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Solving Intuitionistic Fuzzy Wave Equation by a Finite Difference Method

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Abstract

In this paper, a finite difference method to solve "Intuitionistic Fuzzy Wave equation" has been considered. We first explain the necessary notations, definations and the proposed method. Finally the application examples are presented with numerical results.
Keywords: Intuitionistic Fuzzy Number, Fuzzy Wave Equation, Finite difference Scheme.

1. INTRODUCTION

One of the fruitful ways of modelling uncertainty and imprecision in particular quantities for certain real-life problems are Intuitionistic Fuzzy Partial Differential Equations (IFPDEs). IFPDEs were recently utilised in a number of areas, including physics, biology, chemistry and engineering. We propose a method for approximate solution of IFPDE using Finite Difference Method. The topic of numerical methods for solving fuzzy differential equation and fuzzy partial differential equation has been rapidly growing in recent years. In [7], Buckly and Feuring proposed a method for solutions of an elementary fuzzy partial differential equation. In [1], Allahvironloo used a numerical method to solve fuzzy partial differential equation that was based on Seikala derivatives. In [11], Man et. al. used finite difference method to solve intuitionistic fuzzy heat equation. The method proposed here is based on a finite difference method for solving

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Intuitionistic fuzzy hyperbolic equation. Intuitionistic fuzzy set is very useful in providing a flexible model to elaborate uncertainty and vagueness involved in decision making.

At first we describe some definitions and notations which we will use in the paper, and then we specify the type of intuitionistic fuzzy wave equation that we wish to solve. In the next section we solve the intuitionistic fuzzy wave equation by explicit method. Intuitionistic fuzzy set is the generalised form of fuzzy set. So we may obtain better results using intuitionistic fuzzy partial differential equations rather than fuzzy partial differential equations.

In this work, we solve wave equation with intuitionistic fuzzy parameters using finite difference method. We propose a new approach for finding the numerical solution of intuitionistic fuzzy partial differential equation, which has been applied for the solution of the proposed model.

2. PRELIMINARIES

Definition 2.1. Intuitionistic Fuzzy Sets (IFS): Let D be a finite set of elements. An IFS $\tilde{A}$ in D is defined as

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle \mid x \in D\}, \tag{2.1}$$

where the functions $\mu_{\tilde{A}} : D \mapsto [0,1]$ and $\nu_{\tilde{A}} : D \mapsto [0,1]$ respectively represent the degrees of membership and non-membership of the element $x \in D$ to the set $\tilde{A}$ such that $\mu_{\tilde{A}}(x) \in [0,1],\ \nu_{\tilde{A}}(x) \in [0,1],\ 0 \le \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \le 1.$

Definition 2.2. (α, β)**-cuts:** A subset (α, β)-cut of D, generated by IFS $\tilde{A}$, where $\alpha \in [0,1],\ \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \le 1$ defined as

$$\tilde{A}_{\alpha,\beta} = \{x \in D : \mu_{\tilde{A}}(x) \ge \alpha,\ \nu_{\tilde{A}}(x) \le \beta\} \tag{2.2}$$

$\tilde{A}_{\alpha,\beta}$ is a crisp set of elements which belong to $\tilde{A}$ at least to the degree of α and which does not belong to $\tilde{A}$ at most to the degree of β.

Definition 2.3. Intuitionistic Fuzzy Number (IFN): An IFS $\tilde{A}$ on real line is called IFN if it satisfies the following conditions

1. there exists $x'_o \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x'_0) = 1 \ and \ \nu_{\tilde{A}}(x'_0) = 0$
2. the membership function $\mu_{\tilde{A}}$ is convex
   i.e., $\mu_{\tilde{A}}(\lambda x'_1 + (1-\lambda)x'_2) \ge \min\{\mu_{\tilde{A}}(x'_1),\ \mu_{\tilde{A}}(x'_2);\ \forall x'_1, x'_2 \in \mathbb{R},\ \lambda \in [0,1]\}$

3. the non-membership function $\nu_{\tilde{A}}$ is concave
   i.e., $\nu_{\tilde{A}}(\lambda x_1' + (1-\lambda)x_2') \leq \max\{\nu_{\tilde{A}}(x_1'),\ \nu_{\tilde{A}}(x_2')\};\ \forall x_1', x_2' \in \mathbb{R},\ \lambda \in [0,1]\}.$

