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Primitive Idempotents of Some Minimal Abelian Codes of Length $2p^n$

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Abstract

Let p and l are distinct odd primes, $o\ (l)_p = f$ and gcd $\left(\frac{l^f - 1}{p}, p\right) = 1$.Then explicit expressions for all the primitive idempotents in $F_l(H_1 \times H_2), H_1$ and H_2 are abelian groups of order 2 and exponent p^n respectively, are obtained.

Keywords: Group characters, Primitive roots, Abelian codes.

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1. Introduction

Let F_l be a field of odd prime order l and $k \geq 1$ be an integer such that gcd (l, k) = 1. It is well known that a cyclic code of a given length k over F_l is an ideal of the semisimple ring $R_k = \frac{F_l\,[x]}{<x^k - 1>}$. Since, every ideal in R_k is the direct sum of its minimal ideals, therefore, to describe the complete set of ideals (codes over F_l) in R_k, it is sufficient to find its complete set of primitive idempotents. Many authors obtained primitive idempotents in $R_k \cong F_l G$, where G is cyclic group of order k. Therefore, a natural case of more general on the group G arises i.e. G is an abelian group. A minimal abelian code over a finite field is a minimal ideal of the group algebra $F_l G$, where G is an abelian group (Berman and Camion). In view of above discussion abelian codes are the generalization of cyclic codes. Bhandari and Grover [3] determined explicit expressions of all the primitive idempotents of the group algebra $F_l\left(C_{p^r} \times C_{p^n}\right), r \leq n$, where $o(l)_p = f$ and gcd $\left(\frac{l^f - 1}{p}, p\right) = 1$..

In this paper, we consider p and l are distinct odd primes, $o\,(l)_p = f$ and gcd $\left(\frac{l^f-1}{p}, p\right) = 1$. In Section 2, some results on finite fields are discussed. In Section 3, (Theorem 3.1), the explicit expressions for all primitive idempotents in $F_l(H_1 \times H_2), H_1$ and H_2 are abelian groups of order 2 and exponent p^n respectively, are obtained.

2 Some Results on Finite Fields

Lemma 2.1.

(i) Let p and l be two distinct odd primes $o\,(l)_p = f$ and p does not divide $\frac{l^f-1}{p}$. Then, $o\,(l)_{p^n} = fp^{n-1}$. Further, $o\,(l)_{2p^n} = fp^{n-1}$.

(ii) Let p and l be distinct odd primes, $ef = \varphi\,(p)$ and p does not divide $\frac{l^f-1}{p}$. If $o\,(l)_{2p^n} = fp^{n-1}$, $n \geq 1$ be an integer, then $o\,(l)_{2p^{n-j}} = \frac{\varphi(2p^{n-j})}{e} = fp^{n-j}$ for all $j;\ 0 \leq j \leq n-1$.

(iii) The set $\{g^k, g^k l, \dots, g^k l^{fp^{n-j}-1}\}$ forms a reduced residue system modulo p^{n-j}, where g is primitive root modulo $\mathrm{p^{n-j}}, 0 \leq \mathrm{k} \leq \mathrm{e}-1$ and $0 \leq \mathrm{j} \leq \mathrm{n}-1$.

Lemma 2.2. Let F_l be the field containing l elements and α be the primitive $2p^n$th root of unity. Let $K = F_l(\alpha)$ and $\sigma : K \to K$ is an $F_l -$ automorphism defined by $\sigma(\alpha) = \alpha^l$, then

(i) $K = F_{l^{fp^{n-1}}}$.

(ii) The Galois group $G(K/F_l) =< \sigma >$.

Proof. To prove (i) part, we need to show that $\alpha \in F_{l^{fp^{n-1}}}$. For this it is sufficient to prove $\alpha^{l^{fp^{n-1}}} = \alpha$.By Lemma 2.1,$o\,(l)_{2p^n} = fp^{n-1}$ gives that $l^{fp^{n-1}} \equiv 1 \pmod{2p^n}$. Thus, $l^{fp^{n-1}} = 1 + \lambda 2p^n$, for some integer λ.So,

$$\alpha^{l^{fp^{n-1}}} = \alpha^{1+\lambda 2p^n} = \alpha.\alpha^{\lambda 2p^n}.$$

As, α is primitive $2p^n$th root of unity so, $\alpha^{2p^n} = 1$.Thus, $\alpha^{l^{fp^{n-1}}} = \alpha$.

