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Balanced Rank Distribution Labeling of Crown and Wheel Graphs

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Abstract

A balanced rank distribution of a simple graph G of order n is defined with the following constraints; (i) For a given $k \geq n$, there exist an injective function $f : V(G) \to \{1, 2, ..., k\}$ that gives the vertex labeling of G. (ii) There exists an onto function $\phi : E(G) \to B \subset N$ defined by $\phi(uv) = \left\lceil \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rceil$ or $\left\lfloor \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rfloor$ on the minimum bounded set $B = er(G)$ of edge labelings, called the edge ranks. Then G is said to have a balanced rank distribution labeling if (i) the cardinality of $er(G)$ is less than or equal to the minimum degree of G and (ii) a weakly balanced rank distribution labeling if $er(G)$ lies between minimum and maximum degrees of G. Further, the balanced rank distribution number of G denoted by $brd(G)$ is the minimum $k \geq n$ for which the defined rank distribution labelings exist. In this paper, we proved that the crown graph $C_n \odot K_1$ admits a weakly balanced rank distribution labeling for $n \geq 3$ and the wheel graph W_n admits a balanced rank distribution labeling for $4 \leq n \leq 10$ and weakly balanced rank distribution labeling for $n \geq 11$. Further the balanced rank distribution number of the graphs $C_n \odot K_1$ and W_n have also been obtained for the given positive integer n.

Keywords: Balanced rank distribution labeling; Strongly and Weakly balanced rank distribution graphs; Balanced rank distribution number; Crown graphs and Wheel graphs.

AMS Subject Classification: 05C78

1. INTRODUCTION

In this paper, we consider the graph G as finite, simple and undirected. The number of vertices adjacent to a vertex u of G is called degree of a vertex and is denoted by $d(u)$. The maximum and minimum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. For a real value a, $\lfloor a \rfloor$, $[a]$ and $\lceil a \rceil$ respectively denotes the floor function, the greatest integer function and the ceiling function associated with a. For standard terminology and notations we follow J.A.Bondy and U.S.R.Murthy [1]. Graph labeling techniques have many applications in interconnection and communication networks. In wireless communication networks and interconnection network problems, cycles and wheels are important structures. In most of the graph labeling techniques, the labeling of crown and wheel graphs were investigated (Refer [4,5,6,7,8]). For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

The corona product $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. The crown $C_n \odot K_1$ is a planar graph with $2n$ vertices and $2n$ edges. It is obtained from a cycle C_n by attaching a pendant edge at each vertex of the n-cycle. For a positive integer n, W_n denotes the wheel graph of n vertices formed by connecting a single universal vertex to the $n-1$ vertices of a cycle C_{n-1}.

Any interconnection network can be modeled and analyzed like a graph structure in which the processors are the nodes and the connection between the processors are the links. Though we have studied many types of graph labeling techniques in literature, some of the network topologies in interconnection networks may require a new type of labeling to explain their network flow. For example, if we give different input values to the nodes due to certain constraints but we want to maintain a constant output produced by any two linked nodes in the network then the following graph labeling technique called balanced rank distribution [3] will be useful. The node weights(labels) may be the time taken by the individual processors to complete the task assigned(of course processors with varying capacity) and the edge weights(labels) may be the time taken to produce the combined output produced by the two linked nodes. In parallel processors, usually, we want to get the output at the same time from different processing units of different capacities so that we can minimize the time required for the final output. A threshold value (k) in the network for the maximum time utilized by the processors may also be fixed. Is it possible to define a graph labeling that satisfies all these constraints in a network? The affirmative answer to this question is achieved by the definition of balanced rank distribution labeling of graphs given by the authors in [3].

For simple undirected connected graph G of order n and a given positive integer $k \geq n$, we can define an injective function $f : V(G) \to \{1, 2, ..., k\}$ and an onto function $\phi : E(G) \to B \subset N$ by $\phi(uv) = \left\lceil \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rceil$ or $\left\lfloor \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rfloor$ and for any two edges uv and wx of G, $\phi(uv) = \phi(wx)$ only if $\left| \left(\frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right) - \left(\frac{f(w)}{d(w)} + \frac{f(x)}{d(x)} \right) \right| < 1$. Here, $B = \{i| \text{ min } \phi(E) \leq i \leq \text{ max } \phi(E)\}$, we have many B's for a given graph G and we choose the one with minimum cardinality. Such a set B is denoted by $er(G)$ and the elements of $er(G)$ are called the edge ranks of G and f is known as the balanced rank distribution labeling of G. Note that $|er(G)| \leq \delta(G) \leq \Delta(G)$ and if
(i) $|er(G)| = 1$, then G is said to be a *Strongly balanced rank distribution* graph
(ii) $\delta(G) < |er(G)| \leq \Delta(G)$, then G is said to be a *Weakly balanced rank distribution* graph
(iii) $|er(G)| > \Delta(G)$, then G is said to be a non-balanced rank distribution graph.
The balanced rank distribution number of a graph G denoted by $\mathbf{brd}(G)$ is the minimum k such that $f : V(G) \to \{1, 2, ..., k\}$ is a balanced or weakly balanced rank distribution labeling of G where the induced edge label set B has minimum cardinality.
In [3], the authors studied the existence of balanced rank distribution labeling in ladder graphs, complete graphs and complete bipartite graphs. In this paper, the balanced rank distribution labeling of crown and wheel graphs have been investigated.