Definition 2.4. Triangular Intuitionistic Fuzzy Number (TIFN): A triangular intuitionistic fuzzy number $\tilde{A}$ is denoted by $\tilde{A} = \langle (p_1, p_2, p_3), (p_1', p_2, p_3') \rangle$. An IFN $\tilde{A}$ is called TIFN if its membership and non-membership functions follow the following rules:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, \quad x \leq p_1 \\ \frac{x-p_1}{p_2-p_1},\ p_1 \leq x \leq p_2 \\ \frac{p_3-x}{p_3-p_2},\ p_2 \leq x \leq p_3 \\ 0, \quad x \geq p_3 \end{cases} \tag{2.3}$$

and

$$\nu_{\tilde{A}}(x) = \begin{cases} 1, \quad x \leq p_1' \\ \frac{p_2-x}{p_2-p_1'},\ p_1' \leq x \leq p_2 \\ \frac{x-p_2}{p_3'-p_2},\ p_2 \leq x \leq p_3' \\ 1, \quad x \geq p_3' \end{cases} \tag{2.4}$$

where $p_1' \leq p_1 \leq p_2 \leq p_3 \leq p_3'$.
(α, β)-cut of $\tilde{A} = \langle (p_1, p_2, p_3), (p_1', p_2, p_3') \rangle$ may be represented as

$$\tilde{A}_{\alpha,\beta} = \langle [\underline{A}(\alpha), \overline{A}(\alpha)], [\underline{A'}(\beta), \overline{A'}(\beta)] \rangle,$$