(ii) By the given condition $\sigma : K \to K$ defined by $\sigma(\alpha) = \alpha^l$ for some $\alpha \in G(K/F_l)$. We claim that $o(\sigma) = fp^{n-1}$. Let $o(\sigma) = t$, for some integer $t \geq 1$.

If $t = 1$, then $\sigma = I$, which is not possible. Hence, $t > 1$.Now, it is easy to see, $\sigma^2(\alpha) = \sigma\sigma(\alpha) = \sigma(\alpha^l) = \alpha^{l^2}$, consequently $\sigma^k(\alpha) = \alpha^{l^k}$implies $\sigma^{fp^{n-1}}(\alpha) = \alpha^{l^{fp^{n-1}}}$.

As, $o(l)_{2p^n} = fp^{n-1}$, so $l^{fp^{n-1}} \equiv 1 \bmod (2p^n)$ so $\sigma^{fp^{n-1}}(\alpha) = \alpha^{l^{fp^{n-1}}} = \alpha$. But, $o(\sigma) = t$, gives that t divide fp^{n-1}.

On the other hand, $\sigma^t = I$, implies $\sigma^t(\alpha) = I(\alpha) = \alpha$. This gives $\alpha^{l^t} = \alpha$, implies $\alpha^{l^t-1} = 1$. Thus, $l^t \equiv 1 \pmod{2p^n}$ which gives fp^{n-1}, divides t as $o(l)_{2p^n} = fp^{n-1}$.

Hence, $t = fp^{n-1}$.Also, $o\,(G(K/F_l)) = [K{:}\,F_l]$. But by part (i) $K = F_{l^{fp^{n-1}}}$. Therefore, $o\big(G(K/F_l)\big) = \left[F_{l^{fp^{n-1}}} : F_l\right] = fp^{n-1}$. Hence, $G(K/F_l) =< \sigma >$.

Corollary 2.3. Let F_l be the field containing l elements and α be the primitive p^nth root of unity. Let $K = F_l(\alpha)$ and $\sigma : K \to K$ is an $F_l -$ automorphism defined by $\sigma(\alpha) = \alpha^l$, then

(i) $K = F_{l^{fp^{n-1}}}$.

(ii) The Galois group $G(K/F_l) =< \sigma >$.

Notations 2.4.

Let $A_t = \sum_{i=0}^{f-1} \alpha^{p^{n-1}g^t l^i}$ and $L = \{1, l, l^2, \dots, l^{f-1}\}$.

Definition 2.5. Let $F = \mathbb{F}_{q^m}$ be a finite extension of the field $K = \mathbb{F}_q$ then the **trace function** $Tr_{F/K}\,(\alpha)$ of α over K is defined by $Tr_{F/K}\,(\alpha) = \alpha + \alpha^q + \ \dots + \alpha^{q^{m-1}}$.

Definition 2.6. Let G be a group and V a finite-dimensional vector space over a field K. Let $T : G \to GL\,(V)$ be a representation of G over K. Then, the **character** χ of G afforded by the representation T is the mapping $\chi : G \to K$ given by $\chi(g) = tr\,(T_g)$for all $g \in G$.

Theorem 2.7. Let $F = \mathbb{F}_{q^m}$ be a finite extension of the field $K = \mathbb{F}_q$. Then the **trace function** $Tr_{F/K}$ satisfies the following properties:

(i) $Tr_{F/K}(\alpha + \beta) = Tr_{F/K}(\alpha) + Tr_{F/K}\,(\beta)$ for all $\alpha, \beta \in F$;

(ii) $Tr_{F/K}(c\alpha) = cTr_{F/K}\,(\alpha)$ for all $c \in K, \alpha \in F$;

(iii) $Tr_{F/K}$ is a linear transformation from F onto K, where both F and K are viewed as vector spaces over K;

(iv) $Tr_{F/K}(a) = ma$ for all $a \in K$;

(v) $Tr_{F/K}(\alpha^q) = Tr_{F/K}\,(\alpha)$ for all $\alpha \in F$.

Observe that

$$A_t = \sum_{i=0}^{f-1} \alpha^{p^{n-1}g^t l^i} = Tr_{\frac{F_{l^f}}{F_l}}\big(\beta^{g^t}\big), \alpha^{p^{n-1}} = \beta$$

Then, A_t are elements of F_l and can be evaluated by using properties of Gaussian periods, $0 \leq \mathrm{t} \leq \mathrm{e} - 1$.

