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Intuitionistic Fuzzy $k-$ ideals of $\Gamma-$ Semirings

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Abstract

In this paper, we introduce the notions of anti fuzzy ideal, anti fuzzy $k-$ ideal and intuitionistic fuzzy $k-$ ideal of $\Gamma-$ semirings. We investigate their properties and connections with $k-$ ideals and fuzzy $k-$ ideals.
Keywords: $\Gamma-$ semiring, fuzzy ideal, $k-$ ideal, anti fuzzy $k-$ideal and intuitionistic fuzzy $k-$ ideal.
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1. INTRODUCTION

As a generalization of the ring, the notion of a $\Gamma-$ ring was introduced by Nobusawa [11] in 1964. The notion of $\Gamma-$ semigroup was introduced by Sen[15] in 1981 as a generalization of $\Gamma-$ groups. Murali Krishna Rao [12] in 1995 introduced the notion of $\Gamma-$semiring as a generalization of $\Gamma-$ ring, ring, ternary semiring and semiring. The important reason for development of $\Gamma-$ semiring is a generalization of results of rings, $\Gamma-$ rings, semirings, semigroup and ternary semirings. Later on, much has been developed on this concept by different researchers.
The concept of fuzzy set of a set X was introduced by L. A. Zadeh [17] as a function from X into [0,1]. The concept of fuzzy ideals in a ring was introduced by W. L. Liu [10]. In 1972, Rosenfield [14] was first who applied the concept of fuzzy set to group structures and introduced the notion of fuzzy subgroups and ideals of a semigroup. In 1993, Ahsan et.al [1] first studied the fuzzy semirings and semimodules. In 2011, Dutta and Goswami [6] studied "Operations on fuzzy ideals of $\Gamma-$ semirings" and in 2018 Rao [13] investigated the "Fuzzy prime ideals in ordered $\Gamma-$ semirings".

Atanassov introduced the concept of the idea of intuitionistic fuzzy sets which constitute a generalization of the notion of fuzzy sets [3, 4]. K. H. Kim and Park M.A.[9] studied the intuitionistic fuzzification of the concept of several ideals in a semigroups and investigate some properties of such ideals. M. Akram and W. A. Dudek [2] introduced the notion of intuitionistic fuzzy left $k-$ideals in semiring. The prime motivation for this paper is [7, 8]. In this paper, we apply the concept of intuitionistic fuzzy set to $\Gamma-$semirings. We introduced the notion of anti fuzzy right ideal, anti fuzzy right $k-$ideal and intuitionistic fuzzy right $k-$ideal in $\Gamma-$semirings. We investigate the their properties and connections with right $k-$ideals, fuzzy right $k-$ideals, anti fuzzy right $k-$ideals. We describe various methods of constructions of intuitionistic fuzzy right ideals and intuitionistic fuzzy right $k-$ideals.

2. PRELIMINARIES AND EXAMPLES

Recall from [5, 6, 12, 14] that if (R, +) and (Γ, +) be two commutative semigroups then R is called a $\Gamma-$ semiring if there exists a mapping R $\times$ Γ $\times$ R $\rightarrow$ R denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying(i) $x\alpha(y+z) = x\alpha y + x\alpha z$. (ii) $(y+z)\alpha x = y\alpha x + z\alpha x$. (iii) $x(\alpha+\beta)z = x\alpha z + x\beta z$. (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Let A and B be semirings and let $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings. Further, let M be a $\Gamma-$ ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring. A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$. A non empty subset S of a $\Gamma-$ semiring R is said to be a sub $\Gamma-$ semiring of R if $(S, +)$ is a sub semigroup of $(R, +)$ and $x\gamma y \in S$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Let X be a non-empty set. A mapping $\mu : X \rightarrow [0, 1]$ is called fuzzy subset of X. The compliment of μ is denoted by $\bar{\mu}$ is also the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Remark 2.1. *Throughout this paper, R will denote a $\Gamma-$ semiring with zero element "0" and identity element "1" unless otherwise stated.*

