Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 19, Number 1 (2023), pp. 135-150

http://www.ripublication.com/gjpam.htm

M/M/1 Queueing Systems with a Secondary Duty

Song Chew

Department of Mathematics and Statistics,
Southern Illinois University Edwardsville, Edwardsville, IL 62026, USA
Correspondence: schew@siue.edu; Tel.: +1-618-650-3016

Abstract

The study of queueing systems has always been an active area of research. Applications of queueing systems may often be found in traffic control, manufacturing and production systems, healthcare, etc. This paper investigates a novel notion of M/M/1 queueing systems. In our systems, we consider serving a customer as a primary duty. During an idle period, the system may undergo a routine repair service, or the server may perform a side task. We consider carrying out activities during idle periods as a secondary duty. The objective is to conduct steady-state analysis of our systems. To this end, we model our systems as a birth-death process, and recursively solve sets of flow balance equations to establish the steady-state distribution for the number of customers in the system. We then determine the average numbers of customers, and the average amounts of time spent by a customer, in system and in queue. To support the validity of our results, we undertake a simulation study of our systems. Simulation results strongly validate our theoretical results.

Keywords: queueing systems, simulation, operations research, optimization.

1. INTRODUCTION AND MOTIVATION

It is common that we typically will have to wait at a bank, a fast-food restaurant, or any other places where waiting is required for service. For the most part, we do not know how long we must wait in the queue. To answer this question, mathematical models for queueing systems have been developed so that this and other questions may be answered. The study of queueing systems has been rigorously carried out since the early twentieth century. Erlang is among the pioneer researchers in queueing theory in the early twentieth century. Erlang [1] studies the operations of telephone systems. The commonly used notation for queueing systems ($a/b/c$, where a describes the arrival process, *b* indicates the service process, and *c* specifies the number of servers, in queueing systems) is first proposed by Kendall [2]. Morse [3] writes the first textbook on the subject of queueing systems. Applications of queueing systems may be widely found in traffic control, manufacturing and production systems, healthcare, etc.

Queueing models are often applied to study traffic flow. See, for example, [4], Kerbache and Macgregor use an expansion method on queueing networks to investigate into traffic control. Heidemann utilizes queueing theory to approximate the queue length and waiting time at intersections [5–7]. [8] plans and schedules transportation vehicle fleet in a congested traffic environment.

In manufacturing and production systems, researchers have been actively applied queueing theory. For example, see [9–11] for the ways queueing techniques are employed to analyze manufacturing systems. Most recently, a host of researchers actively investigate into a variant, $M^{[X]}/G(a, b)/c$ systems, of the standard M/G/1 system. Batch arrivals and machine repairs are often imposed on these systems; hence, results obtained from their studies are most applicable to production systems; see, for example, [12–17].

Effective healthcare delivery and access is yet another field where queueing analysis is commonly used. For example, Worthington employs queueing models to study waiting times in hospitals [18], and Christie and Levary use simulation in planning the transportation of patients to hospitals following disasters [19]. In [20], Harper and Shahani model the utilization of beds in hospitals. On the other hand, a queueing approach is employed to manage medical appointments and to identify good nursing levels [21, 22]. The reference [23] may provide a further review on use of queueing models in healthcare operations.

Researchers have most recently actively studied a number of variants of M/M/1 queueing systems. See, for example, [24], where continuous service is imposed on the server, and [25–27] impose various restrictions on the systems involving encouraged arrivals. On the other hand, [28, 29] study a variant of M/M/1 queueing systems with various forms of vacation imposed on the server.

The standard M/M/1 queueing system is a system where the interarrival and service times are independently and exponentially distributed with a single server. In this system, the server serves customers one by one when there exist customers in the system. The server becomes idle when there are not any customers present in the system. When a customer arrives into the system during an idle period, the customer enters into service immediately since there is not any customer waiting in queue. However, during a busy period, a new customer arriving into the system does not preempt the current active service and hence must wait in the queue for their service. We in this paper present a novel notion of the standard M/M/1 queueing system.

We call the systems under study the M/M/1 queueing system with a secondary duty. The considered system deems serving customers as a primary duty for the server. However, during an idle period, the server performs a side task. *Carrying out a side activity during an idle period is considered to be a secondary duty for the server.* Specifically, during a busy period, the server provides service to customers one by one (primary duty) as does the server in the standard M/M/1 queueing system. Upon service completion of all customers, the server immediately starts to perform a side task (secondary duty) as an idle period begins. If the time it takes the server to complete the side task is shorter than the idle period (the side task is completed before the next customer arrives), the server becomes completely idle and must now wait to serve the next customer arriving into the system. On the other hand, if the time it takes the server to complete the side task is longer than the idle period (the next customer arrives into the system before the side task is completed), a newly arriving customer does not preempt the ongoing service of the side task and must now wait in the queue for its service. Upon completion of the side task, the server must now immediately attend to a waiting customer from the queue. To be sure, we assume that there is one server in our systems, and that customers arrive into the system in accordance with a Poisson process. We further assume that service times for the primary duty and the secondary duty are independently and exponentially distributed with typically different rates. If the two rates are equal, our considered systems essentially degenerate into the standard M/M/1 queueing system with a smaller effective service rate. As follows, we provide two motivating examples that may be modeled using our systems.

1. A cashier checks out customers when there are customers waiting at the counter. However, when all customers are checked out, the cashier may leave the counter to work on some chores, such as cleaning or stocking. The cashier does not return to the counter until the chores are completed.
2. A machine processes jobs when there are jobs waiting in line. When all jobs are completed, the machine undergoes a routine repair service. The machine does not become operational until the repair service is completed.