Definition 2.8. Let G be a finite abelian group of exponent n and F a finite field of order l, where l is a prime such that $\gcd(n, l) = 1$. Let $K = F\,(\xi_n)$ where ξ_n is primitive nth root of unity, C be set of all characters of G and $G(K, F_l)$ be the Galios group. Then $\{\sigma o \chi\}$ is orbit of χ, where $\sigma \epsilon\, G(K, F_l)$ and $\chi \epsilon\, C$.

3 Primitive Idempotents in F_lG

The following Theorem 3.1 and Proposition 3.2 are given in Grover

Theorem 3.1. Let G be a finite abelian group of exponent n and F a finite field of order l, where l is a prime such that $\gcd(n,l) = 1$. Let $K = F\ (\xi_n)$ where ξ_n is primitive nth root of unity and $C_1 = \{\chi_{1,1}, \chi_{1,2}, \dots, \chi_{1,l_1}\}, \dots, C_r = \{\chi_{r,1}, \chi_{r,2}, \dots, \chi_{r,l_r}\}$ be the orbits of the set C, which is the set of all characters of G under $G(K, F_l)$, then complete set of primitive idempotents of group algebra FG is given by

$$e_i = \frac{1}{|G|}\sum_{g\in G} Tr_{F_{\xi_{d_i}}/F}\ ((\chi_{i1}\ (g^{-1}))g$$

for

$$1 \le i \le r, d_i = [G : \text{Ker}\ \chi_{i,1}].$$

Theorem 3.2. Let $H_1$and H_2 be two abelian groups of exponents $p^n, (n \ge 1)$and $q^m, (m \ge 1)$, where p and q are distinct odd primes. Let l be an odd prime with $\gcd(l, pq) = 1$ and F_l a finite field with l elements. Let $(l)_p = f$ and $(l)_q = g$ such that $\gcd\left(\frac{l^f-1}{p}, p\right) = 1$ if $n > 1$ and $\gcd\left(\frac{l^g-1}{q}, q\right) = 1$ if $m > 1$. Let $\gcd(f,g) = s$ and $G = H_1 \times H_2$, then all the primitive idempotents of the group algebra F_lG are of the form $Tr_{F_{l^s}/F_l}\ (eH_1eH_2)$, where eH_1 and eH_2 are primitive idempotents in $F_{l^s}H_1$ and $F_{l^s}H_2$, respectively.

In Theorem 3.1, it is observed that if $C_1 = \{\chi_{11,\dots,}\chi_{1l_1}\},\dots, C_r = \{\chi_{r1,\dots,}\chi_{1l_r}\}$ be all the orbits of C under the action of Galios group $G\ (K/F_l)$, where C is set of all the characters of G.Then, the number of primitive idempotents in F_lG is equals to the number of orbits of C. Therefore, to obtain the primitive idempotents F_lG, we need to find all the orbits of set of characters of G.

3.3. Characters of H_1

Let H_1 be abelian group of exponent 2 and K_1 is smallest field extension of F_l containing 2nd root of unity, then $K_1 = F_l$ and Galois group $GF\ (K_1/F_l) = \{I\}$.

The characters of H_1 are defined as $\phi_1(x) = 1$ and $\phi_2(x) = -1$.

Then the orbits of set of characters of H_1 are $\{Io\phi_1\}$ and $\{Io\phi_2\}$.

Let us denote $Io\phi_1 = \phi_{1,1}$ and $Io\phi_1 = \phi_{1,2}$.

3.4. Characters of H_2

Let H_2 be abelian group of exponent p^n and K_2 is smallest field extension of F_l containing p^nth root of unity, then by Corollary 2.3

(i) $K = F_{l^{fp^{n-1}}}$.

(ii) The Galois group $G(K/F_l) =< \sigma >$.

The characters of H_2 are defined as $\psi_1(y) = 1$ and $\psi_{2,j,k,s}(y) = \alpha^{p^{n-j}g^k l^s}, 0 \le k \le e-1, 0 \le s \le fp^{j-1} - 1, 1 \le j \le n$.