2. BALANCED RANK DISTRIBUTION LABELING OF CROWN GRAPHS

In this section, we have studied the balanced rank distribution labeling of the crown graphs $C_n \odot K_1$ for the given positive integer $n \geq 3$.
Notation:
Let $G = C_n \odot K_1$ be a crown graph of order $2n$ and $k \geq 2n$ be the given positive integer. In G, a pendant edge is joined to each vertex of a cycle C_n. Let the vertices of the cycle C_n in G be $v_1, v_2, ..., v_n$ and its respective pendant vertices be $u_1, u_2, ..., u_n$.

$$i.e., V(G) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$$
$$E(G) = \{v_i v_{i+1}, 1 \leq i \leq n-1\} \cup \{v_1 v_n\} \cup \{v_i u_i, 1 \leq i \leq n\}$$

Here $|V(G)| = 2n$, $|E(G)| = 2n$, $\Delta(G) = 3$ and $\delta(G) = 1$.
In order to find the balanced rank distribution number of crown graphs $C_n \odot K_1$, we need to find the minimum value of k such that the vertex labeling $f : V(G) \to \{1, 2, ..., k\}$ induces an edge labeling with minimum number of edge ranks.

Lemma 2.1. *The crown graph $C_n \odot K_1$ has a weakly balanced rank distribution labeling for the given odd positive integer n and for the positive integer $k \geq 2n$. Further, the balanced rank distribution number obtained for various values of n is as*

follows:
(i) $brd(C_3 \odot K_1) = 6$.
(ii) $brd(C_5 \odot K_1) = 13$.
(iii) $brd(C_7 \odot K_1) = 22$.
(iv) $brd(C_9 \odot K_1) = 32$.
(v) $brd(C_n \odot K_1) = \frac{9n-17}{2}$, *for* $n \geq 11$.

Proof. Let $G = C_n \odot K_1$. Now, we will analyze the balanced rank distribution of G according to the given values of n and k.
(i) $n = 3$ and $k \geq 6$.
Define the labeling $f : V(C_3 \odot K_1) \to \{1, 2, ..., k\}$ by

$$f(v_i) = k - 3 + i,\ 1 \leq i \leq n$$
$$f(u_i) = m + 3 - i,\ 1 \leq i \leq n$$

where $k \equiv j$ (mod 3), $j = 0, 1, 2$, and $m = \frac{k-3-j}{3}$.
This definition of f induces an edge labeling with edge rank set $\mathbf{er}(C_3 \odot K_1) = \left\{\frac{2k-6+j}{3}, \frac{2k-3+j}{3}\right\}$ for all $k \geq 6$. Here, $|er(C_3 \odot K_1)| = 2 < \Delta(C_3 \odot K_1)$.
Hence, $C_3 \odot K_1$ is a weakly balanced rank distribution graph and $brd(C_3 \odot K_1) = 6$.
Define the labeling $f(u_i)$ of pendant vertices by

$$f(u_i) = \begin{cases} m + n - i, & i = 1, 3, 5, ..., n \\ m + i - 1, & i = 2, 4, ..., n-1 \end{cases} \tag{1}$$

(ii) $n = 5$ and for any value of $k \geq 10$, vertex labeling f does not induce an edge labeling with single edge rank and so $|er(G)| = 1$ is not possible. Hereafter, the labeling $f(u_i)$ of the pendent vertices of G can be obtained by defining a suitable value for m in Eq. (1). One can verify that the minimum k such that f induces 2 edge ranks is 13 and defined as

$$f(v_i) = \begin{cases} k - 10 + 2i, & i = 1, 3, 5 \\ k + 2 - 2i, & i = 2, 4 \end{cases} \tag{2}$$

where $k \equiv j$ (mod 3), $j = 0, 1, 2$, and $m = \frac{k-9-j}{3}$.
The definition of f given in Eqs.(1) and (2) induces an edge labeling with edge rank set

$$er(C_5 \odot K_1) = \begin{cases} \left\{\frac{2k-9}{3}, \frac{2k-6}{3}\right\}, & for\ k \equiv 0 (mod\ 3) \\ \left\{\frac{2k-8}{3}, \frac{2k-5}{3}\right\}, & for\ k \equiv 1 (mod\ 3) \\ \left\{\frac{2k-10}{3}, \frac{2k-7}{3}\right\}, & for\ k \equiv 2 (mod\ 3) \end{cases}$$

Therfore, $C_5 \odot K_1$ has a weakly balanced rank distribution with three edge ranks for the values of k from 10 to 12 and it has weakly balanced rank distribution with two edge ranks for $k \geq 13$. Thus $brd(C_5 \odot K_1) = 13$.
(iii) $n = 7$ and $k \geq 14$.
Consider the vertex labeling of pendant vertices as given in Eq. (1).
From k = 14 to 17, $C_7 \odot K_1$ has a non balanced rank distribution and from k = 18 to 21, it has a weakly balanced rank distribution with three edge ranks. Thus, from k =14 to 21, there is no vertex labeling f that induces an edge labeling with $|er(G)| = 2$. So, take $k \geq 22$ and define $f(v_i)$ that induces an edge labeling to produce two edge ranks. According to $k \equiv 0, 1, 2\ (mod\ 3)$, the vertex labelings of the cycle in G, i.e., $f(v_i)$, values of m which is the index used in the labeling of u_i's and the respective elements of $er(G)$ are given in Table 2.1.