In this paper, we conduct the steady-state analysis of, and determine common performance measures of the M/M/1 queueing system with a secondary duty. In addition, we carry out simulation studies to validate our theoretical results.

The remainder of the paper is organized as follows. Section 2 describes our systems in great details. Section 3 derives the steady-state distribution of the M/M/1 queueing system with a secondary duty. Section 4 determines the average numbers of customers and the average times for our systems. We then conduct a simulation study to validate our theoretical results in Section 5. Last but not least, we offer some concluding remarks in Section 6.

2. DESCRIPTION OF OUR SYSTEMS

This section describes the M/M/1 queueing system with a secondary duty in great details. As is the case for the standard M/M/1 queueing system, customers arrive into our considered systems in accordance with a Poisson process having rate equal to λ. Furthermore, service times for customers with a single server is independently, identically, and exponentially distributed with rate equal to μ. That the server provides service to customers is thought to be a primary duty for the server. After all customer are served, the server immediately embarks on undertaking a side task as soon as an idle period begins. We assume that service times for side tasks are independently, identically, and exponentially distributed with rate equal to β. *Carrying out a side activity during an idle period is considered to be a secondary duty for the server.* If the time it takes the server to complete the side task is shorter than the idle period (the side task is completed before the next customer arrives into the system), the server becomes completely idle and must now wait to serve the next arriving customer. On the other hand, if the time it takes the server to complete the side task is longer than the idle period (the next customer arrives into the system before the side task is completed), a newly arriving customer does not preempt the ongoing service of the side task and must now wait in the queue for its service. Upon completion of the side task, the server must now immediately attend to a waiting customer from the queue. Most notably, we assume that the server always immediately undertakes a side task as soon as each idle period begins.

We model the aforementioned dynamics of our systems as a birth-death process. Generally, we define the state of our systems to be the number of customers present in the system in conjunction with the status of the server with respect to the secondary duty. Specifically, we employ an ordered pair (i, j), $i = 0, 1, 2,\ldots, j = 0, 1$, to represent a state of the system. In a state (i, j), component i ($i = 0, 1, 2,\ldots$) indicates the number of customers existing in the system, whereas component j ($j = 0, 1$) reveals the status of the server: 0 means that the server does not currently undertake a side task but 1 means that the server currently undertakes a side task. In a nutshell, the component j in a state tells us the status of the server with respect to the secondary duty. If the server does not currently work on a secondary duty, then $j = 0$; however, if the server currently takes up a secondary task, then $j = 1$. To be clear, state (0, 0) says that there is not any customer residing in the system and the server is completely idle (The server does not serve any customer since there is not any customer in the system, nor does the server carry out any side task since the server has just completed one.). State (1, 0) represents that there is 1 customer in the system with 0 customers in the queue, and that the server currently attends to the customer (primary duty) but not to a side task (secondary duty). Likewise, (2, 0) indicates that there are 2 customers in system with 1 in queue and 1 currently being served by the server; (3, 0) means 3 customers in system with 2 in queue and 1 currently at service; and so forth. On the other hand, (0, 1) tells us that there is not any customer in the system, and that the server currently undertakes a side task (during an idle period). State (1, 1) reveals that there is 1 customer in the system *but must now wait in the queue* while the server works on the side task; (2, 1) specifies that there are 2 customers in system but both must now wait in queue while the server attends to the side task; and so on. We now may define the state space of the M/M/1 queueing system with a secondary duty as $\Omega = \{(i, j) \mid i = 0, 1, 2,\ldots; j = 0, 1\}$.

Figure 1 depicts a birth-death process with relevant rates for our systems. We may see that the system makes a transition from states (i, j) to $(i + 1, j)$, $i = 0, 1, 2,\ldots, j = 0, 1$, since customers arrive into the system according to a Poisson process of rate λ. Upon completion of service, a customer leaves the system, rendering a transition from $(i, 0)$ to $(i - 1, 0)$, $i = 2, 3,\ldots$, with exponential rate μ. Noticeably, the system transitions from (1, 0) to (0, 1) with rate μ in that the server has just completed serving the last customer and now immediately taken up a side task as soon as an idle period begins. Observe also that system states go from $(i, 1)$ to $(i, 0)$, $i = 0, 1, 2,\ldots$, as the server has just completed its side task with exponential rate β and now immediately started to serve a waiting customer if the queue is not empty. We are now in a position to derive the steady-state distribution for our systems in Section 3.

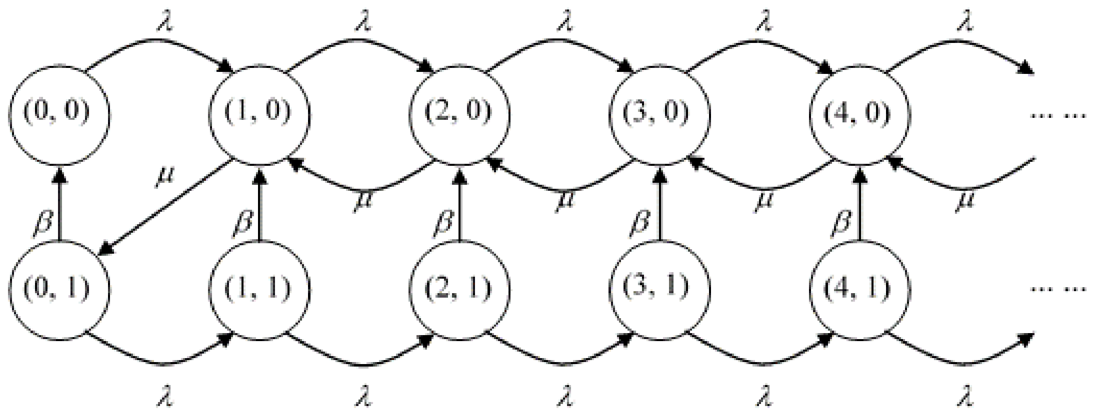

Figure 1. A rate diagram for an M/M/1 queueing system with a secondary duty.