3. INTUITIONISTIC FUZZY $k-$IDEALS OF $\Gamma-$ SEMIRINGS

Let X be a non-empty set. A mapping μ : X $\rightarrow$ [0,1] is called fuzzy subset of X. An upper level set of a fuzzy set μ denoted by $U(\mu, t)$ is defined as $U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ and a lower level set of a fuzzy set μ denoted by $L(\mu, t)$ is defined as $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ for all $t \in [0, 1]$. As an generalization of the notion of fuzzy sets in X, Atanassov [3] introduced the concept of an intuitionistic fuzzy set(IFS) defined on a non empty set X as $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in X\}$, here the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership and degree of non membership respectely and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. Hence, for each fuzzy set μ we have an institutionistc fuzzy set $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$. For our convenience, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set (IFS) $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in X\}$, where $\lambda_A(x) = 1 - \mu_A(x) = \bar{\mu_A}(x)$, for all $x \in X$, the IFS A is a fuzzy set. Hence the notion of intuitionistc fuzzy set(IFS) theory is a generalization of fuzzy set theory.

Remark 3.1. *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistc fuzzy sets in X. We define*

(i) $A \subseteq B$ *if and only if* $(\mu_A(x) \leq \mu_B(x), \lambda_A(x) \geq \lambda_B(x))$ *for all* $x \in X$.

(ii) $A = B$ *if and only if* $A \supseteq B$ *and* $A \subseteq B$.

(iii) $A \cap B = (\mu_A \wedge \mu_B, \lambda_A \vee \lambda_B)$.

(iv) $A \cup B = (\mu_A \vee \mu_B, \lambda_A \wedge \lambda_B)$.

(v) $0 = (0, 1)$ *and* $1 = (1, 0)$.

Definition 3.2. *Let R be a $\Gamma-$ semiring. A fuzzy set μ of R is said to be fuzzy right (left) ideal if*

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(x\alpha y) \geq \mu(x) \qquad (\mu(x\alpha y) \geq \mu(y))$ *for all* $x, y \in R$ *and* $\alpha \in \Gamma$.

Definition 3.3. *Let R be a $\Gamma-$ semiring. A fuzzy set μ of R is said to be an anti fuzzy right (left) ideal if*

(i) $\mu(x + y) \leq \mu(x) \vee \mu(y)$

(ii) $\mu(x\alpha y) \leq \mu(x) \qquad (\mu(x\alpha y) \leq \mu(y))$ *for all* $x, y \in R$ *and* $\alpha \in \Gamma$.

Definition 3.4. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set(IFS) $A = \{< x, \mu_A(x), \lambda_A(x) > \mid x \in R\}$ is called an intuitionistic fuzzy right (left) ideal of R if*

(i) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$

(ii) $\mu_A(x\alpha y) \geq \mu_A(x) \qquad (\mu_A(x\alpha y) \geq \mu_A(y))$

(iii) $\lambda_A(x+y) \leq \lambda_A(x) \vee \lambda_A(y)$

(iv) $\lambda_A(x\alpha y) \leq \lambda_A(x) \quad (\lambda_A(x\alpha y) \leq \lambda_A(y))$ *for all* $x, y \in R$ *and* $\alpha \in \Gamma$.

A is called an intuitionistic fuzzy ideal of R if it is both intuitionistic fuzzy right and left ideal of R. Moreover , it is clear that if R is a $\Gamma-$ semiring with zero element and A is an intuitionistic fuzzy ideal of R then $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in R$.

