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Existence of Three Weak Solutions for the Stationary Kirchhoff-type Problem in a Variable Exponent Sobolev Space

Junichi Aramaki*

Division of Science, Faculty of Science and Engineering,
Tokyo Denki University, Hatoyama-machi, Saitama 350-0394, Japan.
Email ID: E-mail address: aramaki@hctv.ne.jp

Abstract

In this paper, we consider a mixed boundary value problem for the stationary Kirchhoff-type equation containing $p(\cdot)$-Laplacian. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least three weak solutions under some hypotheses on given functions and the values of parameters, applying the Ricceri theorem.
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1. INTRODUCTION

In this paper, we consider the following Kirchhoff-type problem

$$\begin{cases} -M(\Phi(u))\mathrm{div}\,[S_t(x,|\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x)] = \lambda f_0(x,u(x)) + \mu f_1(x,u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ M(\Phi(u))S_t(x,|\boldsymbol{\nabla} u(x)|^2)\frac{\partial u}{\partial \boldsymbol{n}}(x) = \lambda g_0(x,u(x)) + \mu g_1(x,u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

*Corresponding Author.

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $S(x,t)$ is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$ and $S_t = \partial S/\partial t$. Then the operator $u \mapsto \operatorname{div}[S_t(x,|\nabla u(x)|^2)\nabla u(x)]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x) = \operatorname{div}[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$. This generality brings about difficulties and requires some conditions. The function $M = M(s)$ defined in $[0,\infty)$ satisfies the following condition (M).

(M) $M : [0,\infty) \to [0,\infty)$ is a monotone increasing and continuously differentiable (C^1 for short) function, and there exist $0 < m_0 \le m_1 < \infty$ and $\alpha \ge 1$ such that

$$m_0 s^{\alpha-1} \le M(s) \le m_1 s^{\alpha-1} \text{ for all } s \ge 0.$$

Furthermore, the function $\Phi(u)$ is defined by

$$\Phi(u) = \frac{1}{2}\int_\Omega S(x,|\nabla u(x)|^2)dx. \tag{1.3}$$

Thus we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ for $i = 0, 1$ are Carathéodory functions, and λ, μ are real parameters. The first equation in (1.1) is non-local in the sense that the equation is not a pointwise identity according to the term $M(\Phi(u))$.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [35]), in electrorheological fluids (Diening [15], Halsey [20], Mihăilescu and Rădulescu [26], Růžička [28]).

For physical motivation to the problem (1.1), we consider the case where $\Gamma = \Gamma_1$ and $p(x) = 2$. Then the equation

$$M(\|\nabla u\|^2_{\boldsymbol{L}^2(\Omega)})\Delta u(x) = f(x,u(x)) \tag{1.4}$$

is the Kirchhoff equation with the Dirichlet boundary condition, which arises in nonlinear vibration, namely

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2_{\boldsymbol{L}^2(\Omega)})\Delta u = f(x,u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x). \end{cases} \tag{1.5}$$

Equation (1.4) is the stationary counterpart of (1.5). Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho u_{tt} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [24]. This equation extends the classical d'Alembert wave equation by considering the effect of the changes in the length of the strings during the vibrations, where L, h, E, ρ and ρ_0 are constants.

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet or the Steklov boundary condition, (for example, see Arosio and Pannizi [9], Cavalcante and Cavalcante [11], Corrêa and Figueiredo [13], D'Ancona and Spagnolo [14], He and Zou, [21], Yücedaĝ [31, 32], Alessa [1], Ali [2], Hsini et al. [22], Avci [10], Khiddi and Sbai [23]).

However, since we can not find a paper associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). We are convinced of the reason for existence of this paper.

Under some assumptions on f_i, g_i $(i = 0, 1)$ and parameters λ and μ in (1.1), we show the existence of at least three weak solutions using at least three critical points theorem of Ricceri [27] associated with corresponding to the energy functional. Here functions f_1 and g_1 represent perturbation terms.

The paper is organized as follows. Section 2 consists of three subsections. In Subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Subsection 2.2, we introduce a Carathéodory function $S(x, t)$ satisfying the structure conditions and some properties. In Subsection 2.3, we consider the known properties of the associate functionals. Section 3 is devoted to the setting of problem (1.1) rigorously and give a main theorem (Theorem 3.2) and its corollary (Corollary 3.3) on the existence of at least three weak solutions. The proofs of Theorem 3.2 and its Corollary 3.3 are given in Section 4.

2. PRELIMINARIES

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{0,1}$-boundary Γ. Moreover, we assume that Γ satisfies (1.2).

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \ldots, a_d)$ and $\boldsymbol{b} = (b_1, \ldots, b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^d a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

2.1. Variable exponent Lebesgue and Sobolev spaces

In this subsection, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. See Diening et al. [16], Fan and Zhang [17], Kováčik and Rácosník

[25] and references therein for more detail. Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Define $\mathcal{P}(\Omega) = \{p : \Omega \to [1, \infty); p \text{ is a measurable function}\}$. Define

$$p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x) \text{ and } p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

For any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m \ge 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \le m\},$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. Define

$$W_0^{m,p(\cdot)}(\Omega) = \text{ the closure of the set of } W^{m,p(\cdot)}(\Omega)\text{-functions with compact supports in } \Omega.$$

The following three propositions are well known (see Fan et al. [19], Wei and Chen [29], Fan and Zhao [18], Zhao et al. [34], Yücedağ [30]).

