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Krull-Schmidt Theorem Fails for Invertible Lattices over a Discrete Valuation Ring (D.V.R.)

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Abstract

Let us suppose that p be a prime number greater than 3, and let N be the semi-direct product of a group H of order p and a cyclic group C of order $p-1$, is the subgroup of H. Let R be the localization of the ring of integers Z at p. We will show that the Krull-Schmidt Theorem does not hold good for the category of invertible $R[N]$-lattices.

Key words: Invertible $R[N]$-lattice, Semi direct product of groups, Krull-Schmidt Theorem, Permutation lattices, cyclotomic polynomial.

1. Introduction:

Let G be a finite group and let $R[G]$ be the group ring of G with coefficients in a Dedekind domain R. An $R[G]$-lattice M is regarded as a finitely generated R-torsion-free $R[G]$-module. M is said to be a permutation lattice if it is R-free and has an R-basis permuted by G. M is said to be an invertible or a permutation projective lattice, if it is a direct summand of a permutation lattice. This was arisen by a question of A. Merkurjev about the existence of a category of invertible lattices over a discrete valuation ring D.V.R., for which the Krull-Schmidt Theorem does not hold good. The question arose in the study of the problem of the uniqueness of a direct sum decomposition of the motive of a projective homogeneous variety into indecomposable objects in the category of Chow motives. This category contains a subcategory equivalent to the category of invertible lattices for a certain finite group. Failure of Krull-Schmidt theorem for this subcategory implies failure of uniqueness of direct sum decompositions [1].

Let us suppose that p be a prime number greater than 3, and let N be the semi-direct product of a group H of order p by a cyclic group C of order $p-1$, where C is conjugate of H. Let R be the localization of the ring of integers, Z, at the prime number p. Thus we have tried to show that the Krull-Schmidt Theorem fails for the category of invertible $R[N]$ –lattices in this research paper.

2. Discussion on Invertible *Z*[*N*] lattices and failure of Krull-Schmidt Theorem

Let us suppose that $N = H \times C$ such that the $Z[N]$-lattice $Z[N]/H$ is isomorphic to $Z[C]$, and is isomorphic to $Z[H]$ as, $Z[H] \cong Z[N]$ and $Z[N]/C \cong Z[H]$.

Let us suppose that I_H be the augmentation ideal of $Z[H]$. We will get the following $Z[N]$ exact sequences:

$$0 \to I_H \to Z[H] \to Z \to 0.$$

We are tensoring by I_H over Z and putting $V = I_H \otimes I_H$, then we get

$$0 \to V \to Z[H] \otimes I_H \to I_H \to 0. \qquad (1)$$

We can check that $Res^{N}_{C}\ I_H \cong Z[C]$, and so the following isomorphisms are given by Frobenius reciprocity:

$$Z[H] \otimes I_H \cong Z[N] \otimes_{Z[C]} Res^{N}_{C}\ I_H \cong Z[N] \otimes_{Z[C]} Z[C] \cong Z[N].$$

Therefore, the sequence (1) becomes

$$0 \to V \to Z[N] \to I_H \to 0. \qquad (2)$$

Now for any $Z[N]$-lattice M, we will let $M^* = Hom\,(M,Z)$. Let us suppose that q be a prime number different from p, and also let C_q be a q-Sylow subgroup of N. Without loss of generality, we may assume that C_q is contained in C. So $H^1(C_q, I_H) \cong H^1(C_q, Z[C]) = 0$. We have also $H^1(H, I_H) \cong Z/pZ$, and hence $H^1(H, I_H^*) \cong Z/pZ$. Let us suppose that there be a α which generates $H^1(H, I_H^*)$. We have a Proposition 12.5 in [2], there exists a $Z[N]$-lattice W and an exact sequence

$$0 \to I_H^* \to W^* \to Z[N]/H \to 0$$

such that the image of α in $H^1(H, W^*)$ is 0, and hence W^* is H^1-trivial since $H^1(N, I_H^*)$ injects into $H^1(H, I_H^*)$. Since N is meta-cyclic this implies that W^* is invertible by [5]. After Dualizing the above sequence we obtain new sequence

$$0 \to Z[N]/H \to W \to I_H \to 0 \qquad (3)$$

with invertible $Z[N]$ – lattice W.

Lemma 1. There is an isomorphism of $Z[N]$-lattices as

$$V \oplus W \cong Z[N] \oplus Z[N]/H.$$

Proof. We form the following pullback diagram with sequences (2) and (3)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & Z[N] & \to & I_H & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & M & \to & W & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Z[N]/H & \to & Z[N]/H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We have $(I_H)^K = 0$ for all subgroups K of N so that V is H^1-trivial. Since N is metacyclic, this implies that V is invertible by [5], we have also obtained from [5] Lemma 9, section 1], that the middle horizontal and vertical sequences will split, to give

$$V \oplus W \cong Z[N] \oplus Z[N]/H.$$

Remark 2. For a $Z[N]$-lattice M, let M_p denote its localization at the prime number p, and let us suppose that M denote its p-adic completion.

Remark 3. We have

$$Z[N]/H \cong \hat{Z}[x]/(x^{p-1} - 1) \cong \bigoplus_{k=1}^{p-1} \hat{Z}[x]/(x - \theta^k) \cong \bigoplus_{k=1}^{p-1} Z_k$$

where $Z_k \cong \hat{Z}[x]/(x - \theta^k)$ and θ is a primitive $(p-1)^{th}$ root of 1 in $\hat{Z}$. So Z_k is a $\hat{Z}[N]$-module of $\hat{Z}$-rank 1 with trivial H-action and such that if c generates C, then $c.1 = \theta^k$. Therefore,

$$\hat{Z}[N] \cong \hat{Z}[H] \otimes \hat{Z}[C] \cong \bigoplus_{k=1}^{p-1} \hat{Z}[H] \otimes Z_k.$$

It is noted that for each k, $\hat{Z}[H] \otimes Z_k$ is indecomposable since $Res_{H}^{N}\, \hat{Z}[\mathrm{H}] \otimes Z_k \cong \hat{Z}[H]$, and $\hat{Z}[H]$ is an indecomposable $\hat{Z}[H]$-module by [6].

