Lemma 4.3, $(13, 5), (14, 4), (15, 3), (16, 2)$ and $(17, 1)$ are not suitable pairs for the required decomposition in $K_{12,18}$.
Case (xix) $a = 18$, $b = 0$.
By Theorem 2.1, $K_{12,18}$ has 18 copies of S_{12}. Thus, $K_{12,18}$ has a $(S_{12}, B_{12})_{\{a,b\}}$-decomposition where $(a, b) \notin \{(1, 17), (11, 7), (13, 5), (14, 4), (15, 3), (16, 2), (17, 1)\}$. □

Lemma 4.10. *The graph $K_{12,n}$ where $1 \leq n \leq 17$, has $\{aS_{12}, bB_{12}\}$ - decomposition.*

Proof. Table 2 gives the set of admissible pairs (a, b) and the corresponding forms of edge disjoint union of $K_{12,n}$ into subgraphs belong to $\{K_{6,6}, K_{6,8}, K_{6,10}\}$. Then the proof follows from Theorem 2.1, Lemma 4.6, Lemma 4.7 and Lemma 4.8.

Table 2: Decomposition of $K_{12,n}(1 \leq n \leq 17)$ into $K_{6,6}$, $K_{6,8}$ and $K_{6,10}$

$G = K_{m,n}$	Admissible pairs (a, b)	Edge disjoint union of subgraphs of G
$K_{12,1}$	$(1, 0)$	$K_{12,1}$
$K_{12,2}$	$(2, 0)$	$K_{12,2}$
$K_{12,3}$	$(3, 0)$	$K_{12,3}$
$K_{12,4}$	$(4, 0)$	$K_{12,4}$
$K_{12,5}$	$(5, 0)$	$K_{12,5}$
$K_{12,6}$	$(0, 6)$	$2\ K_{6,6}$
	$(6, 0)$	$K_{12,6}$
$K_{12,7}$	$(1, 6)$	$K_{12,1} \oplus K_{6,6}(2)$
	$(7, 0)$	$K_{12,7}$
$K_{12,8}$	$(0, 8)$	$K_{6,8}(2)$
	$(2, 6)$	$K_{12,2} \oplus K_{6,6}(2)$
	$(8, 0)$	$K_{12,8}$
$K_{12,9}$	$(1, 8)$	$K_{12,1} \oplus K_{6,8}(2)$
	$(3, 7)$	$K_{12,3} \oplus K_{6,6}(2)$
	$(9, 0)$	$K_{12,9}$
$K_{12,10}$	$(0, 10)$	$K_{6,10}(2)$
	$(2, 8)$	$K_{12,2} \oplus K_{6,8}(2)$
	$(4, 6)$	$K_{12,4} \oplus K_{6,6}(2)$
	$(12, 0)$	$K_{12,10}$
$K_{12,11}$	$(1, 10)$	$K_{12,1} \oplus K_{6,10}(2)$
	$(3, 8)$	$K_{12,3} \oplus K_{6,8}(2)$
	$(5, 6)$	$K_{12,5} \oplus K_{6,6}(2)$
	$(11, 0)$	$K_{12,11}$
$K_{12,12}$	$(0, 12)$	$K_{6,6}(4)$
	$(2, 10)$	$K_{12,2} \oplus K_{6,10}(2)$
	$(4, 8)$	$K_{12,4} \oplus K_{6,8}(2)$
	$(6, 6)$	$K_{12,6} \oplus K_{6,6}(2)$
	$(12, 0)$	$K_{12,12}$
$K_{12,13}$	$(1, 12)$	$K_{12,1} \oplus K_{6,6}(4)$
	$(3, 10)$	$K_{12,3} \oplus K_{6,10}(2)$
	$(5, 8)$	$K_{12,5} \oplus K_{6,8}(2)$
	$(7, 6)$	$K_{12,7} \oplus K_{6,6}(2)$
	$(13, 0)$	$K_{12,13}$
$K_{12,14}$	$(0, 14)$	$K_{6,6}(2) \oplus K_{6,8}(2)$

	$(2,12)$	$K_{12,2} \oplus K_{6,6}(4)$
	$(4,10)$	$K_{12,4} \oplus K_{6,10}(2)$
	$(6,8)$	$K_{12,6} \oplus K_{6,8}(2)$
	$(8,6)$	$K_{12,8} \oplus K_{6,6}(2)$
	$(14,0)$	$K_{12,14}$
$K_{12,15}$	$(1,14)$	$K_{12,1} \oplus K_{6,6}(2) \oplus K_{6,8}(2)$
	$(3,12)$	$K_{12,3} \oplus K_{6,6}(4)$
	$(5,10)$	$K_{12,5} \oplus K_{6,10}(2)$
	$(7,8)$	$K_{12,7} \oplus K_{6,8}(2)$
	$(9,6)$	$K_{12,9} \oplus K_{6,6}(2)$
	$(15,0)$	$K_{12,15}$
$K_{12,16}$	$(0,16)$	$K_{6,8}(4)$
	$(2,14)$	$K_{12,2} \oplus K_{6,6}(2) \oplus K_{6,8}(2)$
	$(4,12)$	$K_{12,4} \oplus K_{6,6}(4)$
	$(6,10)$	$K_{12,6} \oplus K_{6,10}(2)$
	$(8,8)$	$K_{12,8} \oplus K_{6,8}(2)$
	$(10,6)$	$K_{12,10} \oplus K_{6,6}(2)$
	$(16,0)$	$K_{12,16}$
$K_{12,17}$	$(1,16)$	$K_{12,1} \oplus K_{6,8}(4)$
	$(3,14)$	$K_{12,3} \oplus K_{6,6}(2) \oplus K_{6,8}(2)$
	$(5,12)$	$K_{12,5} \oplus K_{6,6}(4)$
	$(7,10)$	$K_{12,7} \oplus K_{6,10}(2)$
	$(9,8)$	$K_{12,9} \oplus K_{6,8}(2)$
	$(11,6)$	$K_{12,11} \oplus K_{6,6}(2)$
	$(17,0)$	$K_{12,17}$