Then the orbits of set of characters of H_2 are $\{\sigma o\psi_1\}$ and $\{\sigma o\psi_{2,j,k,s}\}$, where $\sigma \epsilon G(K/F_l)$. Let us denote $\sigma o\psi_1 = \tau_{1,1}$ and $\sigma o\psi_{2,j,k,s} = \tau_{2,j,k}$. Then, the orbits of set of characters of H_2 are $\tau_{1,1}(y) = 1$ and $\tau_{2,j,k}(x) = \alpha^{p^{n-j}g^k l^s}$, $0 \le k \le e-1, 1 \le j \le n-1$. The total number of orbits of set of characters of H_2 are $en+1$.

Proposition 3.5. Let $G = H_1 \times H_2$, where H_1 and H_2 be any two abelian groups of exponents 2 and p^n respectively. Let α be the primitive $2p^n$th root of unity and $K = F(\alpha)$. Then any character χ of G can be expressed uniquely as a direct product of characters of H_1 and H_2 and each orbit of set of all characters of G can be obtained uniquely, as products of orbits of sets of characters of H_1 and H_2.

3.6 The $2en+2$ orbits of set of characters of G with values in K are given below:
Let $\mathrm{g} \in \mathrm{G}, \mathrm{g} = \mathrm{xy}$, where $\mathrm{x} \in \mathrm{H}_1, \mathrm{y} \in \mathrm{H}_2$, then

1. $\chi_{1,1}(x) = \chi_{1,1}(y) = 1$,
2. $\chi_{1,2}(x) = -1, \chi_{1,2}(y) = 1$.
   For $0 \le k \le e-1, 1 \le j \le n-1$,
3. $\chi_{3,j,k}(x) = 1, \chi_{3,j,k}(y) = \alpha^{p^{n-j}g^k l^s}$.
4. $\chi_{4,j,k}(x) = -1, \chi_{4,j,k}(x) = \alpha^{p^{n-j}g^k l^s}$.

Theorem 3.7. Let $H_1$and H_2 be two abelian groups of order 2 and exponents $p^n, n \ge 1$, respectively. Let l be an odd prime and F_l a finite field with l elements. Let $G = H_1 \times H_2$, then all the $2en+2$ primitive idempotents of the group algebra F_lG are given by

1. $\theta_1(x,y) = \frac{1}{|G|}(1+x)\sum_{i=0}^{p^n-1} y^i$.
2. $\theta_2\,(x,y) = \frac{1}{|G|}\,(1-x)\sum_{i=0}^{p^n-1} y^i$.
   For $0 \le k \le e-1, 1 \le j \le n$,
3. $\theta_{3,k,j,}(x,y) = \frac{p^{j-1}}{|G|}(1+x)\left(f\sum_{i\equiv 0 (\mathrm{mod}\, p^j)} y^i + A_0 \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\, p) \in L} y^i + \ldots + A_{e-1} \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\, p) \in g^{e-1}L} y^i\right)$.
4. $\theta_{4,k,j,}(x,y) = \frac{p^{j-1}}{|G|}(1-x)\left(f\sum_{i\equiv 0 (\mathrm{mod}\, p^j)} y^i + A_0 \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\, p) \in L} y^i + \ldots + A_{e-1} \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\, p) \in g^{e-1}L} y^i\right)$.

Proof. Evaluation of $\boldsymbol{\theta_1(x,y)}$:
By Theorem 2.7,$\theta_1(x,y) = \frac{1}{|G|}\sum_{g\in G} Tr_{\frac{F_l}{F_l}}\left(\left(\chi_{1,1}(g^{-1})\right)g\right.$ and by (1) of 2.6
$\chi_{1,1}(x) = \chi_{1,1}(y) = 1$, implies $\theta_1(x,y) = \frac{1}{|G|}\sum_{g\in G} Tr_{F_l/F_l}\,(1)g$.