3. DERIVATION OF THE STEADY-STATE DISTRIBUTION

This section delves into the derivation of the steady-state distribution of the number of customers residing in the M/M/1 queueing system with a secondary duty. To this end, we recursively solve sets of flow balance equations. Aiding the algebraic process, we let $\rho = \frac{\lambda}{\mu}$ and $\eta = \frac{\lambda}{\beta}$. It is imperative that $\rho < 1$, and that $\eta < 1$. The aforementioned traffic intensity requirement is to ensure the stability of the system [30–33]. We further let the probability that the system is in state (i, j), there are i customers in the system and the server is in status j with respect to the secondary duty, be $\pi(i, j)$, i = 0, 1, 2,…, j = 0, 1. We now are ready to employ net rate flows in equilibrium to construct an algebraic process as follows.

At state (0, 0): We have (See Figure 1.)

$$\beta\pi(0,1) = \lambda\pi(0,0),$$
$$\pi(0,1) = \eta\pi(0,0). \quad (1)$$

At state (0, 1): $\mu\pi(1,0) = \beta\pi(0,1) + \lambda\pi(0,1)$,

$$\pi(1,0) = \left(\frac{\beta}{\mu} + \rho\right)\pi(0,1)$$
$$= \left(\frac{\beta}{\mu} + \rho\right)\eta\pi(0,0), \text{ using Equation (1),}$$
$$= (\rho + \rho\eta)\pi(0,0)$$
$$= \rho(1 + \eta)\pi(0,0). \quad (2)$$

We now look at Figure 2. We make a cut, cut H, (blue horizontal dashed line) that partitions the state space Ω into two subsets Ω_1 = {(0. 0), (1, 0), (2, 0),…} and Ω_2 =

{(0. 1), (1, 1), (2, 1),…} where $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cup \Omega_2 = \Omega$. In equilibrium, the total rate flow from Ω_1 to Ω_2 equals that from Ω_2 to Ω_1. That is, net rate flow across the cut equals zero. As such, we have the following.

$\beta\pi(0,1) + \beta\pi(1,1) + \beta\pi(2,1) + \cdots = \mu\pi(1,0)$

$\pi(0,1) + \pi(1,1) + \pi(2,1) + \cdots = \frac{\mu}{\beta}\pi(1,0)$

$\pi(0,1) + \pi(1,1) + \pi(2,1) + \cdots = \frac{\mu}{\beta}\rho(1+\eta)\pi(0,0)$, using Equation (2),

$$\pi(0,1) + \pi(1,1) + \pi(2,1) + \cdots = \eta(1+\eta)\pi(0,0). \quad (3)$$

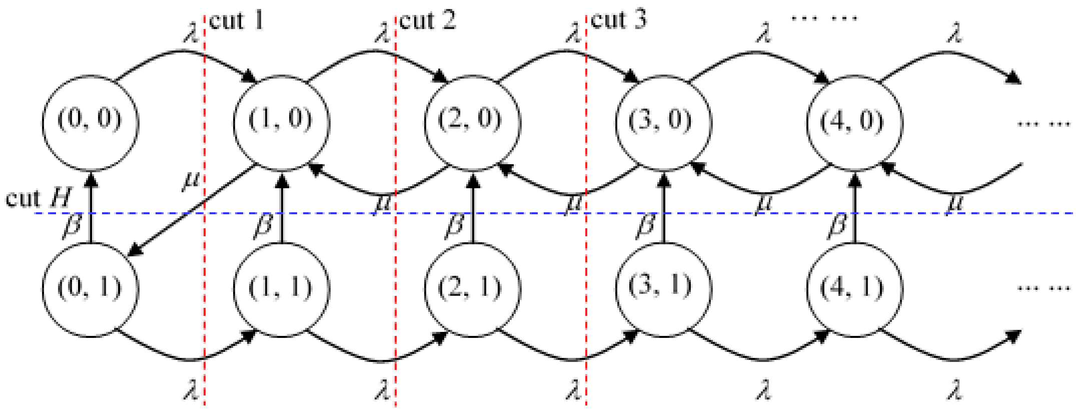

Figure 2. Cuts partition the state space of an M/M/1 queueing system with a secondary duty.

We apply the same approach to partition the state space of the system according to other cuts (red vertical dashed lines) in Figure 2. We may have infinitely many such cuts. In the following, we demonstrate flow balance equations generated by cuts 1, 2, and 3 indicated in Figure 2.