Proposition 2.1. *Let $p \in \mathcal{P}(\Omega)$ and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$. Then we have*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let* $p \in \mathcal{P}_+(\Omega)$*, where*

$$\mathcal{P}_+(\Omega) = \{p \in \mathcal{P}(\Omega); 1 < p^- \le p^+ < \infty\}.$$

(i) *For any* $\varepsilon > 0$*, there exists a constant* $C(\varepsilon) > 0$ *such that*

$$\left| \int_\Omega u(x)v(x)dx \right| \le \varepsilon \rho_{p(\cdot)}(u) + C(\varepsilon)\rho_{p'(\cdot)}(v) \text{ for all } u \in L^{p(\cdot)}(\Omega) \text{ and } v \in L^{p'(\cdot)}(\Omega).$$

(ii) *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, we have*

$$\int_\Omega |u(x)v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, $p'(\cdot)$ *is the conjugate exponent of* $p(\cdot)$*, that is,* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For $p \in \mathcal{P}(\Omega)$, define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain with* $C^{0,1}$*-boundary and let* $p \in \mathcal{P}_+(\Omega)$ *and* $m \ge 0$ *be an integer. Then we have the following:*

(i) *The spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{m,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If* $q(\cdot) \in \mathcal{P}_+(\Omega)$ *and satisfies* $q(x) \le p(x)$ *for all* $x \in \Omega$*, then* $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$*, where* $\hookrightarrow$ *means that the embedding is continuous.*

(iii) *If* $q(x) \in \mathcal{P}_+(\Omega)$ *satisfies that* $q(x) < p^*(x)$ *for all* $x \in \Omega$*, then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

We say that $p \in \mathcal{P}(\Omega)$ belongs to $\mathcal{P}^{\log}(\Omega)$ if p has the $\log$-Hölder continuity in Ω, that is, $p : \Omega \to \mathbb{R}$ satisfies that there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \text{ for all } x, y \in \Omega.$$

We also write $\mathcal{P}_+^{\log}(\Omega) = \{p \in \mathcal{P}^{\log}(\Omega); 1 < p^- \le p^+ < \infty\}$.

Proposition 2.4. *If* $p \in \mathcal{P}_+^{\log}(\Omega)$ *and* $m \ge 0$ *is an integer, then* $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ *is dense in* $W_0^{m,p(\cdot)}(\Omega)$.

For the proof, see [16, Corollary 11.2.4].

Next we consider the trace. Let Ω be a domain of $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and $p \in \mathcal{P}_+(\overline{\Omega})$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1_{\text{loc}}(\Gamma)$. We define

$$\text{Tr}(W^{1,p(\cdot)}(\Omega)) = (\text{Tr}W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then $(\text{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. More precisely, see [16, Chapter 12]. In the later we also write $F\big|_\Gamma = g$ by $F = g$ on Γ. Moreover, we denote

$$(\text{Tr}W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)\} \text{ for } i = 1, 2$$

equipped with the norm

$$\|g\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)}; f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f\big|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\text{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in \mathcal{P}_+(\Gamma) := \{q \in \mathcal{P}(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \text{ satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.5. *We have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} \ge 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

Proposition 2.6. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. If $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \leq C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}.$$

In particular, if $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_1)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_1)} \leq C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}$.

For $p \in \mathcal{P}_+(\overline{\Omega})$, define

$$p^{\partial}(x) = \begin{cases} \frac{(d-1)p(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \geq d. \end{cases}$$

Proposition 2.7. *Let $p \in \mathcal{P}_+(\overline{\Omega})$. Then if $q(x) \in \mathcal{P}_+(\overline{\Omega})$ satisfies $q(x) < p^{\partial}(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \textit{ for } u \in W^{1,p(\cdot)}(\Omega).$$

Define a basic space of this paper by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We show the following Poincaré type inequality (cf. Ciarlet and Dinca [12]).

Lemma 2.8. *Let $p \in \mathcal{P}_+^{\log}(\Omega)$. Then there exists a constant $C = C(\Omega, d, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \textit{ for all } u \in X.$$

In particular, $\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [5, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Lemma 2.8.

2.2. A Carathéodory function

Let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ be fixed. Let $S(x,t)$ be a Carathéodory function on $\Omega \times [0,\infty)$ and assume that for a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C([0,\infty))$ satisfies the following structure conditions: there exist positive constants $0 < s_* \le s^* < \infty$ such that for a.e. $x \in \Omega$

$$S(x,0) = 0 \text{ and } s_* t^{(p(x)-2)/2} \le S_t(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \quad (2.3a)$$

$$s_* t^{(p(x)-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \quad (2.3b)$$

$$S_{tt}(x,t) < 0 \text{ when } 1 < p(x) < 2 \text{ and } S_{tt}(x,t) \ge 0 \text{ when } p(x) \ge 2 \text{ for } t > 0, \quad (2.3c)$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.3a), we have

$$\frac{2}{p(x)} s_* t^{p(x)/2} \le S(x,t) \le \frac{2}{p(x)} s^* t^{p(x)/2} \text{ for } t \ge 0. \quad (2.4)$$

We introduce two examples.

Example 2.9. (i) When $S(x,t) = \nu(x)\frac{1}{p(x)} t^{p(x)/2}$, where ν is a measurable function in Ω satisfying $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. in Ω, the function $S(x,t)$ satisfies (2.3a)-(2.3c). If $\nu(x) \equiv 1$, this example corresponds to the $p(x)$-Laplacian.

(ii) As an another example, we can take

$$g(t) = \begin{cases} ae^{-1/t} + a & \text{for } t > 0, \\ a & \text{for } t = 0, \end{cases}$$

where $a > 0$ is a constant. Then we can see that $S(x,t) = \nu(x) g(t) \frac{1}{p(x)} t^{p(x)/2}$ satisfies (2.3a)-(2.3c) if $p(x) \ge 2$ for all $x \in \overline{\Omega}$, (cf. Aramaki [4]).

We have the following estimate of S_t.

Lemma 2.10. *Under the hypotheses* (2.3a)-(2.3c)*, there exists a constant* $c > 0$ *depending only on* s_* *and* p^+ *such that for any* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$,

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^{p(x)} & \textit{when } p(x) \ge 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p(x)-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \textit{when } 1 < p(x) < 2. \end{cases}$$

For the proof, see Aramaki [3, Lemma 3.6].

2.3. The properties of the functional Φ, J and K

Proposition 2.11. *The functional* Φ *on* X *defined by* (1.3) *has the following properties.*
(i) $\Phi \in C^1(X, \mathbb{R})$.
(ii) *The functional* Φ *is sequentially weakly lower semicontinuous, coercive on* X*, that is,*

$$\lim_{\|u\|_X \to \infty} \frac{\Phi(u)}{\|u\|_X} = \infty$$

and bounded on every bounded subset of X.
(iii) $\Phi \in \mathcal{W}_X$*, that is, if* $u_n \to u$ *weakly in* X *and* $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$*, then* $\{u_n\}$ *has a subsequence converging to* u *strongly in* X.
(iv) *The Fréchet derivative* Φ' *of* Φ *is strictly monotone on* X*, bounded on every bounded subset of* X *and coercive in the sense that*

$$\lim_{\|u\|_X \to \infty} \frac{\langle \Phi'(u), u\rangle_{X^*,X}}{\|u\|_X} = \infty.$$

(v) Φ' *is of* (S_+)*-type, that is,* $u_n \to u$ *weakly in* X *and*

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle_{X^*,X} \le 0$$

imply $u_n \to u$ *strongly in* X.
(vi) $\Phi' : X \to X^*$ *is a homeomorphism.*

For the proof, see [8, Proposition 3.2, 3.3] and Aramaki [7, 6, 5].