Definition 3.5. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is called an intuitionistic fuzzy ideal of R if it is both intuitionistic fuzzy right and left ideal.*

Definition 3.6. *Let R be a $\Gamma-$ semiring and $A = (\mu_A, \lambda_A)$ be an IFS of R. Let $s, t \in [0, 1]$. Then the set $R_A^{(s,t)} = \{x \in R \mid \mu_A(x) \geq s, \lambda_A(x) \leq t\}$ is called $(s,t)-$ level set of $A = (\mu_A, \lambda_A)$. The set $\{(s,t) \in Im(\mu_A) \times Im(\lambda_A) \mid s + t \leq 1\}$ is called image of $A = (\mu_A, \lambda_A)$.*

The following theorems are proved in [16]

Theorem 3.7. *Let R be a $\Gamma-$ semiring. A fuzzy set μ of R is a fuzzy right (left) ideal if and only if $U(\mu, t)$ is a right (left) ideal of R for all $t \in [0, 1]$.*

Theorem 3.8. *Let R be a $\Gamma-$ semiring. A fuzzy set μ of R is an anti fuzzy right (left) ideal if and only if $L(\mu, t)$ is a right (left) ideal of R for all $t \in [0, 1]$.*

Theorem 3.9. *Let R be a $\Gamma-$ semiring. A fuzzy set μ is a fuzzy right (left) ideal of R if and only if $1 - \mu = \bar{\mu}$ is an anti fuzzy right (left) ideal of R.*

Theorem 3.10. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R is an intuitionistic fuzzy right ideal of R if and only if $R_A^{(s,t)}$ is a right ideal of R for all $s, t \in [0, 1]$.*

Let $A = \{< x, \mu_A(x), \lambda_A(x) > \mid x \in R\}$ be an intuitionistic fuzzy set in a $\Gamma-$ semiring R. An upper level set of an intuitionistic fuzzy set μ_A denoted by $U_A(\mu_A, t)$ is defined as $U_A(\mu_A, t) = \{x \in R \mid \mu_A(x) \geq t\}$ and a lower level set of a fuzzy set μ_A denoted by $L_A(\mu_A, t)$ is defined as $L_A(\mu_A, t) = \{x \in R \mid \mu_A(x) \leq t\}$ for all $t \in [0, 1]$.

Definition 3.11. *A right (left) ideal I of a $\Gamma-$ semiring R is called $k-$ideal of R if for $x, y \in R, x + y \in I$ and $y \in I$ implies that $x \in I$.*

Definition 3.12. *A fuzzy ideal μ of a $\Gamma-$ semiring R is called fuzzy k-ideal if for all $x, y \in R$, $\mu(x) \geq \mu(x + y) \wedge \mu(y)$. OR*

A fuzzy ideal μ of R is said to be k-fuzzy ideal of R if $\mu(x + y) \geq \lambda$, $\mu(y) \geq \lambda$ then $\mu(x) \geq \lambda$ for all x, $y \in R$, $\lambda \in [0, 1]$.

Definition 3.13. *An anti fuzzy ideal λ of a $\Gamma-$ semiring R is called anti fuzzy k-ideal if for all $x, y \in R$, $\lambda(x) \leq \lambda(x+y) \vee \lambda(y)$*

Definition 3.14. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy ideal $A = \{< x, \mu_A(x), \lambda_A(x) > \mid x \in R\}$ in R is said to be an intuitionistic fuzzy $k-$ ideal in R if*

(i) $\mu_A(x) \geq \mu_A(x+y) \wedge \mu_A(y)$

(ii) $\lambda_A(x) \leq \lambda_A(x+y) \vee \lambda_A(y)$ *for all $x, y \in R$.*

Definition 3.15. *A is called an intuitionistic fuzzy $k-$ideal if A is an intuitionistic fuzzy ideal .*

Every intuitionistic fuzzy $k-$ ideal is an intuitionistic fuzzy ideal. But an intuitionistic fuzzy ideal need not be an intuitionistic fuzzy $k-$ ideal.