Theorem 2. Let us suppose that R denote the localization of Z at the prime number p. Then the Krull-Schmidt Theorem does not hold good for invertible $R[N]$-lattices.

Proof. We have from [1] theorem 2.3

$$\hat{V} \cong \left(\bigoplus_{k=2}^{p-1} \hat{\mathrm{Z}}[\mathrm{H}] \otimes \mathrm{Z}_k \right) \oplus Z_1$$

Therefore by Lemma 1 and from Remark 3, we get

$$\hat{W} \cong \left(\bigoplus_{k=2}^{p-1} \mathrm{Z}_k \right) \otimes \hat{Z}[H] \oplus Z_1$$

Let us suppose that Q be the field of rational numbers and for each k dividing $p-1$, let ω_k be a primitive k^{th} root of unity over Q. Then

$$Q[N]/H \cong \bigoplus_{k/p-1} Q(\omega_k)$$

and the $Q(\omega_k)$ are the irreducible components of $Q[N]/H$. Now $Z[N]/H$ is isomorphic to $Z[C]$ as a $Z[N]$-module, and since $R[C]$ is a maximal R-order in $Q[C]$ we have

$$R[N]/H \cong \bigoplus_{k/p-1} R[\omega_k]$$

by [6]. Therefore, $R[N] \cong \bigoplus_{k/p-1} R[H] \otimes R[\omega_k]$.

Now for each k we have $\hat{Z}[\omega_k] \cong \hat{Z}[x]/\phi_k(x)$

where $\phi_k(x)$ is the k^{th} cyclotomic polynomial. As above we let $\theta \in Q_p$ be a primitive $(p-1)^{th}$ root of unity over Q, where Q_p is the completion of Q at the prime number p. We take $\omega_k = \theta^{(p-1)/k}$ and let us suppose $J_k = \{i \in Z: 1 \le i < k, (i, k)=1\}$. Then $\phi_k(x) = \prod_{j\in J_k}\left(x-\omega_k^j\right)$. Therefore,

$$\hat{Z}[\omega_k] = \bigoplus_{j\in J_k} Z_j.$$

Consequently, we have

$$\hat{V} \cong \Big(\bigoplus_{k/p-1,k\neq p-1} \hat{Z}[H] \otimes \hat{Z}[\omega_k]\Big) \oplus \Big(\bigoplus_{k\in J_{p-1},k\neq 1} \hat{Z}[H] \otimes Z_k\Big) \oplus Z_1$$

and

$$\hat{W} \cong \Big(\bigoplus_{k/p-1,k\neq p-1} \hat{Z}[\omega_k]\Big) \oplus \Big(\bigoplus_{k\in J_{p-1},k\neq 1} Z_k\Big) \oplus \hat{Z}[H] \otimes Z_1$$

To simplify notation we set

$$M = \bigoplus_{k/p-1,k\neq p-1} R[H] \otimes R[\omega_k] \text{ and } M' = \bigoplus_{k/p-1,k\neq p-1} R[\omega_k]$$

Since $\hat{Z}$ is a simple R-module [4] we so have $\hat{V}/\hat{M} \cong V_p^*/M$. Similarly, $\hat{W}/\hat{M}^t \cong W_p^*/M^t$. Therefore, the $R[N]$-lattices $S = V_p/M$ and $S' = W_p/M^t$ have the property that

$$\hat{S} = \Big(\bigoplus_{k\in J_{p-1},k\neq 1} Z_k \otimes \hat{Z}[H]\Big) \oplus Z_1$$

and

$$\hat{S}' = \Big(\bigoplus_{k\in J_{p-1},k\neq 1} Z_k\Big) \oplus \hat{Z}[H]) \otimes Z_1$$

We have $\hat{V} = \hat{M} \oplus \hat{S}$ and $\hat{W} = \hat{M}' \oplus \hat{S}'$, which implies that $V_p = M \oplus S$ and $W_p = M' \oplus S'$, [6]. Since $V_p \oplus W_p = R[N]/H \oplus R[N]$ by Lemma 1, also S and S' are invertible $R[N]$-lattices. Since

$$\hat{Z}[w_p - 1] = \bigoplus_{k\in J_{p-1}} Z_k,$$

we get

$$\hat{S} \oplus \hat{S}' \cong \hat{Z}[H] \otimes \hat{Z}[\omega_{p-1}] \oplus \hat{Z}[\omega_{p-1}],$$

and so by [6],

$$\hat{S} \oplus \hat{S}' \cong R[H] \otimes R[\omega_{p-1}] \oplus R[\omega_{p-1}].$$

Now $R[\omega_{p-1}]$ is indecomposable since $Q(\omega_{p-1})$ is irreducible, but it is not a direct summand of either S or S', but $\hat{Z}[\omega_{p-1}]$ would be a direct summand of $\hat{S}$ or $\hat{S}'$ which is a contradiction.

We have found that the prime number p be greater than 3, is necessary for if $p = 3$, then $S = Z[\omega_2]$ and if $p = 2$, then $S = R = Z[\omega_1]$. Hence, it has been seen that Krull-Schmidt theorem fails for invertible lattices over a D.V.R.

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