From now on we suppose the following conditions. For $i = 0, 1$,

(f_i) A Carathéodory function $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|f_i(x,t)| \le C_{1,i} + C_{2,i}|t|^{\alpha_i(x)-1} \text{ for a.e } x \in \Omega \text{ and all } t \in \mathbb{R},$$

where $C_{1,i}$ and $C_{2,i}$ are non-negative constants and $\alpha_i \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies that $\alpha_i(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(g_i) A Carathéodory function $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|g_i(x,t)| \le D_{1,i} + D_{2,i}|t|^{\beta_i(x)-1} \text{ for a.e } x \in \Gamma_2 \text{ and all } t \in \mathbb{R},$$

where $D_{1,i}$ and $D_{2,i}$ are non-negative constants and $\beta_i \in \mathcal{P}_+^{\log}(\overline{\Gamma_2})$ satisfies that $\beta_i(x) < p^{\partial}(x)$ for all $x \in \overline{\Gamma_2}$.

We want to solve the problem (1.1). For this purpose, we consider the functional on X defined by

$$I(u) = \Psi(u) - \lambda J(u) - \mu K(u) \text{ for } u \in X,$$

where

$$\Psi(u) = (\widehat{M}\circ\Phi)(u) = \widehat{M}(\Phi(u)), \tag{2.5}$$

$$J(u) = \int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma, \tag{2.6}$$

$$K(u) = \int_\Omega F_1(x,u(x))dx + \int_{\Gamma_2} G_1(x,u(x))d\sigma. \tag{2.7}$$

Here for every $i = 0, 1$,

$$\widehat{M}(t) = \int_0^t M(s)ds \text{ for } t \geq 0, \tag{2.8}$$

$$F_i(x,t) = \int_0^t f_i(x,s)ds \text{ for } (x,t) \in \Omega\times\mathbb{R}, \tag{2.9}$$

$$G_i(x,t) = \int_0^t g_i(x,s)ds \text{ for } (x,t) \in \Gamma_2\times\mathbb{R}. \tag{2.10}$$

Proposition 2.12. *Assume the f_i and g_i $(i = 0, 1)$ satisfy (f_i) and (g_i), respectively. Then the following holds.*

(i) *$J, K \in C^1(X,\mathbb{R})$.*

(ii) *$J', K' : X \to X^*$ are sequentially weakly-strongly continuous, namely, if $u_n \to u$ weakly in X, then $J'(u_n) \to J'(u)$ and $K'(u_n) \to K'(u)$ strongly in X^*, so J' and K' are compact operators. Moreover, J and K are sequentially weakly continuous.*

For the proof, see [8, Proposition 3.4].

3. THE MAIN THEOREM

We introduce the notion of weak solutions for the problem (1.1).

Definition 3.1. We say $u \in X$ is a weak solution of (1.1), if

$$\begin{aligned} M(\Phi(u)) &\int_\Omega S_t(x,|\nabla u(x)|^2)\nabla u(x)\cdot\nabla v(x)dx \\ &= \lambda\left(\int_\Omega f_0(x,u(x))v(x)dx + \int_{\Gamma_2} g_0(x,u(x))v(x)d\sigma\right) \\ &+ \mu\left(\int_\Omega f_1(x,u(x))v(x)dx + \int_{\Gamma_2} g_1(x,u(x))v(x)d\sigma\right) \text{ for all } v\in X. \end{aligned} \tag{3.1}$$

We are in a position to state the main theorem.

Theorem 3.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying (1.2), and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ verify*

$$\alpha p^+ < \frac{(d-1)p^-}{d-p^-} \text{ if } p^- < d. \tag{3.2}$$

Assume that functions f_0 and g_0 satisfy (f_0) and (g_0). Moreover, suppose that

$$\max\left\{\limsup_{t\to 0}\frac{\operatorname{ess\,sup}_{x\in\Omega}F_0(x,t)}{|t|^{\alpha p^+}},\limsup_{|t|\to\infty}\frac{\operatorname{ess\,sup}_{x\in\Omega}F_0(x,t)}{|t|^{\alpha p^-}}\right\}\le 0, \tag{3.3}$$

$$\max\left\{\limsup_{t\to 0}\frac{\operatorname{ess\,sup}_{x\in\Gamma_2}G_0(x,t)}{|t|^{\alpha p^+}},\limsup_{|t|\to\infty}\frac{\operatorname{ess\,sup}_{x\in\Gamma_2}G_0(x,t)}{|t|^{\alpha p^-}}\right\}\le 0 \tag{3.4}$$

and

$$\sup_{u\in X}J(u)>0. \tag{3.5}$$

Set

$$\theta=\inf\left\{\frac{\Psi(u)}{J(u)};u\in X \text{ with } J(u)>0\right\}. \tag{3.6}$$

Then for each compact interval $[a,b]\subset(\theta,\infty)$, there exists $r>0$ with the following property: for every $\lambda\in[a,b]$ and any functions f_1 and g_1 satisfying (f_1) and (g_1), there exists $\delta>0$ such that for each $\mu\in[0,\delta]$, problem (1.1) *has at least three weak solutions whose norms are less than r.*

Now we state a corollary of Theorem 3.2. Assume that

(f_0') A Carathéodory function $f_0:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|f_0(x,t)|\le C_{1,0}+C_{2,0}|t|^{\alpha_0(x)-1} \text{ for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R},$$

where $C_{1,0}$ and $C_{2,0}$ are non-negative constants, and $\alpha_0\in\mathcal{P}_+^{\log}(\Omega)$ satisfies

$$\alpha_0^+<\alpha p^- \text{ and } \lim_{t\to 0}\frac{|f_0(x,t)|}{|t|^{\alpha p^+-1}}=0 \text{ uniformly for a.e. } x\in\Omega. \tag{3.7}$$