Example 3.16. *Let $\mathbb{N}$, the set of non negative integers be a $\Gamma-$ semiring and A be an intuitionistic fuzzy set in $\mathbb{N}$ defined by*

$$A(x) = \begin{cases} (0.3, 0.6) & \text{if x is odd} \\ (0.5, 0.4) & \text{if x is non zero even} \\ (1, 0) & \text{if x=0,} \qquad\qquad \text{for any } x \in \mathbb{N} \end{cases}$$

Then A is an intuitionistic fuzzy $k-$ ideal in $\mathbb{N}$.

Example 3.17. *Let $\mathbb{N}$, the set of non negative integers be a $\Gamma-$ semiring and B be an intuitionistic fuzzy set in $\mathbb{N}$ defined by*

$$B(x) = \begin{cases} (1, 0) & \text{if } x \geq 7 \\ (0.5, 0.4) & \text{if } 5 \leq x < 7 \\ (0, 1) & \text{if } 0 \leq x < 5. \end{cases}$$

Then it can be easily verified that B is an intuitionistic fuzzy ideal in $\mathbb{N}$ but B is not an intuitionistic fuzzy $k-$ ideal in $\mathbb{N}$.

Theorem 3.18. *Let R be a $\Gamma-$ semiring. A fuzzy set μ is a fuzzy right $k-$ ideal of R if and only if $\bar{\mu} = 1 - \mu$ is an anti fuzzy $k-$ ideal of R.*

Proof. Let μ be a fuzzy $k-$ ideal in R. Let $x, y \in R$. Then $\mu(x) \geq \mu(x+y) \wedge \mu(y)$ implies $-\mu(x) \leq -(\mu(x+y) \wedge \mu(y))$. Thus $-\mu(x) \leq -\mu(x+y) \vee (-\mu(y))$. Now, $\bar{\mu}(x) = 1 - \mu(x) \leq (1 - \mu(x+y)) \vee (1 - \mu(y)) = \bar{\mu}(x+y) \vee \bar{\mu}(y)$. Similarly, we can prove $\bar{\mu}(x\alpha y) = 1 - \mu(x\alpha y) \leq \mu(x)$, for all $\alpha \in \Gamma$. So by Theorem 3.9, $\bar{\mu} = 1 - \mu$ is an anti fuzzy $k-$ ideal . By similar argument we can prove the converse part. □

Theorem 3.19. *Let R be a $\Gamma-$ semiring. A fuzzy set μ in R is a fuzzy $k-$ ideal if and only if $U(\mu, t)$ is a $k-$ ideal in R for all $t \in [0, 1]$.*

Proof. Let μ be a fuzzy $k-$ ideal in R and $t \in [0, 1]$. If there exists $x, y \in R$ such that $x + y, y \in U(\mu, t)$ and $x \notin U(\mu, t)$. Then $\mu(x + y) \wedge \mu(y) \geq t > \mu(x)$, a contradiction. Therefore, by Theorem 3.7 $U(\mu, t)$ is a k-ideal in R. Conversely, let $x, y \in R$ be such that $\mu(x) < \mu(x+y) \wedge \mu(y)$, then $x+y, y \in U(\mu, t)$ and $x \notin U(\mu, t)$ where $t = \mu(x + y) \wedge \mu(y)$. This is a contradiction. Therefore by Theorem 3.7 , μ is a fuzzy $k-$ ideal of R. □

Let A be a non empty subset of a $\Gamma-$ semiring R. Then characteristic function of A is a fuzzy subset of R and is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Corollary 3.20. *Let R be a $\Gamma-$ semiring. A ideal I in R is a $k-$ ideal if and only if χ_I is a fuzzy $k-$ ideal of R.*

Proof. The proof follows from Theorem 3.19 □

Theorem 3.21. *Let R be a $\Gamma-$ semiring. A fuzzy set λ in R is an anti fuzzy $k-$ ideal if and only if $L(\lambda, t)$ is a $k-$ ideal in R for all $t \in [0, 1]$.*