□

Lemma 4.11. *The graph $K_{12s,n}$ where $s \geq 1$ and $n \geq 18$, has $\{aS_{12}, bB_{12}\}$ - decomposition.*

Proof. Clearly,

$$K_{12s,n} = K_{12,n}(s). \tag{9}$$

So, we have to prove $K_{12,n}$ admits $\{aS_{12}, bB_{12}\}$ - decomposition, when $n \geq 18$.
If $n = 18$, then by Lemma 4.9, the graph $K_{12,18}$ admits a multi - decomposition $\{aS_{12}, bB_{12}\}$.
If $n = 19$, then the graph $K_{12,19}$ can be written as an edge disjoint union of $K_{12,18}$ and $K_{12,1}$.

$$\text{i.e., } K_{12,19} = K_{12,18} \oplus K_{12,1}.$$

From Lemma 4.9 and Theorem 2.1, the graph $K_{12,19}$ admits a multi - decomposition $\{aS_{12}, bB_{12}\}$. The admissible pairs (a, b) for which $\{aS_{12}, bB_{12}\}$ - decomposition are $(1, 18), (3, 16), (4, 15), ..., (13, 6), (19, 0)$.
If $n = 20$, then the graph $K_{12,20}$ can be written as an edge disjoint union of $K_{12,18}$ and $K_{12,2}$.

$$\text{i.e., } K_{12,20} = K_{12,18} \oplus K_{12,2}.$$

From Lemma 4.9 and Theorem 2.1, the graph $K_{12,20}$ admits a multi - decomposition $\{aS_{12}, bB_{12}\}$. The admissible pairs (a, b) for which $\{aS_{12}, bB_{12}\}$ - decomposition are $(0, 20), (2, 18), (3, 17), (4, 16), ..., (14, 6), (20, 0)$.
In general,

$$K_{12,n} \cong \begin{cases} K_{12,n-1} \oplus K_{12,1}, \text{if n is odd} \\ K_{12,n-2} \oplus K_{12,2}, \text{if n is even.} \end{cases} \tag{10}$$

Here, the admissible pairs (a, b) for which the multi - decomposition of $K_{12,n}$ into $\{aS_{12}, bB_{12}\}$ are given below:

$$(a, b) \in \begin{cases} (1, n-1), (3, n-3), (4, n-4), ..., (n-6, 6), (n, 0), \text{if } n \text{ is odd} \\ (0, n), (2, n-2), (3, n-3), (4, n-4), ..., (n-6, 6), (n, 0), \text{if } n \text{ is even.} \end{cases}$$

The multi - decomposition of $K_{12,n}$ (for $n \geq 18$) into $\{aS_{12}, bB_{12}\}$ can be proved using induction hypothesis and Equation 10 for the admissible pairs (a, b) stated above.
Hence, the graph $K_{12s,n}$ has $\{aS_{12}, bB_{12}\}$ - decomposition by Equation 9 and Theorem 2.6. □

Lemma 4.12. *Whenever $l \equiv 0$(mod 8) or $l \equiv 4$(mod 8) and $l/2 \leq n \leq l$, $K_{l,n}$ has $\{aS_l, bB_l\}$ - decomposition.*

Proof. We know that, $K_{l,n} = K_{l,n-a} \oplus K_{l,a}$.
By Lemma 2.2, there exist a B_l - decomposition in $K_{l,n-a}$, whenever $l/2 \leq n - a < l$.
Further, $K_{l,a}$ can be decomposed into a copies of S_l.
Hence, $K_{l,n}$ has $\{aS_l, bB_l\}$ - decomposition such that $a + b = n$. □

Corollary 4.12.1. *If $n \geq l$, then $K_{l,n}$ has $\{aS_l, bB_l\}$ - decomposition.*

Proof. Let $n \equiv t(\text{mod } l)$, i.e., $n = sl + t$, where the positive integers $s \geq 1$ and $0 \leq t < l$.

$$\begin{aligned} \therefore K_{l,n} &= K_{l,sl+t} \\ &= K_{l,sl} \oplus K_{l,t} \\ &= K_{l,l}(s) \oplus K_{l,t}. \end{aligned}$$