Across cut 1:

$\mu\pi(1,0) = \lambda\pi(0,0) + \lambda\pi(0,1)$

Across cut 2:

$\mu\pi(2,0) = \lambda\pi(1,0) + \lambda\pi(1,1)$

Across cut 3:

$\mu\pi(3,0) = \lambda\pi(2,0) + \lambda\pi(2,1)$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ (4)

We next sum Equations (4) to yield

$\mu\pi(1,0) + \mu\pi(2,0) + \mu\pi(3,0) + \cdots = \lambda\pi(0,0) + \lambda\pi(0,1) + \lambda\pi(1,0) + \lambda\pi(1,1) + \lambda\pi(2,0) + \lambda\pi(2,1) + \cdots$

$\mu(\pi(1,0)+\pi(2,0)+\pi(3,0)+\cdots)=\lambda[(\pi(0,0)+\pi(1,0)+\pi(2,0)+\cdots)+(\pi(0,1)+\pi(1,1)+\pi(2,1)+\cdots)]$

$\mu(\pi(1,0)+\pi(2,0)+\pi(3,0)+\cdots)=\lambda(1)$ since total probability equals 1 [30–33],
$\pi(1,0)+\pi(2,0)+\pi(3,0)+\cdots=\rho.$ (5)

At this moment, we are ready to determine the value of $\pi(0,0)$. To see this, the following equation reveals that
$\pi(0,0)+\rho+\eta(1+\eta)\pi(0,0)=\pi(0,0)+(\pi(1,0)+\pi(2,0)+\pi(3,0)+\cdots)+(\pi(0,1)+\pi(1,1)+\pi(2,1)+\cdots)$, using Equations (3, 5),
$\pi(0,0)+\rho+\eta(1+\eta)\pi(0,0)=1$, since the sum of total probability equals 1,
$(1+\eta(1+\eta))\pi(0,0)=1-\rho$
$\pi(0,0)=\frac{1-\rho}{1+\eta+\eta^2}.$ (6)

Having obtained the value of $\pi(0,0)$, we may now find the values of $\pi(0,1)$ and $\pi(1,0)$. It follows that $\pi(0,1)=\eta\frac{1-\rho}{1+\eta+\eta^2}$, and that $\pi(1,0)=\rho(1+\eta)\frac{1-\rho}{1+\eta+\eta^2}$, based upon Equations (1, 2), respectively.
To derive the remaining steady-state probabilities, we first focus our attention on states (i, 1), i = 1, 2,…. See Figure 1 for discussion below.
At state (1, 1):
$\lambda\pi(0,1)=\beta\pi(1,1)+\lambda\pi(1,1)$
$\pi(1,1)=\frac{\eta}{1+\eta}\pi(0,1)$
$\pi(1,1)=\eta\left(\frac{\eta}{1+\eta}\right)\frac{1-\rho}{1+\eta+\eta^2}$, knowing that $\pi(0,1)=\eta\frac{1-\rho}{1+\eta+\eta^2}$.

At state (2, 1):
$\lambda\pi(1,1)=\beta\pi(2,1)+\lambda\pi(2,1)$
$\pi(2,1)=\frac{\eta}{1+\eta}\pi(1,1)$
$\pi(2,1)=\eta\left(\frac{\eta}{1+\eta}\right)^2\frac{1-\rho}{1+\eta+\eta^2}.$

At state (3, 1):
$\lambda\pi(2,1)=\beta\pi(3,1)+\lambda\pi(3,1)$
$\pi(3,1)=\frac{\eta}{1+\eta}\pi(2,1)$
$\pi(3,1)=\eta\left(\frac{\eta}{1+\eta}\right)^3\frac{1-\rho}{1+\eta+\eta^2}.$
⋮ ⋮ ⋮ ⋮
⋮ ⋮ ⋮ ⋮

Repeating the above process produces the following general results.

$$\pi(i,1) = \eta\left(\frac{\eta}{1+\eta}\right)^i \frac{1-\rho}{1+\eta+\eta^2}, i = 0,1,2, \ldots. \quad (7)$$

We now proceed to determining the last set of steady-state probabilities, $\pi(i, 0)$, $i = 2, 3, \ldots$. To this end, we return to Equations (4). Recall that $\pi(0,0) = \frac{1-\rho}{1+\eta+\eta^2}$ (Equation (6)), and that $\pi(1,0) = \rho(1+\eta)\frac{1-\rho}{1+\eta+\eta^2}$. Equations (4) give rise to the following.

$\mu\pi(1,0) = \lambda\pi(0,0) + \lambda\pi(0,1)$

$\pi(1,0) = \rho(\pi(0,0) + \pi(0,1))$

$\pi(1,0) = \rho(1+\eta)\frac{1-\rho}{1+\eta+\eta^2}$, by Equations (6, 7),

$\pi(1,0) = \left[\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{\eta}{1+\eta}\right)\right]\frac{1-\rho}{1+\eta+\eta^2}$, where $\frac{(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{\eta}{1+\eta}\right) = 1$.

Now, look at

$\mu\pi(2,0) = \lambda\pi(1,0) + \lambda\pi(1,1)$

$\pi(2,0) = \rho(\pi(1,0) + \pi(1,1))$

$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{\eta}{1+\eta}\right) + \eta\left(\frac{\eta}{1+\eta}\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$,

by the above result and Equation (7),

$$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{\eta}{1+\eta} + \frac{\rho(1+\eta)-\eta}{\rho(1+\eta)}\left(\frac{\eta}{1+\eta}\right)\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{\eta}{1+\eta} + \left(1 - \frac{\eta}{\rho(1+\eta)}\right)\left(\frac{\eta}{1+\eta}\right)\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho - \frac{1}{\rho}\left(\frac{\eta}{1+\eta}\right)^2\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$

$= \left[\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2\right)\right]\frac{1-\rho}{1+\eta+\eta^2}$.