$k \geq 22$	$f(v_i)$	m	$er(G)$
$\equiv 0$ (mod 3)	$\{k-13, k-2, k-8, k-7, k-4, k-11, k\}$	$\frac{k-15}{3}$	$\left\{\frac{2k-15}{3}, \frac{2k-12}{3}\right\}$
$\equiv 1$ (mod 3)	$\{k-13, k-2, k-9, k-7, k-4, k-11, k\}$	$\frac{k-16}{3}$	$\left\{\frac{2k-14}{3}, \frac{2k-11}{3}\right\}$
$\equiv 2$ (mod 3)	$\{k-13, k-1, k-8, k-6, k-3, k-11, k\}$	$\frac{k-14}{3}$	$\left\{\frac{2k-13}{3}, \frac{2k-10}{3}\right\}$

Table 2.1: Vertex labelings of C_7 and edge rankings of $C_7 \odot K_1$

Therefore, $C_7 \odot K_1$ is a weakly balanced rank distribution graph and $brd(C_7 \odot K_1) = 22$.
(iv) $n = 9$ and $k \geq 18$.
Consider the vertex labeling of pendant vertices as given in Eq. (1).
From k = 18 to 25, $C_9 \odot K_1$ has a non balanced rank distribution and from k =26 to 31, it has a weakly balanced rank distribution with three edge ranks. Thus, from k =18 to 31, any vertex labeling f doesn't induce an edge labeling with $|er(G)| = 2$. So, take $k \geq 32$ and define the weakly balanced rank distribution labeling. According to $k \equiv 0, 1, 2\ (mod\ 3)$, the vertex labeling $f(v_i)$, the value of m to be used to find the labeling $f(u_i)$ from Eq. (1) and the elements of $er(G)$ are given in Table 2.2.

$k \geq 32$	$f(v_i)$	m	$er(G)$
$\equiv 0$ (mod 3)	$\{k-19, k-2, k-14, k-7,$ $k-10, k-11, k-5, k-16, k\}$	$\frac{k-21}{3}$	$\left\{\frac{2k-21}{3}, \frac{2k-18}{3}\right\}$
$\equiv 1$ (mod 3)	$\{k-20, k-2, k-15, k-7,$ $k-10, k-12, k-5, k-17, k\}$	$\frac{k-22}{3}$	$\left\{\frac{2k-20}{3}, \frac{2k-17}{3}\right\}$
$\equiv 2$ (mod 3)	$\{k-19, k-1, k-14, k-6,$ $k-9, k-11, k-4, k-16, k\}$	$\frac{k-20}{3}$	$\left\{\frac{2k-19}{3}, \frac{2k-16}{3}\right\}$

Table 2.2: Vertex labelings of C_9 and edge rankings of $C_9 \odot K_1$

Therefore, $C_9 \odot K_1$ is a weakly balanced rank distribution graph and $brd(C_9 \odot K_1) = 32$.
(v) $n \geq 11$ and $k \geq 22$.
For any $k \geq 22$, no vertex labeling induces an edge labeling that produces $|er(G)| = 1$ or 2. We seek the minimum value of k for which the vertex labeling induces an edge labeling with $|er(G)| = 3$.
Consider $f(u_i)$ as given in Eq. (1) and define the vertex labeling $f(v_i)$ as

$$f(v_i) = \begin{cases} k - 3n + 7, & for\ i = 1 \\ k - 3n + 3, & for\ i = 3, 5, 7, ..., n - 2 \\ k + 5 - 3i, & for\ i = 2, 4, 6, ..., n - 1 \\ k, & for\ i = n \end{cases} \tag{3}$$

Using the Eqs. (1) and (3), the vertex labeling of $C_n \odot K_1$, for $n \geq 11$ is shown in Fig. 2.1.