where

$$\underline{A}(\alpha) = p_1 + \alpha(p_2 - p_1),$$

$$\overline{A}(\alpha) = p_3 - \alpha(p_3 - p_2),$$

$$\underline{A}'(\beta) = p_2 - \beta(p_2 - p_1')$$

and

$$\overline{A}'(\beta) = p_2 + \beta(p_3' - p_2).$$

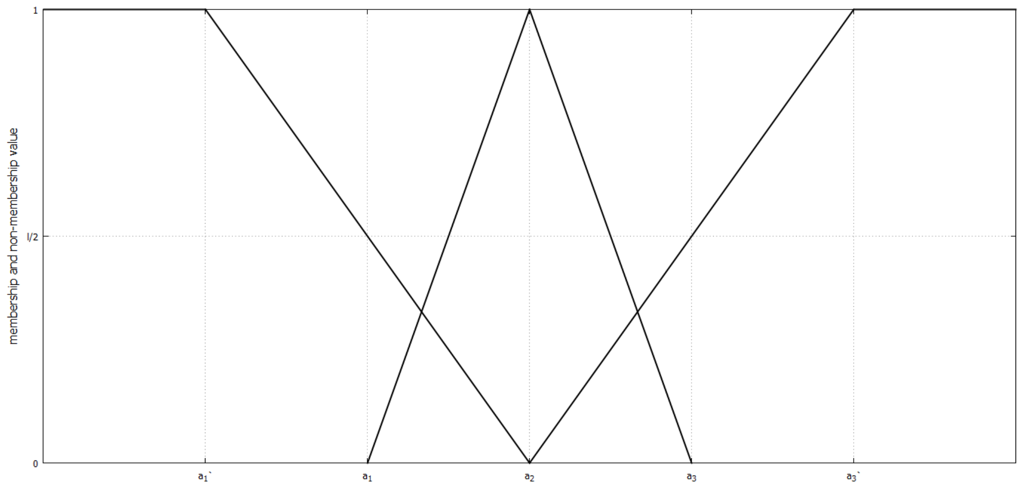

Figure 1: $\langle (p_1, p_2, p_3), (p'_1, p_2, p'_3) \rangle$

Definition 2.5. Fuzzy Arithmetic :
As discussed above , fuzzy numbers may be transformed into an interval through parametric form.
So, for any arbitrary fuzzy number $\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$,
$\tilde{y} = [\underline{y}(\alpha), \overline{y}(\alpha)]$, and scalar k, we have the interval based fuzzy arithmetic as
i) $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$ and $\overline{x}(\alpha) = \overline{y}(\alpha)$
ii) $\tilde{x} + \tilde{y} = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)]$
iii) $\tilde{x} - \tilde{y} = [\underline{x}(\alpha), \overline{x}(\alpha)] - [\underline{y}(\alpha), \overline{y}(\alpha)] = [\underline{x}(\alpha) - \overline{y}(\alpha), \overline{x}(\alpha) - \underline{y}(\alpha)]$
iv) $\tilde{x} \times \tilde{y} = [\text{min}(\underline{x}(\alpha) \times \underline{y}(\alpha)),\ \underline{x}(\alpha) \times \overline{y}(\alpha),\ \overline{x}(\alpha) \times \underline{y}(\alpha),\ \overline{x}(\alpha) \times \overline{x}(\alpha)),\ \text{max}(\underline{x}(\alpha) \times \underline{y}(\alpha)),\ \underline{x}(\alpha) \times \overline{y}(\alpha), \overline{x}(\alpha) \times \underline{y}(\alpha),\ \overline{x}(\alpha) \times \overline{x}(\alpha)]$

v)

$$k\tilde{x} = \begin{cases} [k\underline{x}(\alpha), k\overline{x}(\alpha)],\ k \geq 0 \\ [k\overline{x}(\alpha), k\underline{x}(\alpha)],\ k < 0 \end{cases} \tag{2.5}$$

3. FINITE DIFFERENCE METHOD

In this section we solve the intuitionistic fuzzy wave equation by explicit method.
Now we consider the wave equation

$$(D_t^2 - a^2 D_x^2)\tilde{U} = \tilde{0} \tag{3.1}$$

with boundary condition

$$\tilde{U}(0, t) = \tilde{U}(l, t) = \tilde{0}, \tag{3.2}$$

and initial condition

$$\tilde{U}(x,0)=\tilde{f}(x). \tag{3.3}$$

$$\frac{\partial\tilde{U}}{\partial t}(x,0)=\tilde{g}(x), 0\leq x\leq l \tag{3.4}$$

Let us consider $\tilde{U}$ be a fuzzy function of the independent crisp variables x and t. We define the domain

$$I=\{(x,t):0\leq x\leq l, 0\leq t\leq T\}. \tag{3.5}$$

A (α,β)-cuts of $\tilde{U}(x,t)$ and its parametric form will be

$$\tilde{U}(x,t)[\alpha,\beta]=<[\underline{U}(x,t;\alpha),\overline{U}(x,t;\alpha)],\ [\underline{U'}(x,t;\beta),\overline{U'}(x,t;\beta)]> \tag{3.6}$$

We suppose that $\underline{U}(x,t;\alpha),\overline{U}(x,t;\alpha)),\underline{U'}(x,t;\beta)$ and $\overline{U'}(x,t;\beta))$ have continuous partial derivatives with respect to x and t, therefore $(D_t^2-a^2D_x^2)\underline{U}(x,t;\alpha)$, $(D_t^2-a^2D_x^2)\overline{U}(x,t;\alpha)),(D_t^2-a^2D_x^2)\underline{U'}(x,t;\beta)$ and $(D_t^2-a^2D_x^2)\overline{U'}(x,t;\beta))$ are continuous for all $(x,t)\in I$, all $\alpha\in[0,1]$, $\beta\in[0,1]$.
Equation (3.1) can be decomposed as

$$(D_t^2)\underline{U}-a^2(D_x^2)\overline{U}=\underline{0}, \tag{3.7}$$

$$(D_t^2)\overline{U}-a^2(D_x^2)\underline{U}=\overline{0}, \tag{3.8}$$

$$(D_t^2)\underline{U'}-a^2(D_x^2)\overline{U'}=\underline{0}, \tag{3.9}$$

$$(D_t^2)\overline{U'}-a^2(D_x^2)\underline{U'}=\overline{0}. \tag{3.10}$$

We subdivide the x-t plane into sets of equal rectangles of sides $\delta x=h$, $\delta t=k$, by equally spaced grid lines parallel to t-axis, defined by $x_i=ih, i=0,1,2,...$ and equally spaced grid lines parallel to x-axis, defined by $t_j=jk, j=0,1,2,...$
Denote the value of $\tilde{U}$ at the representative mesh point $P(ih,jk)$ by