Proof. Let λ be an anti fuzzy $k-$ ideal in R and $t \in [0, 1]$. Let $x, y \in R$ be such that $x + y, y \in L(\lambda, t)$ and $x \notin L(\lambda, t)$. Then $\lambda(x + y) \vee \lambda(y) \leq t < \lambda(x)$, which is a contradiction. Therefore by Theorem 3.8, $L(\lambda, t)$ is a $k-$ ideal in R. Conversely, let $x, y \in R$ be such that $\lambda(x) > \lambda(x+y) \vee \lambda(y)$, then $x+y, y \in L(\lambda, t)$ and $x \notin L(\lambda, t)$ where $t = \lambda(x + y) \vee \lambda(y)$, which is a contradiction. Therefore by Theorem 3.8 , λ is a fuzzy $k-$ ideal of R. □

Theorem 3.22. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = \{< x, \mu_A(x), \lambda_A(x) > \mid x \in R\}$ of R is an intuitionistic fuzzy $k-$ ideal of R if and only if $U(\mu_A, t)$ is a $k-$ ideal of R and $L(\lambda_A, t)$ is a $k-$ ideal of R for all $t \in [0, 1]$.*

Proof. The proof follows from Theorem 3.19 and Theorem 3.21. □

Theorem 3.23. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of R is an intuitionistic fuzzy $k-$ ideal of R if and only if $R_A^{(s,t)}$ is a $k-$ ideal of R for all $s, t \in [0, 1]$.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal in R. Clearly $R_A^{(s,t)} = U(\mu_A, s) \cap L(\lambda_A, t)$. Then by Theorem 3.19 and Theorem 3.21, $R_A^{(s,t)}$ is a $k-$ideal in R for all $s, t \in [0, 1]$. Conversely, by Theorem 3.10, $A = (\mu_A, \lambda_A)$ in R is an intuitionistic fuzzy ideal in R. Let $x, y \in R$ and $\mu_A(x + y) \wedge \mu_A(y) = s$, $\lambda_A(x + y) \vee \lambda_A(y) = t$. Then $x + y, y \in R_A^{(s,t)}$ implies that $x \in R_A^{(s,t)}$. Thus $\mu_A(x) \geq \mu_A(x + y) \wedge \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(x+y) \vee \lambda_A(y)$. Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy right k-ideal in R. □

Corollary 3.24. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = \{< x, \mu_A(x), \lambda_A(x) > | \ x \in R\}$ of R is an intuitionistic fuzzy $k-$ ideal of R if and only if μ_A is a fuzzy $k-$ ideal of R and λ_A is an anti fuzzy $k-$ ideal of R.*

Proof. The proof follows from Theorem 3.19, Theorem 3.21 and Theorem 3.22. □

Corollary 3.25. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = \{< x, \mu_A(x), (1 - \mu_A)(x) > | \ x \in R\}$ of R is an intuitionistic fuzzy $k-$ ideal of R if and only if μ_A is a fuzzy $k-$ ideal of R .*

Proof. The proof follows from Theorem 3.18 and Corollary 3.24 . □

Corollary 3.26. *Let R be a $\Gamma-$ semiring. An intuitionistic fuzzy set $A = \{< x, (1 - \lambda_A)(x), \lambda_A(x) > | \ x \in R\}$ of R is an intuitionistic fuzzy $k-$ ideal of R if and only if λ_A is an anti fuzzy $k-$ ideal of R.*

Proof. The proof follows from Theorem 3.18l and Corollary 3.24 . □

Theorem 3.27. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of a $\Gamma-$ semiring R such that μ_A is a fuzzy $k-$ ideal of R then $D = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy $k-$ ideal of R.*