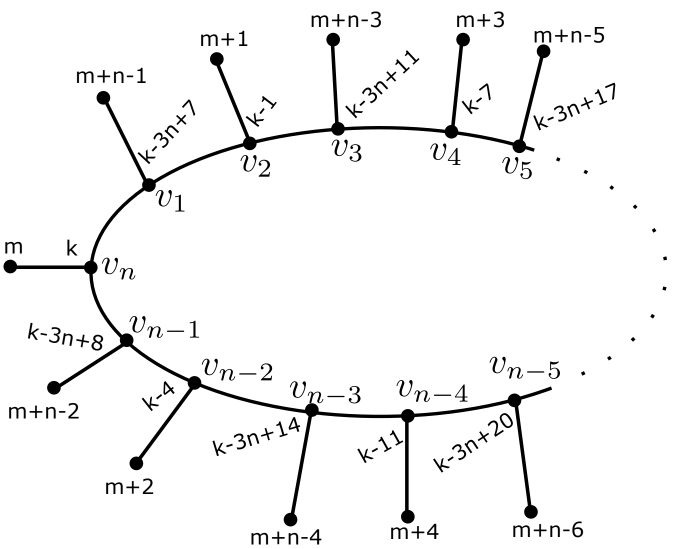

Figure 2.1: Vertex labeling of $C_n \odot K_1$, $n \geq 11$

From Fig. 2.1, we get

$$Max\{|\phi(v_iv_{i+1}) - \phi(v_jv_{j+1})|, |\phi(v_iv_{i+1}) - \phi(v_1v_n)|, |\phi(v_iv_{i+1}) - \phi(v_ju_j)|, |\phi(v_iu_i) - \phi(v_ju_j)|\} < 3,$$

for all $i, j \in \{1, 2, ..., n - 1\}$. Therefore, the vertex labeling defined in Eqs. (1) and (3) induces an edge labeling with $|er(G)| = 3$.
In order to find the balanced rank distribution number, the value of m according to different values of k has to be evaluted. We have the following results from Fig. 2.1.
For $k \equiv 0$(mod 3), $\phi(v_nu_n) = \phi(v_1v_2)$
For $k \equiv 1$(mod 3), $\phi(v_1v_n) = \phi(v_2u_2)$

For $k \equiv 2$(mod 3), $\phi(v_n u_n) = \phi(v_{n-1}v_{n-2})$
Thus, for all $n \geq 11$, we can generalize the values of m as follows:
For $k \equiv j$ (mod 3),

$$m = \frac{k - 3n + 6 - j}{3}, j = 0, 1, 2. \tag{4}$$

If k is minimum then $f(v_1) = f(u_1) + 1$. By substituting the minimum value of m obtained from Eq. (4), we get $k = (9n - 17)/2$. For all $k \geq (9n - 17)/2$ and $n \geq 11$, the edge rank set is generalized as

$$er(C_n \odot K_1) = \left\{ \frac{2k - 3(n-1) + j}{3}, \frac{2k - 3(n-2) + j}{3}, \frac{2k - 3(n-3) + j}{3} \right\} \tag{5}$$

where $k \equiv j$(mod 3), $j = 0, 1, 2$.
Therefore, $brd(C_n \odot K_1) = (9n - 17)/2$ for all odd $n \geq 11$.
Thus, $C_n \odot K_1$ is a non-balanced rank distribution graph for $k < (9n - 17)/2$ and a weakly balanced rank distribution graph for $k \geq (9n - 17)/2$ for all odd $n \geq 11$.

□

Lemma 2.2. *The crown graph $C_n \odot K_1$ has a weakly balanced distribution labeling for the given even positive integer n and for the positive integer $k \geq 2n$. Further, the balanced rank distribution number obtained for various values of n is as follows:*
(i) $brd(C_4 \odot K_1) = 9$.
(ii) $brd(C_n \odot K_1) = \frac{9n-20}{2}$, for $n \geq 6$.

Proof. Let $G = C_n \odot K_1$ be a crown graph where n is even and $k \geq 2n$ be the given positive integer. Now, we will analyze the balanced rank distribution of G according to the given values of n and k.
For all even $n \geq 4$, the vertex labeling of u_i's are defined according to $n \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4) and it is tabulated below:

$n \equiv 0$(mod 4)	$n \equiv 2$(mod 4)	$f(u_i)$
$i = 1, 3, 5, ..., \frac{n}{2} - 1$ & $i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n$	$i = 1, 3, 5, ..., \frac{n}{2}$ & $i = \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n$	m+n-i
$i = 2, 4, 6, ..., \frac{n}{2}$ & $i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 1$	$i = 2, 4, 6, ..., \frac{n}{2} - 1$ & $i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 1$	m+i-1

Table 2.3: Labeling of u_i's in $C_n \odot K_1$, for all even $n \geq 4$

(i) $n = 4$ and $k \geq 8$
If $k = 8$, we can define $f(v_i)$ as

$$f(v_1) = 5,\ f(v_2) = 7,\ f(v_3) = 6,\ f(v_4) = 8$$

The induced edge labeling produces three edge ranks, $er(C_4 \odot K_1) = \{4, 5, 6\}$.
But for all $k \geq 9$, we can define the vertex labeling as

$$f(v_1) = k - 4,\ f(v_2) = k - 1,\ f(v_3) = k - 3,\ f(v_4) = k$$

that induces an edge labeling with two edge ranks. The edge ranks and m values are given below:

$$m = \begin{cases} \frac{k-6}{3}, & for\ k \equiv 0(mod\ 3) \\ \frac{k-7}{3}, & for\ k \equiv 1(mod\ 3) \\ \frac{k-5}{3}, & for\ k \equiv 2(mod\ 3) \end{cases}$$