$$\tilde{U_P}=\tilde{U}(ih,jk)=\tilde{U}_{i,j} \tag{3.11}$$

and also denote the parametric form of intuitionistic fuzzy number $\tilde{U}_{i,j}$, involving the parameters α and β, as

$$\tilde{U}_{i,j}=<[\underline{u}_{i,j}(\alpha),\overline{u}_{i,j}(\alpha)],[\underline{u'}_{i,j}(\beta),\overline{u'}_{i,j}(\beta)]>. \tag{3.12}$$

Then, we have

$$(D_t^2)\tilde{U}(x,t) =< [D_t^2\underline{U}(x,t;\alpha), D_t^2\overline{U}(x,t;\alpha)], [D_t^2\underline{U'}(x,t;\beta), D_t^2\overline{U'}(x,t;\beta)] >, \tag{3.13}$$

$$D_x^2\tilde{U}(x,t) =< [D_x^2\underline{U}(x,t;\alpha), D_x^2\overline{U}(x,t;\alpha)], [D_x^2\underline{U'}(x,t;\beta), D_x^2\overline{U'}(x,t;\beta)] > . \tag{3.14}$$

Following Farajzadeh et al (2010), using Taylor's theorem and definition of standard difference formula we obtain (see definition 2.5)

$$D_x^2\underline{U}(x,t;\alpha)|_{i,j} \simeq \frac{\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j}}{h^2}, \tag{3.15}$$

$$D_x^2\overline{U}(x,t;\alpha)|_{i,j} \simeq \frac{\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j}}{h^2}, \tag{3.16}$$

$$D_x^2\underline{U'}(x,t;\beta)|_{i,j} \simeq \frac{\underline{u'}_{i+1,j} - 2\overline{u'}_{i,j} + \underline{u'}_{i-1,j}}{h^2}, \tag{3.17}$$

$$D_x^2\overline{U'}(x,t;\beta)|_{i,j} \simeq \frac{\overline{u'}_{i+1,j} - 2\underline{u'}_{i,j} + \overline{u'}_{i-1,j}}{h^2}, \tag{3.18}$$

with a leading error of $O(h^2)$. The notation of central-difference approximation for $(D_t^2)U$ at P, we have

$$D_t^2\underline{U}(x,t;\alpha)|_{i,j} \simeq \frac{\underline{u}_{i,j+1} - 2\overline{u}_{i,j} + \underline{u}_{i,j-1}}{k^2}, \tag{3.19}$$

$$D_t^2\overline{U}(x,t;\alpha)|_{i,j} \simeq \frac{\overline{u}_{i,j+1} - 2\underline{u}_{i,j} + \overline{u}_{i,j-1}}{k^2}, \tag{3.20}$$

$$D_t^2\underline{U'}(x,t;\beta)|_{i,j} \simeq \frac{\underline{u'}_{i,j+1} - 2\overline{u'}_{i,j} + \underline{u'}_{i,j-1}}{k^2}, \tag{3.21}$$

$$D_t^2\overline{U'}(x,t;\beta)|_{i,j} \simeq \frac{\overline{u'}_{i,j+1} - \underline{u'}_{i,j} + \overline{u'}_{i,j-1}}{k^2}, \tag{3.22}$$

with a leading error of $O(k^2)$.

Using (3.15), (3.16), (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22) the difference scheme of wave equation is

$$\frac{\underline{u}_{i,j+1} - 2\overline{u}_{i,j} + \underline{u}_{i,j-1}}{k^2} - a^2\frac{\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j}}{h^2} = 0, \tag{3.23}$$

$$\frac{\overline{u}_{i,j+1} - 2\underline{u}_{i,j} + \overline{u}_{i,j-1}}{k^2} - a^2 \frac{\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j}}{h^2} = 0, \tag{3.24}$$

$$\frac{\underline{u'}_{i,j+1} - 2\overline{u'}_{i,j} + \underline{u'}_{i,j-1}}{k^2} - a^2 \frac{\overline{u'}_{i+1,j} - 2\underline{u'}_{i,j} + \overline{u'}_{i-1,j}}{h^2} = 0, \tag{3.25}$$

$$\frac{\overline{u'}_{i,j+1} - \underline{u'}_{i,j} + \overline{u'}_{i,j-1}}{k^2} - a^2 \frac{\underline{u'}_{i+1,j} - 2\overline{u'}_{i,j} + \underline{u'}_{i-1,j}}{h^2} = 0. \tag{3.26}$$