(g_0') A Carathéodory function $g_0:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|g_0(x,t)|\le D_{1,0}+D_{2,0}|t|^{\beta_0(x)-1} \text{ for a.e. } x\in\Gamma_2 \text{ and all } t\in\mathbb{R},$$

where $D_{1,0}$ and $D_{2,0}$ are non-negative constants, and $\beta_0\in\mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies

$$\beta_0^+<\alpha p^- \text{ and } \lim_{t\to 0}\frac{|g_0(x,t)|}{|t|^{\alpha p^+-1}}=0 \text{ uniformly for a.e. } x\in\Gamma_2. \tag{3.8}$$

(h) There exists $\delta_0>0$ such that

$$f_0(x,t)>0 \text{ for } (x,t)\in\Omega\times(0,\delta_0] \text{ and } g_0(x,t)\ge 0 \text{ for } (x,t)\in\Gamma_2\times(0,\delta_0] \tag{3.9}$$

or

$$f_0(x,t)\ge 0 \text{ for } (x,t)\in\Omega\times(0,\delta_0] \text{ and } g_0(x,t)>0 \text{ for } (x,t)\in\Gamma_2\times(0,\delta_0] \tag{3.10}$$

Then we obtain the following corollary of Theorem 3.2.

Corollary 3.3. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ satisfying* (1.2) *and $p\in\mathcal{P}_+^{\log}(\overline{\Omega})$ satisfy* (3.2)*. Assume that (f_0'), (g_0') and (h) holds. Then the conclusion of Theorem 3.2 holds, that is, problem* (1.1) *has at least three weak solutions.*

4. PROOFS OF THEOREM 3.2 AND COROLLARY 3.3

In this section, we give proofs of Theorem 3.2 and Corollary 3.3.

Define an energy functional $I : X \to \mathbb{R}$ by

$$I(v) = \Psi(v) - \lambda J(v) - K(v) \text{ for } v \in X. \tag{4.1}$$

We apply the following result of [27, Theorem 2].

Theorem 4.1. *Let B be a separable, reflexive and real Banach space. Assume that a functional $\Psi : B \to \mathbb{R}$ is coercive, sequentially weakly lower semi-continuous, of C^1-functional belonging to $\mathcal{W}_B$, that is, if $u_n \to u$ weakly in B and $\liminf_{n\to\infty} \Psi(u_n) \le \Psi(u)$, then the sequence $\{u_n\}$ has a subsequence converging to u strongly in B, bounded on every bounded subset of B and the derivative $\Psi' : B \to B^*$ admits a continuous inverse $(\Psi')^{-1} : B^* \to B$. Moreover, assume that $J : B \to \mathbb{R}$ is a C^1-functional with compact derivative, and assume that Ψ has a strictly local minimum $u_0 \in B$ with $\Psi(u_0) = J(u_0) = 0$. Finally, put*

$$\alpha = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\}, \tag{4.2}$$

$$\beta = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)}, \tag{4.3}$$

and assume that $\alpha < \beta$. Then for each compact interval $[a, b] \subset \left(\frac{1}{\beta}, \frac{1}{\alpha}\right)$ (with the conventions $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1-functional $K : B \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation $\Psi'(u) = \lambda J'(u) + \mu K'(u)$ has at least three solutions in B whose norms are less that r.

We give a lemma and some propositions.

Lemma 4.2. *Let $v \in X$. Then we have the following properties.*

(i) *If $\|v\|_X \ge 1$, then*

$$\frac{m_0}{\alpha}\left(\frac{s_*}{p^+}\right)^{\alpha} \|v\|_X^{\alpha p^-} \le \Psi(v) \le \frac{m_1}{\alpha}\left(\frac{s^*}{p^-}\right)^{\alpha} \|v\|_X^{\alpha p^+}.$$

(ii) *If $\|v\|_X < 1$, then*

$$\frac{m_0}{\alpha}\left(\frac{s_*}{p^+}\right)^{\alpha} \|v\|_X^{\alpha p^+} \le \Psi(v) \le \frac{m_1}{\alpha}\left(\frac{s^*}{p^-}\right)^{\alpha} \|v\|_X^{\alpha p^-}.$$

Proof. From (M) and the definition of $\widehat{M}$, we have

$$\frac{m_0}{\alpha}t^\alpha \le \widehat{M}(t) = \int_0^t M(s)ds \le \frac{m_1}{\alpha}t^\alpha \text{ for } t \ge 0.$$

From (2.4),

$$\int_\Omega \frac{s_*}{p(x)}|\boldsymbol{\nabla} v(x)|^{p(x)}dx \le \Phi(v) \le \int_\Omega \frac{s^*}{p(x)}|\boldsymbol{\nabla} v(x)|^{p(x)}dx.$$

Thus we have

$$\left(\frac{s_*}{p^+}\rho_{p(\cdot)}(\boldsymbol{\nabla} v)\right)^\alpha \le \Phi(v)^\alpha \le \left(\frac{s^*}{p^-}\rho_{p(\cdot)}(\boldsymbol{\nabla} v)\right)^\alpha.$$

Since $\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$, the conclusions of (i) and (ii) follow from Proposition 2.1. □

We give the properties of the functional Ψ.

Proposition 4.3. *The functional Ψ has the following properties.*

(i) $\Psi \in C^1(X, \mathbb{R})$ *and* $\Psi'(u) = M(\Phi(u))\Phi'(u)$ *for* $u \in X$.

(ii) Ψ *is sequentially weakly lower semicontinuous on* X.

(iii) Ψ *is coercive on* X, *that is,*

$$\lim_{\|u\|_X \to \infty} \frac{\Psi(u)}{\|u\|_X} = \infty.$$

(iv) Ψ *is bounded on every bounded subset of* X.

(v) $\Psi \in \mathcal{W}_X$, *that is, if* $u_n \to u$ *weakly in* X *and* $\liminf_{n\to\infty} \Psi(u_n) \le \Psi(u)$, *then the sequence* $\{u_n\}$ *has a subsequence converging to* u *strongly in* X.

Proof. (i) Since $\widehat{M}$ is a C^1-function on $[0, \infty)$ and $\Phi \in C^1(X, \mathbb{R})$ by Proposition 2.11, clearly $\Psi \in C^1(X, \mathbb{R})$ and $\Psi'(u) = M(\Phi(u))\Phi'(u)$ for $u \in X$.