$$er(C_4 \odot K_1) = \left\{ \frac{2k-6+j}{3}, \frac{2k-3+j}{3} \right\}$$

Hence, $brd(C_4 \odot K_1) = 9$.
(ii) $n \geq 6$ and $k \geq 2n$
For any $k < (9n-20)/2$, there is no vertex labeling which induces an edge labeling with minimum cardinality of $er(C_n \odot K_1)$. Thus for $k \geq (9n-20)/2$, we have two cases for the vertex labeling which induces the edge labeling with $|er(C_n \odot K_1)|$ = 2 (or) 3.
Case (i): $n = 6, 8$ and 10.
From k = 2n to $k < (9n-20)/2$, we have the following results:
$C_6 \odot K_1$ is a non balanced rank distribution graph for k = 12 to 13 and a weakly balanced rank distribution graph with three edge ranks for k = 14 to 16.
$C_8 \odot K_1$ is a non balanced rank distribution graph for k = 16 to 21 and a weakly balanced rank distribution graph with three edge ranks for k = 22 to 25.
$C_{10} \odot K_1$ is a non balanced rank distribution graph for k = 20 to 29 and a weakly balanced rank distribution graph with three edge ranks for k = 30 to 34.
Now consider $k = (9n-20)/2$, $m = (k+4-3n)/3$ and define the vertex labeling as below:

$n = 6, 10$	$n = 8$	$f(v_i)$
$i = 1$	$i = 1$	$k - 3n + 8$
$i = 2, 4, 6, ..., \frac{n}{2} - 1$	$i = 2, 4, ..., \frac{n}{2}$	$k - 2i + 2$
$i = 3, 5, ..., \frac{n}{2}$	$i = 3, 5, ..., \frac{n}{2} - 1$	$k - 3n + 5 + 2i$
$i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 1$	$i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 1$	$k + 7 - n - 2i$
$i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n$	$i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n$	$k - 2n + 2i$

Table 2.4: Labeling of v_i's in $C_n \odot K_1$, for $n = 6, 8, 10$ when $k = (9n-20)/2$

From Table 2.3 and Table 2.4 the resulting edge labeling produces 2 edge ranks, namely

$$er(G) = \left\{ \frac{2k-3n+5}{3}, \frac{2k-3n+8}{3} \right\}$$

Therefore, $|er(G)| = 2$ for $n = 6, 8$ and 10.
For all $k > (9n-20)/2$, the above vertex labeling does not induce the edge labeling with two edge ranks. Based upon $k \equiv 0$, 1, 2 (mod 3) slight modifications can be made to acheive two edge ranks.

i		$f(v_i)$	
$n = 6, 10$	$n = 8$	$k \equiv 0, 1$(mod 3)	$k \equiv 2$(mod 3)
$i = 1, 3, 5, ..., \frac{n}{2}$	$i = 1, 3, ..., \frac{n}{2} - 1$	$k - 3n + 6 + 2i$	$k - 3n + 6 + 2i$
$i = 2, 4, ..., \frac{n}{2} - 1$	$i = 2, 4, ..., \frac{n}{2}$	$k - 2i + 2$	$k - 2i + 3$
$i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 1$	$i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 1$	$k + 8 - n - 2i$	$k + 8 - n + 2i$
$i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 2$	$i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 2$	$k - 2n + 2i$	$k + 1 - 2n + 2i$
$i = n$	$i = n$	k	k

Table 2.5: Labeling of v_i's in $C_n \odot K_1$, for $n = 6, 8, 10$ when $k > (9n-20)/2$

The values of m and respective edge ranks produced by the vertex labeling defined in Table 2.3 and Table 2.5 are listed below:

$$m = \begin{cases} \frac{k-3n+6}{3}, & for\ k \equiv 0(mod\ 3) \\ \frac{k-3n+5}{3}, & for\ k \equiv 1(mod\ 3) \\ \frac{k-3n+7}{3}, & for\ k \equiv 2(mod\ 3) \end{cases}$$

$$\mathbf{er}(C_n \odot K_1) = \left\{ \frac{2k-3(n-2)+j}{3}, \frac{2k-3(n-3)+j}{3} \right\} \tag{6}$$

Thus, for $k \geq (9n-20)/2$ and for $n = 6, 8$ and 10, we have $|er(C_n \odot K_1)| = 2$.
Case (ii): $n \geq 12$
In this case, defining a vertex labeling for any $k \geq 24$ to induce an edge labeling with $|er(G)| = 2$ is not possible. In order to find the minimum k to obtain $|er(G)| = 3$, first define the vertex labeling in terms of k such that the induced edge labeling must satisfies the condition $|\phi(e_p) - \phi(e_q)| < 3$ for every $e_p, e_q \in E(G)$. One such vertex labeling is defined in Table 2.6.