Proof. Let $x, y \in R$. Since μ_A is a fuzzy $k-$ ideal of R so μ_A is a fuzzy ideal of R. This implies that $\mu_A(x + y) \geq (\mu_A(x) \wedge \mu_A(y))$ and $\mu(x\alpha y) \geq \mu(x) \quad (\mu(x\alpha y) \geq \mu(y))$ for all $x, y \in R$ and $\alpha \in \Gamma$. Now, $\bar{\mu_A}(x+y) = 1-\mu_A(x+y) \leq 1-(\mu_A(x) \wedge \mu_A(y)) = (1 - \mu_A(x)) \vee (1 - \mu_A(y)) = \bar{\mu_A}(x) \vee \bar{\mu_A}(y)$. Further, $\bar{\mu_A}(x\alpha y) = 1 - \mu_A(x\alpha y) \leq 1 - \mu_A(y) = \bar{\mu_A}(y)$. In similar way $\bar{\mu_A}(x\alpha y) \leq \bar{\mu_A}(x)$. Thus, $D = (\mu_A, \bar{\mu_A})$ is an

intuitionistic fuzzy ideal of R.
Let $\mu_A(x) \geq (\mu_A(x+y) \wedge \mu_A(y))$. This implies that $\bar{\mu_A}(x) = 1 - \mu_A(x) \leq 1 - (\mu_A(x+y) \wedge \mu_A(y)) = (1 - \mu_A(x+y)) \vee (1 - \mu_A(y)) = \bar{\mu_A}(x+y) \vee \bar{\mu_A}(y)$. Hence $D = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy $k-$ ideal of R. □

Theorem 3.28. *Let R be a $\Gamma-$ semiring. An IFS $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R if and only if the fuzzy sets μ_A and $\bar{\lambda_A}$ are fuzzy $k-$ ideals of R.*

Proof. Let IFS $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R . This implies that μ_A is a fuzzy $k-$ ideal of R. Let $x, y \in R$ and $\alpha \in \Gamma$. Since $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R, so $\lambda_A(x+y) \leq \lambda_A(x) \vee \lambda_A(y)$, $\lambda_A(x\alpha y) \leq \lambda_A(y)(\lambda_A(x\alpha y) \leq \lambda_A(x))$ and $\lambda_A(x) \leq \lambda_A(x+y) \vee \lambda_A(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$. Now, $\bar{\lambda}(x+y) = 1 - \lambda_A(x+y) \geq 1 - (\lambda_A(x) \vee \lambda_A(y)) = (1 - \lambda_A(x)) \wedge (1 - \lambda_A(y)) = \bar{\lambda_A}(x) \wedge \bar{\lambda_A}(y)$. Again, $\bar{\lambda_A}(x\alpha y) = 1 - \lambda_A(x\alpha y) \geq 1 - \lambda_A(y) = \bar{\lambda_A}(y)$. In similar way $\bar{\lambda_A}(x\alpha y) \geq \bar{\lambda_A}(x)$. Further, $\bar{\lambda_A}(x) = 1 - \lambda_A(x) \geq 1 - (\lambda_A(x+y) \vee \lambda_A(y)) = (1 - \lambda_A(x+y)) \wedge (1 - \lambda_A(y)) = \bar{\lambda_A}(x+y) \wedge \bar{\lambda_A}(y)$. Thus, $\bar{\lambda_A}$ is a fuzzy $k-$ ideal of R. Conversely, suppose that μ_A and $\bar{\lambda_A}$ are fuzzy $k-$ ideals of R. Let $x, y \in R$ and $\alpha \in \Gamma$. Since μ_A is a fuzzy $k-$ ideal of R, so $\mu_A(x+y) \geq (\mu_A(x) \wedge \mu_A(y))$, $\mu(x\alpha y) \geq \mu(x) \quad (\mu(x\alpha y) \geq \mu(y))$ and $\mu_A(x) \geq (\mu_A(x+y) \wedge \mu_A(y))$ for all $x, y \in R$ and $\alpha \in \Gamma$. Again, $\bar{\lambda_A}$ is a fuzzy ideal of R, so $\lambda_A(x+y) = 1 - \bar{\lambda_A}(x+y) \leq 1 - (\bar{\lambda_A}(x) \wedge \bar{\lambda_A}(y)) = (1 - \bar{\lambda_A}(x)) \vee (1 - \bar{\lambda_A}(y)) = \lambda_A(x) \vee \lambda_A(y)$. Also, $\lambda_A(x\alpha y) = 1 - \bar{\lambda_A}(x\alpha y) \leq 1 - \bar{\lambda_A}(y) = \lambda_A(y)$. In similar way $\lambda_A(x\alpha y) \leq \lambda_A(x)$. Again, $\bar{\lambda_A}$ is a fuzzy $k-$ ideal of R, therefore $\lambda_A(x) = 1 - \bar{\lambda_A}(x) \leq (\bar{\lambda_A}(x+y) \wedge \bar{\lambda_A}(y)) = 1 - (\bar{\lambda_A}(x+y)) \vee (1 - \bar{\lambda_A}(y)) = \lambda_A(x+y) \vee \lambda_A(y)$. Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy $k-$ ideal of R. □