Further observe that

$\mu\pi(3,0) = \lambda\pi(2,0) + \lambda\pi(2,1)$

$\pi(3,0) = \rho(\pi(2,0) + \pi(2,1))$

$= \rho\left(\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2\right) + \eta\left(\frac{\eta}{1+\eta}\right)^2\right)\frac{1-\rho}{1+\eta+\eta^2}$,

by the above result and Equation (7),

$$= \rho\left(\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2 + \frac{\rho(1+\eta)-\eta}{\rho(1+\eta)}\left(\frac{\eta}{1+\eta}\right)^2\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2 + \left(1 - \frac{\eta}{\rho(1+\eta)}\right)\left(\frac{\eta}{1+\eta}\right)^2\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \rho\left(\rho + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \frac{1}{\rho}\left(\frac{\eta}{1+\eta}\right)^3\right)\right)\frac{1-\rho}{1+\eta+\eta^2}$$
$$= \left[\rho^3 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^3 - \left(\frac{\eta}{1+\eta}\right)^3\right)\right]\frac{1-\rho}{1+\eta+\eta^2}.$$

⋮ ⋮ ⋮

⋮ ⋮ ⋮

It is obvious that the above process generates a repeating pattern for our last results.

$$\pi(i,0) = \left[\rho^i + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^i - \left(\frac{\eta}{1+\eta}\right)^i\right)\right]\frac{1-\rho}{1+\eta+\eta^2}, i = 0,1,2, ... \quad (8)$$

We have now successfully derived the complete steady-state distribution for our systems. Essentially, Equations (7, 8) represent the steady-state distribution on the state space Ω of our systems. The next section seeks to determine the average numbers of customers and the average times for our systems.

4. DERIVATION OF THE AVERAGE NUMBERS OF CUSTOMERS AND THE AVERAGE TIMES

The aim of this section is to establish the average numbers of customers and the average amounts of time spent by a customer in two subsections. Specifically, Subsection 4.1. deals with the average number L of customers present in the system. The average amount of time W a customer spends in the system is discussed as well in this subsection. On the other hand, Subsection 4.2. handles the average number L_q of customers residing in the queue, as well as the average amount of time W_q a customer spends in the queue. Basically, we employ Equations (7, 8) to determine the average numbers of customers; Little's Law [30–33] is invoked to find the average times.

4-1. The Average Number *L* of Customers and the Average Time *W* in System

We establish the average number L of customers present in the system as follows.

$$L = 0(\pi(0,0) + \pi(0,1)) + 1(\pi(1,0) + \pi(1,1)) + 2(\pi(2,0) + \pi(2,1)) + 3(\pi(3,0) + \pi(3,1)) + \cdots$$
$$= 1\left\{\left[\rho^1 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^1 - \left(\frac{\eta}{1+\eta}\right)^1\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^1\frac{1-\rho}{1+\eta+\eta^2}\right\} +$$
$$2\left\{\left[\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^2\frac{1-\rho}{1+\eta+\eta^2}\right\} +$$
$$3\left\{\left[\rho^3 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^3 - \left(\frac{\eta}{1+\eta}\right)^3\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^3\frac{1-\rho}{1+\eta+\eta^2}\right\} + \cdots,$$ by Equations (7, 8),
$$= \left\{1\rho^1 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(1\rho^1 - 1\left(\frac{\eta}{1+\eta}\right)^1\right) + 1\eta\left(\frac{\eta}{1+\eta}\right)^1\right\}\frac{1-\rho}{1+\eta+\eta^2} +$$

$$\left\{2\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(2\rho^2 - 2\left(\frac{\eta}{1+\eta}\right)^2\right) + 2\eta\left(\frac{\eta}{1+\eta}\right)^2\right\}\frac{1-\rho}{1+\eta+\eta^2} +$$

$$\left\{3\rho^3 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(3\rho^3 - 3\left(\frac{\eta}{1+\eta}\right)^3\right) + 3\eta\left(\frac{\eta}{1+\eta}\right)^3\right\}\frac{1-\rho}{1+\eta+\eta^2} + \cdots$$

$$= \left\{(1\rho^1 + 2\rho^2 + 3\rho^3 + \cdots) + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left((1\rho^1 + 2\rho^2 + 3\rho^3 + \cdots) - \left(1\left(\frac{\eta}{1+\eta}\right)^1 + 2\left(\frac{\eta}{1+\eta}\right)^2 + 3\left(\frac{\eta}{1+\eta}\right)^3 + \cdots\right)\right) + \left(1\eta\left(\frac{\eta}{1+\eta}\right)^1 + 2\eta\left(\frac{\eta}{1+\eta}\right)^2 + 3\eta\left(\frac{\eta}{1+\eta}\right)^3 + \cdots\right)\right\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \left\{\rho(1 + 2\rho^1 + 3\rho^2 + \cdots) + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho(1 + 2\rho^1 + 3\rho^2 + \cdots) - \frac{\eta}{1+\eta}\left(1 + 2\left(\frac{\eta}{1+\eta}\right)^1 + 3\left(\frac{\eta}{1+\eta}\right)^2 + \cdots\right)\right) + \eta\frac{\eta}{1+\eta}\left(1 + 2\left(\frac{\eta}{1+\eta}\right)^1 + 3\left(\frac{\eta}{1+\eta}\right)^2 + \cdots\right)\right\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \left\{\frac{\rho}{(1-\rho)^2} + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\frac{\rho}{(1-\rho)^2} - \frac{\frac{\eta}{1+\eta}}{\left(1-\frac{\eta}{1+\eta}\right)^2}\right) + \frac{\eta\frac{\eta}{1+\eta}}{\left(1-\frac{\eta}{1+\eta}\right)^2}\right\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \left\{\frac{\rho}{(1-\rho)^2} + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\frac{\rho}{(1-\rho)^2} - \eta(1+\eta)\right) + \eta^2(1+\eta)\right\}\frac{1-\rho}{1+\eta+\eta^2}. \quad (9)$$

Equation (9) calculates the average number of customers in the system. In order to determine the average amount of time a customer spends in the system, we may resort to Little's Law for ease of computation. Subsequently, the average time spent in the system by a customer is $W = \frac{L}{\lambda}$.