i		$f(v_i)$
$n \equiv 0$ (mod 4)	$n \equiv 2$ (mod 4)	$k \equiv 0, 1, 2$(mod 3)
$i = 1$	$i = 1$	$k - 3n + 8$
$i = 2, 4, 6, ..., \frac{n}{2}$ & $\frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 3$	$i = 2, 4, 6, ..., \frac{n}{2} - 1$ & $\frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 3$	$k + 5 - 3i$
$i = 3, 5, ..., \frac{n}{2} - 1$ & $\frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 2$	$i = 3, 5, ..., \frac{n}{2}$ & $\frac{n}{2} + 1, \frac{n}{2} + 3, ..., n - 2$	$k - 3n + 2 + 3i$
$i = n - 1$	$i = n - 1$	$k - 3n + 9$
$i = n$	$i = n$	k

Table 2.6: Labeling of v_i's in $C_n \odot K_1$, for all even $n \geq 12$

Clearly this vertex labeling induces an edge labeling with $|er(G)| = 3$.
From Table 3 and Table 6, the vertex labeling of $C_n \odot K_1$ for $n \geq 12$ is shown in Fig. 2.2.

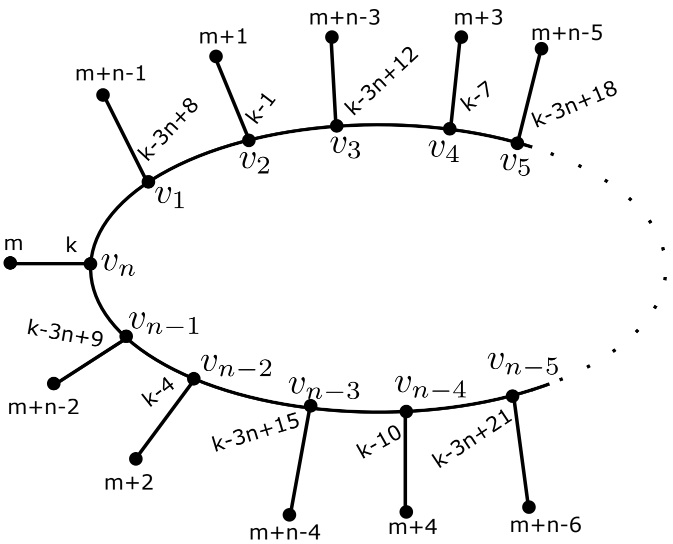

Figure 2.2: Vertex labeling of $C_n \odot K_1$, $n \geq 12$

In order to find the balanced rank distribution number, the value of m according to different values of k has to be evaluted. We have the following results from Fig. 2.2.
For $k \equiv 0$(mod 3), $\phi(v_{n-1}u_{n-1}) = \phi(v_{n-1}v_n)$
For $k \equiv 1$(mod 3), $\phi(v_nu_n) = \phi(v_{n-1}v_{n-2})$
For $k \equiv 2$(mod 3), $\phi(v_nu_n) = \phi(v_3v_4)$
Thus, for all $n \geq 12$, the generalized values of m are given in Eq. (4).
To obtain minimum value of k, let $f(v_1) = f(u_1) + 1$. Here $f(v_1) = k - 3n + 8$ and $f(u_1) = m + n - 1$. By substituting the minimum value of m obtained from Eq. (4),

we get $k = (9n - 20)/2$.
For all $k \geq (9n - 20)/2$ and $n \geq 12$, the edge rank set is generalized in Eq. (5).
Therefore,

$$brd(C_n \odot K_1) = \frac{9n - 20}{2}, \quad \text{for all } n \geq 12$$

Thus, $C_n \odot K_1$ is a non-balanced rank distribution graph for $k < (9n - 20)/2$ and a weakly balanced rank distribution graph for $k \geq (9n - 20)/2$ for all even $n \geq 12$. □

The results obtained from Lemma 2.1 and Lemma 2.2 are summarized in the following theorem.

Theorem 2.1. *A crown graph $C_n \odot K_1$ of order $2n$ and $k \geq 2n$ is a weakly balanced rank distribution graph and its balanced rank distribution number is given by*

$$brd(C_n \odot K_1) = \begin{cases} 6, & for\ n = 3, \\ \frac{9n-19}{2}, & for\ n = 5, 7, \\ \frac{9n-17}{2}, & for\ odd\ n \geq 9. \end{cases}$$

and

$$brd(C_n \odot K_1) = \begin{cases} 9, & n = 4, \\ \frac{9n-20}{2}, & for\ even\ n \geq 6. \end{cases}$$

Example 2.1.
Case (i): n is odd
For $n = 11$ and $k = 41$, the balanced rank disribution labeling of $C_{11} \odot K_1$ is given in Fig. 2.3.