Corollary 3.29. *Let R be a $\Gamma-$ semring. An IFS $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R if and only if $D = (\mu_A, \bar{\mu_A})$ and $E = (\bar{\lambda_A}, \lambda_A)$ are intuitionistic fuzzy $k-$ ideals of R.*

Theorem 3.30. *Let R be a $\Gamma-$ semring. Then $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R if and only if for any $t \in [0,1], U_A(\mu_A, t) \neq \phi, U_A(\bar{\lambda_A}, t) \neq \phi$, $U_A(\mu_A, t)$ and $U_A(\bar{\lambda_A}, t)$ are $k-$ ideals of R.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R. So by theorem 3.28, μ_A and $\bar{\lambda_A}$ are fuzzy $k-$ ideals of R. Thus μ_A and $\bar{\lambda_A}$ are fuzzy ideals of R. Also, for any $t \in [0,1], U_A(\mu_A, t) \neq \phi, U_A(\bar{\lambda_A}, t) \neq \phi$, $U_A(\mu_A, t)$ and $U_A(\bar{\lambda_A}, t)$ are ideals of R. Further let $y \in R$ and $x, x+y \in U_A(\mu_A, t)$. This implies that $\mu_A(x) \geq t$ and $\mu_A(x+y) \geq t$. Since μ_A is fuzzy $k-$ ideal of R so $\mu_A(y) \geq \mu_A(x+y) \wedge \mu_A(x) \geq t \wedge t = t$. Thus $y \in U_A(\mu_A, t)$. Therefore $U_A(\mu_A, t)$ is a $k-$ ideals of R. Similarly we can prove that $U_A\bar{\lambda_A}, t)$ is a $k-$ ideal of R. Conversely suppose that for any $t \in [0,1], U_A(\mu_A, t) \neq \phi, U_A(\bar{\lambda_A}, t) \neq \phi$, $U_A(\mu_A, t)$ and $U_A(\bar{\lambda_A}, t)$ are $k-$ ideals of R. This implies that μ_A and $\bar{\lambda_A}$ are fuzzy ideals of R. Let $x, y \in R$ and $\mu_A(y) = r_1, \mu_A(x+y) = r_2, r_i \in [0,1], t = r_1 \wedge r_2 = min(r_1, r_2)$. So $\mu_A(y) = r_1 \geq t$. This implies that $y \in U_A(\mu_A, t)$. Similarly $x + y \in U_A(\mu_A, t)$. But $U_A(\mu_A, t)$ is a $k-$ ideal of R, so $x \in U_A(\mu_A, t)$. This implies that $\mu_A(x) \geq t = r_1 \wedge r_2 = \mu_A(x+y) \wedge \mu_A(y)$. Therefore, μ_A is a fuzzy $k-$ ideal of R. Similarly we can prove that $\bar{\lambda_A}$ is a fuzzy $k-$ ideal of R. By theorem 3.28, $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R. □