(ii) Let $u_n \to u$ weakly in X. Since Φ is sequentially weakly lower semicontinuous on X from Proposition 2.11, we have $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$. Since $\widehat{M}$ is an increasing and continuous function, we have

$$(\widehat{M} \circ \Phi)(u) \le \widehat{M}(\liminf_{n\to\infty} \Phi(u_n)) = \widehat{M}(\lim_{N\to\infty} \inf_{n\ge N}\{\Phi(u_n)\}) = \lim_{N\to\infty} \widehat{M}(\inf_{n\ge N}\{\Phi(u_n)\}).$$

Since $\inf_{n\ge N}\{\Phi(u_n)\} \le \Phi(u_m)$ for all $m \ge N$, we have $\widehat{M}(\inf_{n\ge N}\{\Phi(u_n)\}) \le \widehat{M}(\Phi(u_m))$ for all $m \ge N$. Therefore, we have $\widehat{M}(\inf_{n\ge N}\{\Phi(u_n)\}) \le \inf_{n\ge N} \widehat{M}(\Phi(u_n))$. Thus we see that $(\widehat{M} \circ \Phi)(u) \le \liminf_{n\to\infty} \widehat{M}(\Phi(u_n))$.

(iii) Let $\|u\|_X > 1$. By Lemma 4.2, $\Psi(u) = (\widehat{M} \circ \Phi)(u) \geq \frac{m_0}{\alpha}\left(\frac{s_*}{p^+}\right)^\alpha \|u\|_X^{\alpha p^-}$. By (M), $\alpha p^- > 1$, so

$$\lim_{\|u\|_X \to \infty} \frac{\Psi(u)}{\|u\|_X} = \infty.$$

(iv) Let $\|u\|_X \leq C$. Since Φ is bounded on every bounded subset of X by Proposition 2.11, $0 \leq \Phi(u) \leq C_1$ for some constant $C_1 > 0$. Since $\widehat{M}$ is continuous on $[0, \infty)$, $\widehat{M} \circ \Phi(u)$ is bounded.

(v) Let $u_n \to u$ weakly in X and $\liminf_{n\to\infty}(\widehat{M} \circ \Phi)(u_n) \leq (\widehat{M} \circ \Phi)(u)$. Then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that

$$\lim_{n'\to\infty}(\widehat{M} \circ \Phi)(u_{n'}) = \liminf_{n\to\infty}(\widehat{M} \circ \Phi)(u_n) = \widehat{M}(\lim_{n'\to\infty} \Phi(u_{n'})) \leq (\widehat{M} \circ \Phi)(u).$$

Since $\widehat{M}$ is a strictly monotone increasing function on $[0, \infty)$, we have $\lim_{n'\to\infty} \Phi(u_{n'}) \leq \Phi(u)$. Since $\Phi \in \mathcal{W}_X$ from [8, Proposition 3.2 (iii)], the sequence $\{u_{n'}\}$ has a subsequence converging to u strongly in X. □

Next, we study the properties of Ψ'.

Proposition 4.4. *The Fréchet differential $\Psi' : X \to X^*$ of Ψ has the following properties.*

(i) *The operator Ψ' is strictly monotone.*

(ii) *The operator Ψ' is bounded on every bounded subset of X.*

(iii) *The operator Ψ' is coercive, that is,*

$$\lim_{\|u\|_X \to \infty} \frac{\langle \Psi'(u), u\rangle_{X^*,X}}{\|u\|_X} = \infty.$$

(iv) *The operator Ψ' is of (S_+)-type, that is, if $u_n \to u$ weakly in X and*

$$\limsup_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle_{X^*,X} \leq 0,$$

then $u_n \to u$ strongly in X.

Proof. (i) Let $u_1, u_2 \in X$ and $u_1 \neq u_2$. Then we have

$$\begin{aligned}
&\langle \Psi'(u_1) - \Psi'(u_2), u_1 - u_2\rangle_{X^*,X} \\
&= \left\langle \int_0^1 \frac{d}{d\tau}\Psi'(\tau u_1 + (1-\tau)u_2)d\tau, u_1 - u_2 \right\rangle_{X^*,X} \\
&= \int_\Omega \int_0^1 \frac{d}{d\tau}\big[M(\Phi(\tau u_1 + (1-\tau)u_2))S_t(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\
&\quad \times(\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)) \cdot (\nabla u_1(x) - \nabla u_2(x))\big]d\tau dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)\int_\Omega S_t(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\ &\quad |\nabla u_1(x) - u_2(x)|^2 dx d\tau \\ I_2 &= \int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)\int_\Omega 2S_{tt}(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\ &\quad \times\big((\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x))\cdot(\nabla u_1(x) - \nabla u_2(x))\big)^2 dx d\tau \\ I_3 &= \int_0^1 M'(\Phi(\tau u_1 + (1-\tau)u_2))\langle\Phi'(\tau u_1 + (1-\tau)u_2), u_1 - u_2\rangle^2_{X^*,X}. \end{aligned}$$

Since $M' \geq 0$ from the hypothesis (M), we see that $I_3 \geq 0$. If we put $\Omega_1 = \{x \in \Omega; p(x) \geq 2\}$ and $\Omega_2 = \{x \in \Omega; p(x) < 2\}$, then we can write $I_1 + I_2 = L_1 + L_2 + L_3$, where

$$\begin{aligned} L_1 &= \int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)\int_{\Omega_1} S_t(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\ &\quad \times|\nabla u_1(x) - \nabla u_2(x)|^2 dx d\tau \\ L_2 &= \int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)\int_{\Omega_1} 2S_{tt}(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2 \\ &\quad \times\big((\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x))\cdot(\nabla u_1(x) - \nabla u_2(x))\big)^2 dx d\tau \\ L_3 &= \int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)\int_{\Omega_2}\Big\{S_t(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\ &\quad +2S_{tt}(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2) \\ &\qquad \times\big((\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x))\cdot(\nabla u_1(x) - \nabla u_2(x))\big)^2\Big\} dx d\tau. \end{aligned}$$