By Definition 2.4, $Tr_{F_l/F_l}(1) = 1$, which gives

$$\theta_1(x,y) = \frac{1}{|G|}(1+x)\sum_{i=0}^{p^n-1} y^i.$$

Evaluation of $\theta_2(x,y)$:

By Theorem 2.7, $\theta_2(x,y) = \frac{1}{|G|}\sum_{g\in G} Tr_{F_l/F_l}\left(\chi_{1,2}(g^{-1})\right) g$ and by (2) of 2.6

$$\chi_{1,2}(x) = -1, \chi_{1,2}(y) = 1,$$

implies

$$\theta_2(x,y) = \frac{1}{|G|}\sum_{g\in G} Tr_{F_l/F_l}(-1)g.$$

By Definition 2.4, $Tr_{F_l/F_l}(-1) = -1$, which gives

$$\theta_2(x,y) = \frac{1}{|G|}(1-x)\sum_{i=0}^{p^n-1} y^i.$$

Evaluation of $\theta_{3,k,j}(x,y)$, for $0 \le k \le e-1, 1 \le j \le n$:

By Theorem 2.7 and Theorem 3.2,

$$\theta_{3,k,j}(x,y) = \frac{1}{|G|}(1+x)\sum_{i=0}^{p^n-1} Tr_{K/F}\left(\chi_{3,k,j}(y^{-i})\right) y^i,$$

where $\chi_{3,j,k}(y) = \alpha^{p^{n-j}g^k l^s}$ and K is smallest field extension of F_l containing primitive p^jth root of unity.Using Lemma 2.2, we have $K = F_{l^{fp^{j-1}}}$.

Also by (3) of 3.6, $\chi_{3,k,j}(x) = 1, \chi_{3,k,j}(y^{-i}) = \alpha^{-p^{n-j}g^k l^s i}$ which gives that $Tr_{K/F}\left(\chi_{3,k,j}(y^{-i})\right) = Tr_{K/F}(\alpha^{-p^{n-j}g^k l^s i})$. As discussed above, $K = F_{l^{fp^{j-1}}}$, so

$$Tr_{K/F}(\alpha^{-p^{n-j}g^k l^s i}) = Tr_{F_{l^{fp^{j-1}}}/F_l}(\alpha^{-p^{n-j}g^k l^s i}).$$

By Definition 2.4,

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = \sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i},$$

where

$$\beta = \alpha^{-p^{n-j}}.$$

Since α is primitive p^nth root of unity, then β is primitive p^jth root of unity.

Case (i) If $i \equiv 0 (\mathrm{mod}\, p^j)$, then $\beta^{g^k l^s i} = 1$.

This gives that

$$\sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i} = fp^{j-1} \text{ implies } Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = fp^{j-1}.$$

Case (ii) Let $i \not\equiv 0 \pmod{p^j}$.

Sub-case (a) If $i \equiv 0 \pmod{p^{j-1}}$ and $\frac{g^k i}{p^{j-1}} \pmod{p} \in g^t L$, for some $t, 1 \le t \le e-1$. Then, $\sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i} = \sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s}$, where γ is primitive p^{th} root of unity.
Now $o\ (l)_p = f$, so $l^r = l^h$ iff $r \equiv h \pmod{f}$, thus $\sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s} = p^{j-1} \sum_{s=0}^{f-1} \gamma^{g^t l^s}$.
By Notation 2.3, $A_t = \sum_{i=0}^{f-1} \alpha^{p^{n-1} g^t l^i}$, hence $\sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s} = p^{j-1} \sum_{s=0}^{f-1} \gamma^{g^t l^s} = p^{j-1} A_t$.

Sub-case (b): If $i \not\equiv 0 \pmod{p^{j-1}}$, then $i = g^a p^h l^t$, for some, $0 \le a \le e-1, 0 \le h \le j-2$ and $0 \le t \le fp^{j-1}-1$.
Then $\alpha^{-p^{n-j} g^k l^s i} = \alpha^{-p^{n-h} g^b l^{s+t}} = \beta$ (say).

Thus β is primitive p^hth root of unity and $0 \le h \le j-2$ and hence

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j} g^k l^s i}\right) = Tr_{F_{l^{fp^{j-1}}}/F_l}(\beta).$$

Then, as discussed in Section 2, Grover and Bhandari [48], we get

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j} g^k l^s i}\right) = 0.$$

Combining case (i) and case (ii), we get

$$\theta_{3,k,j,}(x,y) = \frac{p^{j-1}}{|G|}(1+x)\left(f \sum_{i \equiv 0 (\text{mod } p^j)} y^i + A_0 \sum_{p - \frac{g^k i}{p^{j-1}} (\text{mod } p) \in L} y^i + \dots + A_{e-1} \sum_{p - \frac{g^k i}{p^{j-1}} (\text{mod } p) \in g^{e-1} L} y^i\right).$$