Theorem 3.31. *Let R be a $\Gamma-$ semring. Then $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ ideal of R if and only if for any $t \in [0,1], U_A(\mu_A, t) \neq \phi, L_A(\bar{\lambda_A}, t) \neq \phi$, $L_A(\mu_A, t)$ and $L_A(\bar{\lambda_A}, t)$ are $k-$ ideals of R.*

Theorem 3.32. *Let R be a $\Gamma-$ semiring with zero element and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy $k-$ideal in R. If $x + y = 0$ then $A(x) = A(y)$ for any $x, y \in R$.*

Proof. Since A is an intuitionistic fuzzy $k-$ideal in R, so $\mu_A(x) \geq \mu_A(x+y) \wedge \mu_A(y) = \mu_A(0) \wedge \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(x+y) \vee \lambda_A(y) = \lambda_A(0) \vee \lambda_A(y)$. Since A is an intuitionistic fuzzy ideal in R , $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in R$. Thus $\mu_A(x) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$. By the similar arguments, $\mu_A(x) \leq \mu_A(y)$ and $\lambda_A(x) \geq \lambda_A(y)$. So $\mu_A(x) = \mu_A(y)$ and $\lambda_A(x) = \lambda_A(y)$. Hence, $A(x) = A(y)$. □

Theorem 3.33. *Let R be a $\Gamma-$ semiring with zero element and A be an intuitionistic fuzzy $k-$ideal in R. Let $R_A = \{x \in R : A(x) = A(0)\}$. Then R_A is a $k-$ideal of R.*

Proof. Let $x, y \in R_A$. Then $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(0)$ and $\lambda_A(x + y) \leq \lambda_A(x) \vee \lambda_A(y) = \lambda_A(0)$. Since A is an intuitionistic fuzzy ideal in R, so $\mu_A(0) \geq \mu_A(x + y)$ and $\lambda_A(0) \leq \lambda_A(x + y)$. Thus $A(x + y) = A(0)$. So $x + y \in R_A$. Now let $r \in R$, $x \in R_A$ and $\alpha \in \Gamma$. Then $\mu_A(r\alpha x) \geq \mu_A(x) = \mu_A(0)$ and $\lambda_A(r\alpha x) \leq \lambda_A(x) = \lambda_A(0)$. Similarly, $\mu_A(0) \geq \mu_A(r\alpha x)$ and $\lambda_A(0) \leq \lambda_A(r\alpha x)$. Thus, $A(r\alpha x) = A(0)$. So $r\alpha x \in R_A$. By the similar argument, it can be shown that $x\alpha r \in R_A$. Hence R_A is an ideal of R. Now suppose $x \in R, a \in R_A$ and $a + x \in R_A$. Then $\mu_A(x) \geq \mu_A(a+x) \wedge \mu_A(a) = \mu_A(0)$ and $\lambda_A(x) \leq \lambda_A(a+x) \vee \lambda_A(a) = \lambda_A(0)$. Similarly, we can see that $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$. Thus $A(x) = A(0)$. So $x \in R_A$. Hence, R_A is a $k-$ideal of R. □

Conclusion: In this paper, we apply the concept of intuitionistic fuzzy set (IFS) to $\Gamma-$ semirings. We introduced the notion of anti fuzzy right ideal, anti fuzzy right $k-$ ideal and intuitionistic fuzzy right $k-$ ideal in $\Gamma-$ semirings. We investigate their properties and various methods of constructions of intuitionistic fuzzy right ideal and intuitionistic fuzzy right $k-$ ideal. The authors of this paper suggests that the future study of intuitionistic fuzzy $k-$ ideals of $\Gamma-$ semirings can be focused on constructing the spectrum of intuitionistic fuzzy $k-$ ideals of $\Gamma-$semirings.

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