4-2. The Average Number L_q of Customers and the Average Time W_q in Queue

We want to obtain the average number L_q of customers in the queue here. The average amount of time a customer spends in the queue will subsequently be determined as well.

$L_q = 0\big(\pi(0,0) + \pi(0,1) + \pi(1,0)\big) + 1\big(\pi(2,0) + \pi(1,1)\big) +$
$2\big(\pi(3,0) + \pi(2,1)\big) + 3\big(\pi(4,0) + \pi(3,1)\big) + \cdots$

$$= 1\left\{\left[\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^2 - \left(\frac{\eta}{1+\eta}\right)^2\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^1\frac{1-\rho}{1+\eta+\eta^2}\right\} +$$
$$2\left\{\left[\rho^3 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^3 - \left(\frac{\eta}{1+\eta}\right)^3\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^2\frac{1-\rho}{1+\eta+\eta^2}\right\} +$$
$$3\left\{\left[\rho^4 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\rho^4 - \left(\frac{\eta}{1+\eta}\right)^4\right)\right]\frac{1-\rho}{1+\eta+\eta^2} + \eta\left(\frac{\eta}{1+\eta}\right)^3\frac{1-\rho}{1+\eta+\eta^2}\right\} + \cdots,$$

by Equations (7, 8),

$$= \left\{1\rho^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(1\rho^2 - 1\left(\frac{\eta}{1+\eta}\right)^2\right) + 1\eta\left(\frac{\eta}{1+\eta}\right)^1\right\}\frac{1-\rho}{1+\eta+\eta^2} +$$
$$\left\{2\rho^3 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(2\rho^3 - 2\left(\frac{\eta}{1+\eta}\right)^3\right) + 2\eta\left(\frac{\eta}{1+\eta}\right)^2\right\}\frac{1-\rho}{1+\eta+\eta^2} +$$
$$\left\{3\rho^4 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(3\rho^4 - 3\left(\frac{\eta}{1+\eta}\right)^4\right) + 3\eta\left(\frac{\eta}{1+\eta}\right)^3\right\}\frac{1-\rho}{1+\eta+\eta^2} + \cdots$$

$$= \Bigg\{(1\rho^2 + 2\rho^3 + 3\rho^4 + \cdots)$$
$$+ \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\Bigg((1\rho^2 + 2\rho^3 + 3\rho^4 + \cdots)$$
$$- \left(1\left(\frac{\eta}{1+\eta}\right)^2 + 2\left(\frac{\eta}{1+\eta}\right)^3 + 3\left(\frac{\eta}{1+\eta}\right)^4 + \cdots\right)\Bigg)$$
$$+ \left(1\eta\left(\frac{\eta}{1+\eta}\right)^1 + 2\eta\left(\frac{\eta}{1+\eta}\right)^2 + 3\eta\left(\frac{\eta}{1+\eta}\right)^3 + \cdots\right)\Bigg\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \Bigg\{\rho^2(1 + 2\rho^1 + 3\rho^2 + \cdots)$$
$$+ \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\Bigg(\rho^2(1 + 2\rho^1 + 3\rho^2 + \cdots)$$
$$- \left(\frac{\eta}{1+\eta}\right)^2\left(1 + 2\left(\frac{\eta}{1+\eta}\right)^1 + 3\left(\frac{\eta}{1+\eta}\right)^2 + \cdots\right)\Bigg)$$
$$+ \eta\frac{\eta}{1+\eta}\left(1 + 2\left(\frac{\eta}{1+\eta}\right)^1 + 3\left(\frac{\eta}{1+\eta}\right)^2 + \cdots\right)\Bigg\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \left\{\frac{\rho^2}{(1-\rho)^2} + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\frac{\rho^2}{(1-\rho)^2} - \frac{\left(\frac{\eta}{1+\eta}\right)^2}{\left(1-\frac{\eta}{1+\eta}\right)^2}\right) + \frac{\eta\frac{\eta}{1+\eta}}{\left(1-\frac{\eta}{1+\eta}\right)^2}\right\}\frac{1-\rho}{1+\eta+\eta^2}$$

$$= \left\{\left(\frac{\rho}{1-\rho}\right)^2 + \frac{\rho\eta(1+\eta)}{\rho(1+\eta)-\eta}\left(\left(\frac{\rho}{1-\rho}\right)^2 - \eta^2\right) + \eta^2(1+\eta)\right\}\frac{1-\rho}{1+\eta+\eta^2}. \quad (10)$$

Equation (10) returns the average number of customers waiting in queue. Based up Little's Law, the average amount of time a customer spends in queue is readily available, $W_q = \frac{L_q}{\lambda}$.
The next section sees a simulation study to validate our theoretical results obtained in the preceding discussions.

5. SIMULATION RESULTS AND DISCUSSION

Section 5 provides a simulation study to validate afore-derived theoretical results. To this end, we essentially compare theoretical values against simulated values for accuracy.
We select a number of theoretical results for validation purposes. Specifically, we may want to know that the fraction of time the server may spend on a primary duty in our systems; notably, the fraction of time spent on the primary duty by the server equals ρ, see Equation (5). That the server conducts a secondary duty during an idle period is an important feature of our systems. We are in particular interested in gaining insight into the fraction of time the server spends on the secondary duty; Equation (3) basically returns this information. There will be time when the server is completely idle. Therefore, the fraction of time the server being idle may be of interest as well; this is given by $\pi(0, 0)$ (Equation (6)). In addition, we also look at the average numbers of customers showing up in system and in queue. Finally, we examine the average amounts of time a customer spends in system and in queue.
We prespecify both ρ and η = 0.25, 0.50, and 0.75 for our simulation study. Throughout the study, λ = 1 customer per minute is adopted. As such, μ and β are obtained accordingly for the study. Recall that $\rho = \frac{\lambda}{\mu}$ and $\eta = \frac{\lambda}{\beta}$. Hence, if ρ = 0.25, then μ = 4 customers per minute. As another example, if η = 0.75, then $\beta = \frac{4}{3}$ jobs per minute. We simulate our systems in ARENA simulation package. Each independent simulation run is administrated for 1,000,000,000 minutes. Figure 3 depicts an ARENA model for our systems. We present and discuss simulation results in the sequel.