By the hypothesis (2.3c), we have $S_{tt}(x, |\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^2)dx \geq 0$ in Ω_1, so $L_2 \geq 0$. Since $S_{tt}(x,t) < 0$ for $(x,t) \in \Omega_2 \times [0,\infty)$,

$$\begin{aligned} &S_{tt}(x, |\tau\nabla u_1(x)+(1-\tau)\nabla u_2(x)|^2)\big((\tau\nabla u_1(x)+(1-\tau)\nabla u_2(x))\cdot(\nabla u_1(x)-\nabla u_2(x))\big)^2 \\ &\geq S_{tt}(x, |\tau\nabla u_1(x)+(1-\tau)\nabla u_2(x)|^2)|\tau\nabla u_1(x)+(1-\tau)\nabla u_2(x)|^2|\nabla u_1(x)-\nabla u_2(x)|^2. \end{aligned}$$

By (2.3b), we have

$$\begin{aligned} I_1 + I_2 \geq &\int_0^1 M(\Phi(\tau u_1 + (1-\tau)u_2)) \\ &\times\int_\Omega s_*|\tau\nabla u_1(x) + (1-\tau)\nabla u_2(x)|^{p(x)-2}|\nabla u_1(x) - \nabla u_2(x)|^2 dx d\tau. \end{aligned}$$

Since if $u_1 \neq u_2$ in X, then $\nabla u_1 \neq \nabla u_2$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, so $I_1 + I_2 > 0$. Hence Ψ' is strictly monotone.

(ii) Let $\|u\|_X \le C$. Since Φ' is bounded on every bounded subset of X from Proposition 2.9 (iv), we have $\|\Phi'(u)\|_{X^*} \le C_1$ for some constant $C_1 > 0$. By Proposition 2.11 (ii), Φ is bounded on every bounded subset of X, so $0 \le \Phi(u) \le C_2$ for some constant $C_2 > 0$. Since M is continuous on the compact interval $[0, C_2]$, $\Psi'(u) = M(\Phi(u))\Phi'(u)$ is bounded.

(iii) Let $\|u\|_X > 1$. We remember

$$\Phi(u) \ge \frac{s_*}{p^+}\int_\Omega |\boldsymbol{\nabla} u(x)|^{p(x)}dx \ge \frac{s_*}{p^+} > 0.$$

Since M is monotone increasing and $M(s) > 0$ for $s > 0$, there exists a constant $c > 0$ such that $M(\Phi(u)) \ge c$ for all $u \in X$ with $\|u\|_X > 1$. Since Φ is coercive from Proposition 2.11 (ii), we have

$$\frac{\langle \Psi'(u), u\rangle_{X^*,X}}{\|u\|_X} = \frac{M(\Phi(u))\langle \Phi'(u), u\rangle_{X^*,X}}{\|u\|_X} \ge c\frac{\langle \Phi'(u), u\rangle_{X^*,X}}{\|u\|_X} \to \infty$$

as $\|u\|_X \to \infty$.

(iv) Let $u_n \to u$ weakly in X and $\limsup_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle_{X^*,X} \le 0$.

If $\inf_n\{M(\Phi(u_n))\} = 0$, then necessarily we have $\alpha > 1$. So there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} M(\Phi(u_{n'})) = 0$. Since M satisfies the condition (M) and $\alpha > 1$, we see that $\lim_{n'\to\infty}\Phi(u_{n'}) = 0$. Since $\frac{s_*}{p^+}\rho_{p(\cdot)}(\boldsymbol{\nabla} u_{n'}) \le \Phi(u_{n'}) \to 0$, it follows from Proposition 2.1 (iv) that $u_{n'} \to 0$ strongly in X, so $u = 0$. By the convergent principle (cf. Zeidler [33, Proposition 10.13 (i)], $u_n \to 0$ strongly in X.

If $\inf_n\{M(\Phi(u_n)) = c > 0$, since $u_n \to u$ weakly in X, $\{u_n\}$ is bounded in X, so $M(\Phi(u_n)) \le C$ for some constant $C > 0$. Thereby, we have

$$M(\Phi(u_n))\langle \Phi'(u_n), u_n - u\rangle_{X^*.X} \ge \begin{cases} c\langle \Phi'(u_n), u_n - u\rangle_{X^*,X} & \text{if } \langle \Phi'(u_n), u_n - u\rangle_{X^*,X} \ge 0, \\ C\langle \Phi'(u_n), u_n - u\rangle_{X^*,X} & \text{if } \langle \Phi'(u_n), u_n - u\rangle_{X^*,X} < 0. \end{cases}$$

Therefore, we have $\lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle_{X^+,X} \le 0$. Since Φ' is of (S_+)-type from Proposition 2.11 (v), we have $u_n \to u$ strongly in X. □

Proposition 4.5. *The operator* $\Psi' : X \to X^*$ *has a continuous inverse* $(\Psi')^{-1} : X^* \to X$.

Proof. Step 1. Ψ' is surjective. In fact, for any $f \in X^*$, define a functional I_0 on X by

$$I_0(v) = \Psi(v) - \langle f, v\rangle_{X^*,X} \text{ for } v \in X.$$

We note that I_0 is sequentially weakly lower semicontinuous on X. Since

$$I_0(v) = \Psi(v) - \langle f, v\rangle_{X^*,X} \ge \frac{m_0}{\alpha}\left(\frac{s_*}{p^+}\right)^\alpha \|v\|_X^{\alpha p^-} - \|f\|_{X^*}\|v\|_X$$

for any $v \in X$ with $\|v\|_X > 1$ and $\alpha p^- > 1$, I_0 is weakly coercive, that is, $I_0(v) \to \infty$ as $\|v\|_X \to \infty$. By [33, Theorem 25.D], I_0 has a minimum $u \in X$, so $I_0'(u) = 0$. Hence $\Psi'(u) = f$.

Step 2. Ψ' is injective. This follows from the strictly monotonicity of Ψ'.

Step 3. By Step 1 and Step 2, $(\Psi')^{-1}$ exists. We show that $(\Psi')^{-1} : X^* \to X$ is continuous. Let $f_n, f \in X^*$ and $f_n \to f$ in X^*. Then there exists $u_n, u \in X$ such that $\Psi'(u_n) = f_n, \Psi'(u) = f$. It suffices to show that $u_n \to u$ in X.