This can be written as

$$\underline{u}_{i,j+1} = r^2(\overline{u}_{i+1,j} + \overline{u}_{i-1,j}) + 2(1 - r^2)\underline{u}_{i,j} - \underline{u}_{i,j-1}, \tag{3.27}$$

$$\overline{u}_{i,j+1} = r^2(\underline{u}_{i+1,j} + \underline{u}_{i-1,j}) + 2(1 - r^2)\overline{u}_{i,j} - \overline{u}_{i,j-1}, \tag{3.28}$$

$$\underline{u'}_{i,j+1} = r^2(\overline{u'}_{i+1,j} + \overline{u'}_{i-1,j}) + 2(1 - r^2)\underline{u'}_{i,j} - \underline{u'}_{i,j-1}, \tag{3.29}$$

$$\overline{u'}_{i,j+1} = r^2(\underline{u'}_{i+1,j} + \underline{u'}_{i-1,j}) + 2(1 - r^2)\overline{u'}_{i,j} - \overline{u'}_{i,j-1}, \tag{3.30}$$

where $r^2 = \frac{a^2k^2}{h^2}$.

Example 3.1. This example can be found in [9]

$$\frac{\partial^2 \tilde{U}}{\partial t^2}(x,t) - 4\frac{\partial^2 \tilde{U}}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \;\; t > 0, \tag{3.31}$$

with the boundary conditions

$$\tilde{U}(0,t) = \tilde{U}(1,t) = 0, \quad t > 0 \tag{3.32}$$

and initial condition

$$\tilde{U}(x,0) = \tilde{f}(x) = \tilde{K}\sin \pi x, \quad 0 \le x \le 1, \tag{3.33}$$

$$\frac{\partial \tilde{U}}{\partial t}(x,0) = 0, 0 \le x \le l \tag{3.34}$$

and

$$\begin{aligned} \tilde{K}[\alpha,\beta] &=< [\underline{K}(\alpha), \overline{K}(\alpha)], [\underline{K}(\beta), \overline{K}(\beta)] > \\ &=< [0.75 + 0.25\alpha, 1.25 - 0.25\alpha], [1 - 0.5\beta, 1 + 0.5\beta] > . \end{aligned} \tag{3.35}$$

The exact solutions for

$$\frac{\partial^2 \underline{U}}{\partial t^2}(x,t;\alpha) - 4\frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha) = 0, \quad 0 < x < 1, \quad t > 0, \tag{3.36}$$

$$\frac{\partial^2 \overline{U}}{\partial t^2}(x,t;\alpha) - 4\frac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha) = 0, \quad 0 < x < 1, \quad t > 0, \tag{3.37}$$

$$\frac{\partial^2 \underline{U'}}{\partial t^2}(x,t;\beta) - 4\frac{\partial^2 \underline{U'}}{\partial x^2}(x,t;\beta) = 0, \quad 0 < x < 1, \quad t > 0, \tag{3.38}$$

$$\frac{\partial^2 \overline{U'}}{\partial t^2}(x,t;\beta) - 4\frac{\partial^2 \overline{U'}}{\partial x^2}(x,t;\beta) = 0, \quad 0 < x < 1, \quad t > 0 \tag{3.39}$$

are, respectively

$$\underline{U}(x,t;\alpha) = \underline{K}(\alpha)\sin(\pi x)cos(2\pi t) \tag{3.40}$$

$$\overline{U}(x,t;\alpha) = \overline{K}(\alpha)\sin(\pi x)cos(2\pi t) \tag{3.41}$$

$$\underline{U'}(x,t;\beta) = \underline{K}(\beta)\sin(\pi x)cos(2\pi t) \tag{3.42}$$

$$\overline{U'}(x,t;\beta) = \overline{K}(\beta)\sin(\pi x)cos(2\pi t) \tag{3.43}$$