By Lemma 2.4 and by Theorem 2.6, there exist a B_l - decomposition in $K_{l,l}(s)$. And by Lemma 4.12, $K_{l,t}$ has $\{aS_l, bB_l\}$ - decomposition. Hence, $K_{l,n}$ (for $n \geq l$) has $\{aS_l, bB_l\}$ - decomposition such that $a + b = n$. □

Corollary 4.12.2. *For $l \equiv 0$(mod 8) or $l \equiv 4$(mod 8), if $m \equiv 0$(mod l) and $n \geq l/2$, then $K_{m,n}$ admits a $\{aS_l, bB_l\}$ - decomposition for some admissible pairs (a, b) such that $l(a + b) = mn$.*

Proof. Clearly, $K_{m,n} = K_{ls,n}$, since $m \equiv 0$(mod *l*) i.e., $m = ls$, where s is any positive integers.
$\therefore K_{ls,n} = K_{l,n}(s)$. Then proof immediately follows from Lemma 4.12 and Theorem 2.6. □

5. MAIN THEOREM

Theorem 5.1. *For the given positive integers $l \equiv 0$(mod 8) or $l \equiv 4$(mod 8), $m \equiv 0$(mod l) and $n \geq l/2$. the graph $K_{m,n}$ admits $\{aS_l, bB_l\}$-decomposition for some of the admissible pairs (a, b) if and only if $l(a + b) = mn$.*

Proof. Proof follows from Theorem 3.1 and Corollary 4.12.2. □

Conclusion: In this paper, the problem of existence of multi-decomposition of $K_{m,n}$ into stars and bowties of size l (for even l) has been investigated and a necessary and sufficient condition for the same has been given.

REFERENCES

[1] Atif A.Abueida and Chester Lian, *On the decompositions of complete graphs into cycles and stars on the same number of edges*, Discuss. Math. Graph Theory, Vol. 34 (2014), 113-125.

[2] Bondy.J.A., Murty. U.S.R.: *Graph Theory with Applications*, Macmillan Press, London(1976).

[3] Chao-Chih Chou, Chin-Mei Fu and Wen-Chung Huang *Decomposition of $K_{m,n}$ into short cycles*, Discrete Math., 197/198 (1999), 195-203.

[4] P. Hemalatha and K. Ramya, *Multi - decomposition of complete graphs into stars and bowties of size 6*, revised version submitted to Discrete Mathematics, Algorithms and Applications.

[5] Fairouz Begas, Mohammed Haddad and Hamaache Kheddouci, *Decomposition of complete multigraphs into cycles and stars*, Discuss. Math. Graph Theory, Vol. 35 (2015), 629 - 639.

[6] Hung Chih Lee, *Decomposition of complete bipartite multigraphs into cycles and stars*, Discrete Math., Vol. 338 (2015), 1362-1369.

[7] M. Illayaraja, K. Sowndhariya and A. Muthusamy, *Decomposition of product graphs into paths and stars on five vertices*, AKCE Int.J.Graphs Comb., Vol. 17 (2020), 777-783.

[8] M. Illayaraja, K. Sowndhariya and A. Muthusamy, *Decomposition of complete bipartite graphs into cycles and stars with four edges*, AKCE Int.J.Graphs Comb., Vol. 17 (2020), 697-702.

[9] S. Jeevadoss, A. Muthusamy, *Decomposition of complete bipartite graphs into paths and cycles*, Discrete Math., Vol. 331 (2014), 98-108.

[10] Michal Tarsi, *Decomposition of complete multigraphs into stars*, Discrete Math., Vol. 26 (1979), 273-278.

[11] Opeyemi Oyeunmi, Abolape Deborah Akwu, *C_4 and C_6 Decomposition of the tensor product of complete graphs*, arXiv:1908.00172v1 [math.CO].

[12] S. Priyadarshini and A. Muthusamy, *Decomposition of complete tripartite graphs into cycles and paths of length three*, Contributions to Discrete Math., Vol. 15 (2020),117-129.

[13] P.Paulraja and R.Srimathi, *Decomposition of complete equipartite graphs into cycles of lengths 3 and 6*, Australas.J.Combin., Vol. 78(2) (2020), 297-313.

[14] R.Sangeetha and A. Muthusamy, *3-Simple 2-Fold 5-Cycle Systems*, ICTCSDM 2016, 357 -361.

[15] Tay. Woei. Shyu, *Decomposition of complete graphs into paths and stars*, Discrete Math., Vol. 310 (2010), 2164-2169.

[16] Tay. Woei. Shyu, *Decomposition of complete graphs into cycles and stars*, Graphs and Combin., Vol. 29 (2013), 301-313.

[17] Tay. Woei. Shyu, *Decomposition of complete bipartite graphs into paths and stars*, Discrete Math., Vol. 313 (2013), 865-871.

[18] Yamamoto.S., Ikeda.H., Shige-Eda.S., Ushio.K., Hamada.N., *On claw decomposition of complete graphs and complete bigraphs*, Hiroshima Math.J., Vol. 5(1), (1975), 33-42.