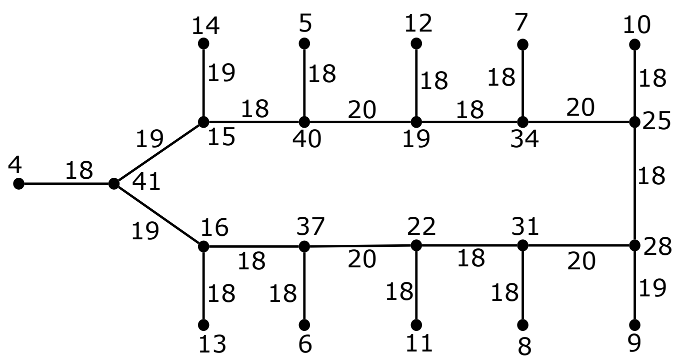

Figure 2.3: Weakly balanced rank distribution of $C_{11} \odot K_1$

Here $|B|$ = 3 = $\Delta(G)$. Hence, $C_{11} \odot K_1$ is a weakly balanced rank distribution graph.

Case (ii):n is even
Subcase (i): $n \equiv 0 \pmod 4$
For $n = 12$ and $k = 44$, the balanced rank disribution labeling of $C_{12} \odot K_1$ is given in Fig. 2.4.

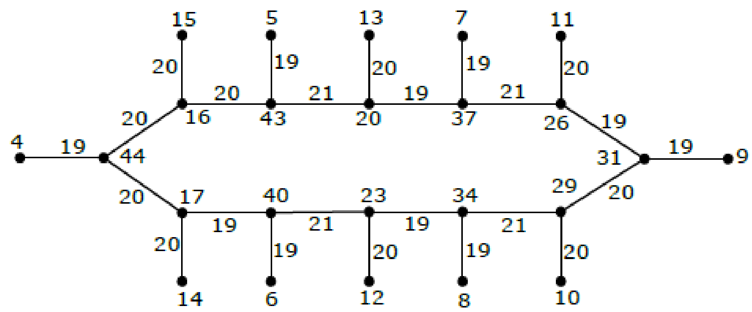

Figure 2.4: Weakly balanced rank distribution of $C_{12} \odot K_1$

Here $|B|$ = 3 = $\Delta(G)$. Hence, $C_{12} \odot K_1$ is a weakly balanced rank distribution graph.
Subcase (ii): $n \equiv 2 \pmod 4$
For $n = 14$ and $k = 53$, the balanced rank disribution labeling of $C_{12} \odot K_1$ is given in Fig. 2.5.

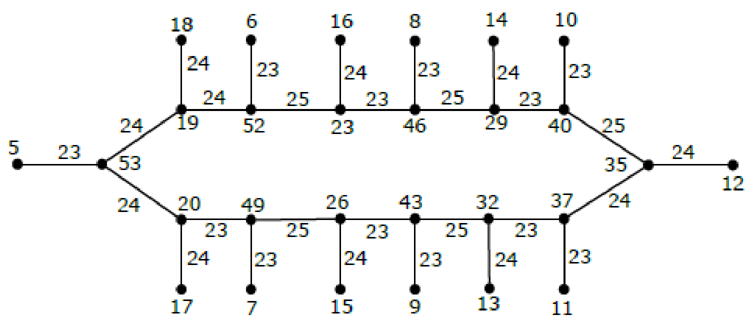

Figure 2.5: Weakly balanced rank distribution labeling of $C_{14} \odot K_1$

Here $|B|$ = 3 = $\Delta(G)$. Hence, $C_{14} \odot K_1$ is a weakly balanced rank distribution graph.

3. BALANCED RANK DISTRIBUTION LABELING OF WHEEL GRAPHS

Theorem 3.1. *The wheel graph W_n of order $n \geq 4$ and $k \geq n$ is a*

(i) balanced rank distribution graph from $n = 4$ to 10 and

(ii) weakly balanced rank distribution graph for all $n \geq 11$.

Further, $brd(W_n) = n$*, for all* $n \geq 4$.

Proof. Let W_n be a wheel graph with n vertices. It has $n-1$ vertices with degree 3 and 1 vertex with degree $n-1$. Let $v_1, v_2, ..., v_{n-1}$ be the vertices of degree 3 and v_n be the vertex of degree $n-1$. Here, $V(W_n) = V_0 \cup V_n$ where $V_0 = \{v_1, v_2, ..., v_{n-1}\}$ and $V_n = \{v_n\}$.
Also, $E(W_n) = X \cup Y$, where $X = \{v_i v_{i+1}\} \cup \{v_1 v_{n-1}\}$, for all $i \in \{1, 2, ..., n-2\}$ and $Y = \{v_j v_n\}$, for all $v_j \in V_0$
For any given $k \geq n$, we define the vertex labeling $f : V(W_n) \to \{1, 2..., k\}$ to induce an edge labeling $\phi : E(W_n) \to B \subset N$ where the edge labeling set B has minimum cardinality.
Case (i): $n = 4$
For $n = 4$, $W_4 = K_4$, the complete graph with 4 vertices. We know that, every complete graph K_n is a balanced rank distribution graph with $brd(K_n) = n$ from [6]. Therefore, $brd(W_4) = 4$ and W_4 is a balanced rank distribution graph.
Case (ii): $n = 5$
For any $k \geq 5$, define $f(v_1) = k-4$, $f(v_2) = k-2$, $f(v_3) = k-3$, $f(v_4) = k-1$ and $f(v_5) = k$. The resulting edge labeling set is $\phi(E(W_n)) = \left\{\left\lceil \frac{7k-6}{12} \right\rceil, \left\lceil \frac{7k-4}{12} \right\rceil\right\}$. $|\phi(E)| = 2$ implies W_5 is a balanced rank distribution graph for $k \geq 5$. Therefore, $brd(W_5) = 5$
Case (iii): $n \geq 6$
Let $k \geq 6$. Label the vertex v_n as k.
In order to get B with minimum cardinality, the cardinality of $\phi(Y)$ should be minimum. To minimize $|\phi(Y)|$, the vertices of V_0 be labeled with the numbers from $k-n+1$ to $k-1$ in a specific order and it satisifies the condition $\min \phi(Y) \leq \phi(X) \leq \max \phi(Y)$. Such a vertex labeling of V_0 is given in the Table 3.1.