Evaluation of $\boldsymbol{\theta_{4,k,j}(x,y)}$ for $\mathbf{0 \le k \le e-1, 1 \le j \le n}$:
By Theorem 2.7 and Theorem 2.2,

$$\theta_{4,k,j}(x,y) = \frac{1}{|G|}(1-x) \sum_{i=0}^{p^n-1} Tr_{K/F}\left(\chi_{4,k,j}(y^{-i})\right) y^i,$$

where $\chi_{4,j,k}(y) = \alpha^{p^{n-j} g^k l^s}$ and K is smallest field extension of F_l containing primitive p^jth root of unity. Using Lemma 2.2, we have $K = F_{l^{fp^{j-1}}}$.
Also by (4) of 3.6, $\chi_{4,k,j}(x) = -1, \chi_{4,k,j}(y^{-i}) = \alpha^{-p^{n-j} g^k l^s i}$
which gives that $Tr_{K/F}\left(\chi_{4,k,j}(y^{-i})\right) = Tr_{K/F}\ (\alpha^{-p^{n-j} g^k l^s i})$.
As discussed above, $K = F_{l^{fp^{j-1}}}$, so

$$Tr_{K/F}\ (\alpha^{-p^{n-j} g^k l^s i}) = Tr_{F_{l^{fp^{j-1}}}/F_l}\ (\alpha^{-p^{n-j} g^k l^s i}).$$

By Definition 2.4,

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = \sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i},$$

where

$$\beta = \alpha^{-p^{n-j}}.$$

Since α is primitive p^nth root of unity, then β is primitive p^jth root of unity.

Case (i) If $i \equiv 0\left(\mathrm{mod}\, p^j\right)$, then $\beta^{g^k l^s i} = 1$.
This gives that

$$\sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i} = fp^{j-1}$$

which implies

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = fp^{j-1}.$$

Case (ii) Let $i \not\equiv 0\left(\mathrm{mod}\, p^j\right)$.

Sub-case (a) If $i \equiv 0\left(\mathrm{mod}\, p^{j-1}\right)$ and $\frac{g^k i}{p^{j-1}}$ $(\mathrm{mod}\, p) \in g^t L$,for some $t, 0 \le t \le e-1$, then $\sum_{s=0}^{fp^{j-1}-1} \beta^{g^k l^s i} = \sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s}$, where γ is primitive p^{th} root of unity.
Now $o\ (l)_p = f$, so $l^r = l^h$ iff $r \equiv h$ (mod f), thus

$$\sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s} = p^{j-1} \sum_{s=0}^{f-1} \gamma^{g^t l^s}.$$

By Notation 2.3,

$$A_t = \sum_{i=0}^{f-1} \alpha^{p^{n-1} g^t l^i}$$

hence,

$$\sum_{s=0}^{fp^{j-1}-1} \gamma^{g^t l^s} = p^{j-1} \sum_{s=0}^{f-1} \gamma^{g^t l^s} = p^{j-1} A_t.$$

In view of above discussion,

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = p^{j-1} A_t.$$

Sub-case (b): If $i \not\equiv 0\left(\mathrm{mod}\, p^{j-1}\right)$, then $i = g^a p^h l^t$,where $0 \le a \le e-1, 0 \le h \le j-2$ and $0 \le t \le fp^{j-1}-1$.
Then $\alpha^{-p^{n-j}g^k l^s i} = \alpha^{-p^{n-h} g^b l^{s+t}} = \beta$ (say).
Then β is primitive p^hth root of unity, $0 \le h \le j-2$ and $Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = Tr_{F_{l^{fp^{j-1}}}/F_l}(\beta)$.

Then, as discussed in Section 2, Grover and Bhandari, we get

$$Tr_{F_{l^{fp^{j-1}}}/F_l}\left(\alpha^{-p^{n-j}g^k l^s i}\right) = 0.$$

Combining case (i) and case (ii), we get

$$\theta_{4,k,j,}(x,y) = \frac{p^{j-1}}{|G|}(1-x)\left(f\sum_{i\equiv 0 (\mathrm{mod}\ p^j)} y^i + A_0 \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\ p)\in L} y^i + \ldots + A_{e-1} \sum_{p-\frac{g^k i}{p^{j-1}} (\mathrm{mod}\ p)\in g^{e-1}L} y^i\right).$$

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