The sequence $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is unbounded, then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\|_X \to \infty$ as $n' \to \infty$. We see that

$$\langle \Psi'(u_{n'}), u_{n'} \rangle_{X^*,X} = \langle f_{n'}, u_{n'} \rangle_{X^*,X} \leq \|f_n\|_{X^*} \|u_{n'}\|_X \leq C_1 \|u_{n'}\|_X$$

because since $f_n \to f$ in X^*, we have $\{f_n\}$ is bounded, so $\|f_n\|_{X^*} \leq C_1$ for some constant $C_1 > 0$. This contradicts the coerciveness of Ψ'.

Since $\{u_n\}$ is bounded in a reflexive Banach space X, there exists a subsequence $\{u_{n''}\}$ of $\{u_n\}$ and $u_0 \in X$ such that $u_{n''} \to u_0$ weakly in X as $n'' \to \infty$. Now we have

$$\begin{aligned} \lim_{n'' \to \infty} \langle \Psi'(u_{n''}), u_{n''} - u_0 \rangle_{X^*,X} &= \lim_{n'' \to \infty} \langle \Psi'(u_{n''}) - \Psi'(u), u_{n''} - u_0 \rangle_{X^*,X} \\ &= \lim_{n'' \to \infty} \langle f_{n''} - f, u_{n''} - u_0 \rangle_{X^*,X} = 0. \end{aligned}$$

Since Ψ' is of (S_+)-type from Proposition 4.4 (iv), we see that $u_{n''} \to u_0$ strongly in X. Since Ψ' is continuous, $\Psi'(u_{n''}) = f_{n''} \to \Psi'(u_0) = f = \Psi'(u)$. Since Ψ' is injective, we have $u_0 = u$. Using again the convergent principle (cf. [33, Proposition 10.13 (i)]), the full sequence $u_n \to u$ strongly in X.

□

Proof of Theorem 3.2.

We note that if $u \in X$ is a critical point of the functional I, that is, $I'(u) = \Psi'(u) - \lambda J'(u) - \mu K'(u) = 0$, then u is a weak solution of (1.1). Under the hypotheses of Theorem 3.2, we derive the hypotheses of Theorem 4.1 with $B = X$ defined by (2.1) and the functionals Ψ, J and K defined by (2.5), (2.6) and (2.7), respectively. Since $\Psi(u) \geq 0$ for all $u \in X$, and $\Psi(u) = 0$ if and only if $u = 0$, we see that Ψ has a strictly local minimum $u = 0$, and by the definitions of F_0 and G_0, clearly $J(0) = 0$, so $\Psi(0) = J(0) = 0$. Moreover, the hypotheses on Ψ and J follows from Propositions 4.3, 4.4 and 4.5.

Fix $\varepsilon > 0$. From (3.3) and (3.4), there exist ρ_1 and ρ_2 with $0 < \rho_1 < 1 < \rho_2$ such that

$$F_0(x,t) \leq \varepsilon |t|^{\alpha p^+} \quad \text{for all } (x,t) \in \Omega \times [-\rho_1, \rho_1], \tag{4.4}$$

$$F_0(x,t) \leq \varepsilon |t|^{\alpha p^-} \quad \text{for all } (x,t) \in \Omega \times \big(\mathbb{R} \setminus [-\rho_2, \rho_2]\big) \tag{4.5}$$

and

$$G_0(x,t) \le \varepsilon |t|^{\alpha p^+} \qquad \text{for all } (x,t) \in \Gamma_2 \times [-\rho_1, \rho_1], \tag{4.6}$$
$$G_0(x,t) \le \varepsilon |t|^{\alpha p^-} \quad \text{for all } (x,t) \in \Gamma_2 \times \big(\mathbb{R} \setminus [-\rho_2, \rho_2]\big). \tag{4.7}$$

Thus we have

$$F_0(x,t) \le \varepsilon |t|^{\alpha p^+} \text{ for all } (x,t) \in \Omega \times \big(\mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])\big)$$

and

$$G_0(x,t) \le \varepsilon |t|^{\alpha p^+} \text{ for all } (x,t) \in \Gamma_2 \times \big(\mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])\big).$$

On the other hand, since f_0 and g_0 satisfy (f_0) and (g_0), respectively, we have

$$|F_0(x,t)| \le C_{1,0}|t| + \frac{C_{2,0}}{\alpha_0(x)}|t|^{\alpha_0(x)} \le C_{1,0}|t| + \frac{C_{2,0}}{\alpha_0^-}|t|^{\alpha_0(x)} \text{ for } (x,t) \in \Omega \times \mathbb{R}$$

and

$$|G_0(x,t)| \le D_{1,0}|t| + \frac{D_{2,0}}{\beta_0(x)}|t|^{\beta_0(x)} \le D_{1,0}|t| + \frac{D_{2,0}}{\beta_0^-}|t|^{\beta_0(x)} \text{ for } (x,t) \in \Gamma_2 \times \mathbb{R}.$$

Hence F_0 is bounded on each bounded subset of $\Omega \times \mathbb{R}$ and G_0 is bounded on each bounded subset of $\Gamma_2 \times \mathbb{R}$.

From the hypothesis (3.2),

$$\alpha p^+ < \frac{dp^-}{d - p^-} \le \frac{dp(x)}{d - p(x)} = p^*(x) \text{ if } p^- < d$$

and

$$\alpha p^+ < \frac{(d-1)p^-}{d - p^-} \le \frac{(d-1)p(x)}{d - p(x)} = p^\partial(x) \text{ if } p^- < d.$$

If we choose $q \in \mathbb{R}$ such that $\alpha p^+ < q < p^\partial(x)$ for all $x \in \Gamma_2$ and $\alpha p^+ < q < p^*(x)$ for all $x \in \Omega$, then we have

$$F_0(x,t) \le \varepsilon |t|^{\alpha p^+} + c|t|^q \text{ for all } (x,t) \in \Omega \times \mathbb{R} \tag{4.8}$$

and

$$G_0(x,t) \le \varepsilon |t|^{\alpha p^+} + c|t|^q \text{ for all } (x,t) \in \Gamma_2 \times \mathbb{R} \tag{4.9}$$

for some constant $c > 0$. Since the embedding mappings $X \hookrightarrow L^{\alpha p^+}(\Omega)$, $L^{\alpha p^+}(\Gamma_2)$, $L^q(\Omega)$, $L^q(\Gamma_2)$ are continuous, there exist positive constants C_{p^+} and C_q such that