n	i	$f(v_i)$
≡0,2(mod 4)	$i = 1, 3, 5, ..., n-1$	$k-n+i$
	$i = 2, 4, 6, ..., n-2$	$k-i$
≡1 (mod 4)	$i = 1, 3, 5, .., \frac{n-3}{2}$ & $\frac{n+3}{2}, \frac{n+7}{2}, ..., n-1$	$k-n+i$
	$i = 2, 4, 6, .., \frac{n-1}{2}$ & $\frac{n+1}{2}, \frac{n+5}{2}, ..., n-2$	$k-i$
≡3 (mod 4)	$i = 1, 3, 5, .., \frac{n-1}{2}$ & $\frac{n+1}{2}, \frac{n+5}{2}, ..., n-1$	$k-n+i$
	$i = 2, 4, 6, .., \frac{n-3}{2}$ & $\frac{n+3}{2}, \frac{n+7}{2}, ..., n-2$	$k-i$

Table 3.1: Labeling of the vertex set V_0

From Table 3.1,

$$\phi(v_1v_n) = \min\phi(Y) = \left\lceil \frac{k}{n-1} + \frac{k-1}{3} \right\rceil \text{ and}$$
$$\phi(v_{n-1}v_n) = \max\phi(Y) = \left\lceil \frac{k}{n-1} + \frac{k-n+1}{3} \right\rceil$$

$$er(W_n) = \left\{ \left\lceil \frac{k}{n-1} + \frac{k-n+1}{3} \right\rceil, \left\lceil \frac{k}{n-1} + \frac{k-n+1}{3} \right\rceil + 1, ..., \left\lceil \frac{k}{n-1} + \frac{k-1}{3} \right\rceil \right\}$$
$$|er(W_n)| = \left[\left(\frac{k}{n-1} + \frac{k-1}{3} \right) - \left(\frac{k}{n-1} + \frac{k-n+1}{3} \right) \right] + 1$$
$$= \left[\left(\frac{n-2}{3} \right) \right] + 1 = \left\lceil \frac{n+1}{3} \right\rceil < n-1 = \Delta(W_n)$$

Therefore, $2 \leq |er(W_n)| \leq \Delta(W_n)$.
Here, $\left\lceil \frac{n+1}{3} \right\rceil = 2$, for $n = 6, 7$. $\left\lceil \frac{n+1}{3} \right\rceil = 3$, for $n = 8, 9, 10$ and $\left\lceil \frac{n+1}{3} \right\rceil > 3$, for $n \geq 11$.
i.e., $|er(W_n)| \leq \delta(W_n)$ for $n = 6$ to 10 & $\delta(W_n) < |er(W_n)| < \Delta(W_n)$ for all $n \geq 11$.
Thus, W_n is a balanced rank distribution graph from $n = 4$ to 10 and a weakly balanced rank distribution graph for all $n \geq 11$ & $k \geq n$. Hence $brd(W_n) = n$. □

Example 3.1.
Consider the balanced rank distribution labeling of W_8 (See Fig. 3.1).

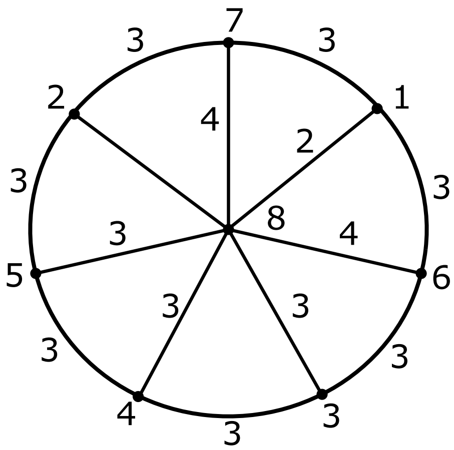

Figure 3.1: Balanced rank distribution labeling of W_8

Here $|B| = 4 < 7 = \Delta(G)$. Thus, W_8 is a balanced rank distribution graph with $brd(W_8) = 8$.

4. CONCLUSION

In this paper, the existence and non-existence of balanced rank distribution labeling of crown and wheel graphs have been investigated and the balanced rank distribution numbers of these graphs have been obtained. The balanced rank distribution labeling of other classes of graphs are yet to be investigated.

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