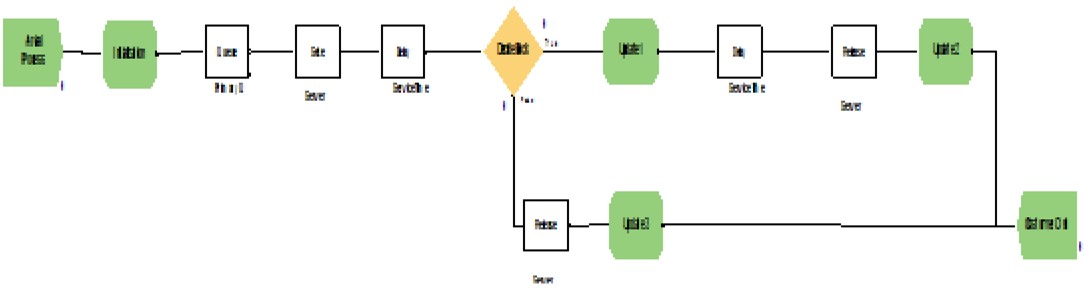

Figure 3. An ARENA model of an M/M/1 queueing system with a secondary duty.

Table 1. Fraction of time on primary duty, secondary duty, and being idle.

ρ	η	**Fraction Time on Primary Duty**		**Fraction Time on Secondary Duty**		**Fraction Time Being Idle**	
		Theoretical	**Simulated**	**Theoretical**	**Simulated**	**Theoretical**	**Simulated**
0.25	**0.25**	0.2500	0.2500	0.1786	0.1786	0.5714	0.5714
	0.50	0.2500	0.2500	0.3214	0.3214	0.4286	0.4286
	0.75	0.2500	0.2500	0.4257	0.4257	0.3243	0.3243
0.50	**0.25**	0.5000	0.5000	0.1190	0.1191	0.3810	0.3810
	0.50	0.5000	0.5000	0.2143	0.2143	0.2857	0.2857
	0.75	0.5000	0.5000	0.2838	0.2838	0.2162	0.2162
0.75	**0.25**	0.7500	0.7500	0.0595	0.0595	0.1905	0.1905
	0.50	0.7500	0.7500	0.1071	0.1071	0.1429	0.1428
	0.75	0.7500	0.7500	0.1419	0.1419	0.1081	0.1081

Table 1 contains both theoretical and simulated values for the fractions of time on a primary duty, on a secondary duty, as well as being idle. It is obvious that simulated values equal theoretical values for various combinations of ρ and η values.
Table 2 shows results for average numbers of customers in system and in queue. It reveals that both theoretical and simulated values are approximately equal.

Table 2. Average numbers of customers in system and in queue.

ρ	η	**Average Number L of Customers in System**		**Average Number L_q of Customers in Queue**	
		Theoretical	**Simulated**	**Theoretical**	**Simulated**
0.25	**0.25**	0.3929	0.3929	0.1429	0.1429
	0.50	0.5476	0.5476	0.2976	0.2976
	0.75	0.7590	0.7590	0.5090	0.5090
0.50	**0.25**	1.0595	1.0596	0.5595	0.5596
	0.50	1.2143	1.2144	0.7143	0.7144
	0.75	1.4257	1.4257	0.9257	0.9257
0.75	**0.25**	3.0595	3.0601	2.3095	2.3101
	0.50	3.2143	3.2152	2.4643	2.4652
	0.75	3.4257	3.4263	2.6757	2.6763

The average amounts of time a customer spends in system and in queue are presented on Table 3. Once again, both theoretical and simulated results are very close, as can be seen on the table.

Table 3. Average times in system and in queue.

ρ	η	Average Time W in System		Average Time W_q in Queue	
		Theoretical	Simulated	Theoretical	Simulated
0.25	**0.25**	0.3929	0.3929	0.1429	0.1429
	0.50	0.5476	0.5476	0.2976	0.2976
	0.75	0.7590	0.7590	0.5090	0.5090
0.50	**0.25**	1.0595	1.0596	0.5595	0.5596
	0.50	1.2143	1.2144	0.7143	0.7144
	0.75	1.4257	1.4258	0.9257	0.9258
0.75	**0.25**	3.0595	3.0601	2.3095	2.3101
	0.50	3.2143	3.2152	2.4643	2.4652
	0.75	3.4257	3.4263	2.6757	2.6763

The above simulation study has provided a strong support for the accuracy of our theoretical results.

6. CONCLUSIONS

In this paper, we looked into a variant of the standard M/M/1 queueing system. In our systems, the server may be slated to carry out a secondary duty as soon as an idle period begins. We dubbed the considered systems M/M/1 queueing systems with a secondary duty. We derived the steady-state distribution for our systems. The average numbers of customers present in the system and in the queue were established as well. Subsequently, Little's Law was applied to obtain the average amounts of time a customer spends in system and in queue. We conducted a simulation study to validate our theoretical results. Our simulation results strongly supported the accuracy of our derived results. Future work may research into such systems with non-Markovian arrival process or nonexponential service times.