$$\|u\|_{L^{\alpha p^+}(\Omega)} \le C_{p^+}\|u\|_X, \|u\|_{L^{\alpha p^+}(\Gamma_2)} \le C_{p^+}\|u\|_X,$$
$$\|u\|_{L^q(\Omega)} \le C_q\|u\|_X \text{ and } \|u\|_{L^q(\Gamma_2)} \le C_q\|u\|_X$$

for all $u \in X$. Thus, from (4.8) and (4.9), there exists a constant $c_1 > 0$ such that

$$\begin{aligned} J(u) &= \int_\Omega F_0(x, u(x))dx + \int_{\Gamma_2} G_0(x, u(x))d\sigma \\ &\leq \varepsilon \int_\Omega |u(x)|^{\alpha p^+} dx + c_1 \int_\Omega |u(x)|^q dx + \varepsilon \int_{\Gamma_2} |u(x)|^{\alpha p^+} d\sigma + c_1 \int_{\Gamma_2} |u(x)|^q d\sigma \\ &\leq 2C_{p^+}^{\alpha p^+} \varepsilon \|u\|_X^{\alpha p^+} + 2c_1 C_q^q \|u\|_X^q. \end{aligned}$$

When $\|u\|_X < 1$, it follows Proposition 2.1 that

$$\frac{J(u)}{\Psi(u)} \leq \frac{2C_{p^+}^{\alpha p^+} \varepsilon \|u\|_X^{\alpha p^+} + 2cC_q^q \|u\|_X^q}{\frac{m_0}{\alpha}\left(\frac{s_*}{p^+}\right)^\alpha \|u\|_X^{\alpha p^+}}.$$

Since $q > \alpha p^+$, we have

$$\limsup_{u \to 0} \frac{J(u)}{\Psi(u)} \leq 2\frac{\alpha}{m_0}\left(\frac{p^+}{s_*}\right)^\alpha C_{p^+}^{\alpha p^+} \varepsilon. \tag{4.10}$$

On the other hand, since the embedding mappings $X \hookrightarrow L^{\alpha p^-}(\Omega), L^{\alpha p^-}(\Gamma_2)$ are continuous, there exists a constant $C_{p^-} > 0$ such that

$$\|u\|_{L^{\alpha p^-}(\Omega)} \leq C_{p^-}\|u\|_X \text{ and } \|u\|_{L^{\alpha p^-}(\Gamma_2)} \leq C_{p^-}\|u\|_X \text{ for all } u \in X.$$

Since F_0 and G_0 are bounded on each bounded subset of $\Omega \times \mathbb{R}$ and $\Gamma_2 \times \mathbb{R}$, respectively, when $\|u\|_X > 1$, it follows from (4.5) and (4.7) that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} J(u) &= \int_{\{x\in\Omega; |u(x)|\leq \rho_2\}} F_0(x, u(x))dx + \int_{\{x\in\Omega; |u(x)|> \rho_2\}} F_0(x, u(x))dx \\ &\quad + \int_{\{x\in\Gamma_2; |u(x)|\leq \rho_2\}} G_0(x, u(x))d\sigma + \int_{\{x\in\Gamma_2; |u(x)|> \rho_2\}} G_0(x, u(x))d\sigma \\ &\leq 2C_1 + 2\varepsilon C_{p^-}^{\alpha p^-} \|u\|_X^{\alpha p^-}. \end{aligned}$$

Hence

$$\limsup_{\|u\|_X \to \infty} \frac{J(u)}{\Psi(u)} \leq 2\frac{m_0}{\alpha}\left(\frac{p^+}{s_*}\right)^\alpha C_{p^-}^{\alpha p^-} \varepsilon. \tag{4.11}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (4.10) and (4.11) that

$$\max\left\{\limsup_{u \to 0} \frac{J(u)}{\Psi(u)}, \limsup_{\|u\| \to \infty} \frac{J(u)}{\Psi(u)}\right\} \leq 0.$$

Therefore, we have $\alpha = 0$ in Theorem 4.1. By the hypothesis (3.5), we have $\beta > 0$ in (4.3). Thus all the hypotheses of Theorem 4.1 hold. If we put $\theta = 1/\beta$, then the conclusion of Theorem 3.2 is verified. This completes the proof of Theorem 3.2.

Proof of Corollary 3.3

From (3.7), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t| < \delta$, then $|f_0(x,t)| \le \varepsilon |t|^{\alpha p^+ - 1}$. Hence, for $|t| < \delta$, $|F_0(x,t)| \le \frac{\varepsilon}{\alpha p^+}|t|^{\alpha p^+}$, so we have

$$\limsup_{t \to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{\alpha p^+}} \le \frac{\varepsilon}{\alpha p^+}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{t \to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{\alpha p^+}} \le 0.$$

On the other hand, since f_0 is bounded on each bounded subset of $\Omega \times \mathbb{R}$ from (f_0'), there exists a constant $C > 0$ such that $|f_0(x,t)| \le C$ for $(x,t) \in \Omega \times [0,1]$. When $|t| > 1$,

$$|f_0(x,t)| \le C_{1,0} + C_{2,0}|t|^{\alpha_0(x)-1} \le C_{1,0} + C_{2,0}|t|^{\alpha_0^+ - 1},$$

so we have $|f_0(x,t)| \le C'_{1,0} + C_{2,0}|t|^{\alpha_0^+ - 1}$ for all $(x,t) \in \Omega \times \mathbb{R}$. Thus $|F_0(x,t)| \le C'_{1,0}|t| + C'_{2,0}|t|^{\alpha_0^+}$ for some constants $C'_{1,0}$ and $C'_{2,0}$. Therefore, since $\alpha_0^+ < \alpha p^-$,

$$\limsup_{|t| \to \infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{\alpha p^-}} \le 0,$$

so (3.3) holds.

Similarly, using (g_0'), we can derive

$$\limsup_{t \to 0} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{\alpha p^+}} \le 0 \text{ and } \limsup_{|t| \to \infty} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{\alpha p^-}} \le 0,$$

so (3.4) holds.

Under (h), since we can easily choose $0 \not\equiv \varphi \in X$ with $0 \le \varphi(x) \le \delta_0$ such that

$$\int_\Omega F_0(x,\varphi(x))dx + \int_{\Gamma_2} G_0(x,\varphi(x))d\sigma > 0,$$

(3.5) holds. Thus, since all the hypotheses of Theorem 3.2 hold, the conclusion of Corollary 3.3 follows from Theorem 3.2.

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