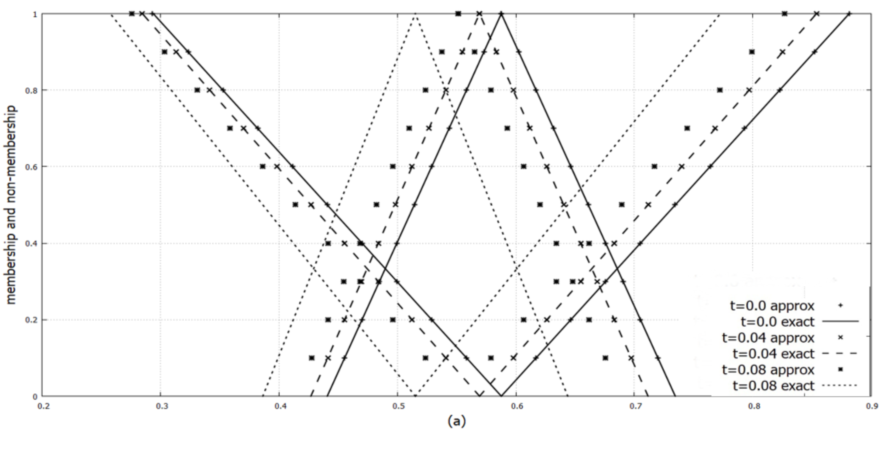

Figure 2: $h = 0.1, k = 0.04$

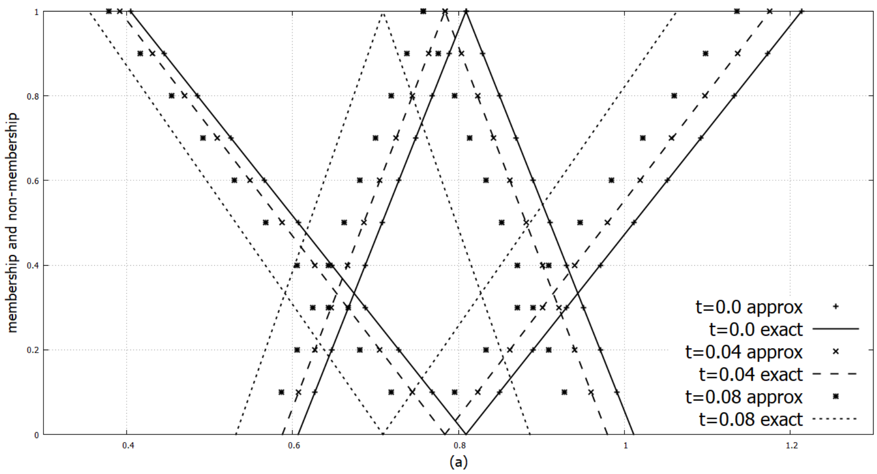

Figure 3: $h = 0.1, k = 0.04$

It is clear that the partial derivatives of $\frac{\partial^2 \overline{U}}{\partial t^2}$ and $\frac{\partial^2 \overline{U}}{\partial x^2}$ exist. Using the equations (3.40), (3.41), (3.42) and (3.43) the exact solutions with $h = 0.1$ and $k = 0.04$, resulting the value of $r = 0.8$, is presented in figures (2) and (3). This figures show the approximate and exact solutions at the point $(0.2, 0), (0.2, 0.04), (0.2, 0.0.08)$ and $(0.3, 0), (0.3, 0.04), (0.3, 0.08)$ respectively. This figures show the displacement of the entire wave at a particular time t. The wave travelling from right to left when the time increasing.

4. CONCLUSION

The intuitionistic fuzzy wave equation can be applied for modeling in physics, engineering and mechanical system. In this paper we applied an finite difference method to solve intuitionistic fuzzy wave equation. If these solutions define $\alpha - cut$ and $\beta - cut$ of a fuzzy number, then the solutions of IFPDE, would exist. Future work may focus on solution of intuitionistic fuzzy wave equation using another method.

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