REFERENCES

[1] Erlang, A.K., 1901, "Sandsynlighedsregning og telefonsamtaler," Nyt tidsskrift for Matematik, **20,** 33–39.

[2] Kendall, D.G., 1953, "Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain," The Annals of Mathematical Statistics, **24,** 338–354.

[3] Morse, P.M., 1958, Queues, Inventories, and Maintenance, Wiley: New York, USA.

[4] Kerbachea, L.; Smith, J.M.,1987, "The generalized expansion method for open finite queueing networks," European Journal of Operational Research, **32,** 448–461.

[5] Heidemann, D., 1991, "Queue length and waiting-time distributions at priority intersections," Transportation Research Part B: Methodological, **25,** 163–174.

[6] Heidemann, D., 1994, "Queue length and delay distributions at traffic signals," Transportation Research Part B: Methodological, **28,** 377–389.

[7] Heidemann, D., 1996, "A queueing theory approach to speed-flow-density relationships," In Transportation and Traffic Theory, Proceedings of the 13th International Symposium on Transportation and Traffic Theory, Lyon, France.

[8] Kerbache, L.; Van Woensel, T., 2005, "Planning and scheduling transportation vehicle fleet in a congested traffic environment," Supply Chain Management-European Perspectives.

[9] Groover, M.P., 1980, Automation, Production Systems, and Computer-aided Manufacturing, Prentice Hall: New Jersey, USA.

[10] Buzacott, J.A.; Shanthikumar, J.G., 1993, Stochastic Models of Manufacturing Systems, Prentice Hall: New Jersey, USA.

[11] Gershwin, S.B., 1994, Manufacturing Systems Engineering, Prentice Hall: New Jersey, USA.

[12] Baba, Y., 1986, "On the $M^{[X]}$/G/1 queue with vacation time," Operations Research Letters, **5,** 93–98.

[13] Krishna Reddy, G. V.; Nadarajan, R.; Arumuganathan, R., 1998, "Analysis of a bulk queue with *N* policy multiple vacations and setup times," Computers Operations Research, **25,** 957–967.

[14] Choudhury, G., 2002, "A batch arrival queue with a vacation time under single vacation policy," Computers and Operations Research, **29,** 1941–1955.

[15] Khalaf, R. F., 2014, "Queueing systems with four different main server's interruptions and a stand-by server," International Journal of Operational Research, **3,** 49–54.

[16] Jeyakumar, S.; Senthilnathan, B., 2017, "Modelling and analysis of a bulk service queueing model with multiple working vacations and server breakdown," RAIRO-Oper. Res, **51,** 485-508.

[17] Ayyappan, G.; Karpagam, S., 2018, "An $M^{[X]}/G(a, b)/1$ queueing system with breakdown and repair, stand-by server, multiple vacation and control policy on request for re-service," Mathematics, **6,** 101–119.

[18] Worthington, D.J., 1987, "Queueing models for hospital waiting lists," Journal of the Operational Research Society, **38,** 413–422.

[19] Christie, P.M.; Levary, R.R.,1998, "The use of simulation in planning the transportation of patients to hospitals following a disaster," Journal of Medical Systems, **22,** 289–300.

[20] Harper, P.R.; Shahani, A.K.,2002, "Modelling for the planning and management of bed capacities in hospitals," Journal of the Operational Research Society, **53,** 11–18.

[21] Green, L.V.; Savin, S., 2008, "Reducing delays for medical appointments: A queueing approach," Operations Research, **56,** 1526–1538.

[22] Yankovic, N.; Green, L.V., 2011, "Identifying good nursing levels: A queuing approach," Operations Research, **59,** 942–955.

[23] Gupta, D., 2013, “Queueing models for healthcare operations,” Handbook of Healthcare Operations Management, Springer: New York, USA.
[24] Chew, S., 2019, “Continuous-service M/M/1 queuing systems,” Applied System Innovation, **2,**16.
[25] Som, B.K.; Seth, S., 2017, “An M/M/1/N queuing system with encouraged arrivals,” Global Journal of Pure and Applied Mathematics, **17,** 3443–3453.
[26] Awasthi, B., 2018, “Performance analysis of M/M/1/k finite capacity queuing model with reverse balking and reverse reneging,” J. Comp. Math. Sci**., 9,** 850-855.
[27] Rao, S.H.; Kumar, V.V.; Kumar, K.S., 2020, “Encouraged or discoursed arrivals of an M/M/1/N queueing system with modified reneging,” Advances in Mathematics: Scientific Journal, **9,** 6641-6647.
[28] Tian, N.; Zhao, X.; Wang, K., 2008, “The M/M/1 queue with single working vacation,” International Journal of Information and Management Sciences, 19, 621-634.
[29] Servi, L.D.; Finn, S.G., 2002, “M/M/1 queues with working vacations (m/m/1/wv),” Performance Evaluation, 50, 41-52.
[30] Winston, W.L.; Goldberg, J.B., 2003, Operations Research: Applications and Algorithms, 4th ed.; Cengage Learning: Massachusetts, USA.
[31] Wagner, H.M., 1975, Principles of Operations Research, 2nd ed.; Prentice Hall: New Jersey, USA.
[32] Bain, L.J.; Engelhardt, M., 1991, Introduction to Probability and Mathematical Statistics, 2nd ed.; Cengage Learning: Massachusetts, USA.
[33] Wackerly, D.; Mendenhall, W.; Scheaffer, R.L., 2007, Mathematical Statistics with Applications, 7th ed.; Duxbury Press: Massachusetts, USA.