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Weak and Strong Convergence for Split Equilibrium Problems and Fixed Point Problems of Asymptotically Nonexpansive Semigroup Operators in Hilbert Spaces*

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Abstract

In this work, we study the extragradient method is defined by P. Kumam et al.. First, we prove weak convergence to a common solution to the split equilibrium problem and fixed point problem of two asymptotically nonexpansive semigroups, second, we modified the method together with the classical shrinking projection algorithm and prove the strongly convergence to the same solution in real Hilbert spaces. Our results is extended some ersults of P. Kumam et al. [2].

Keywords: asymptotically nonexpansive mappings; asymptotically nonexpansive semigroup; fixed point; split equilibrium problems

2000 Mathematics Subject Classification: 47H05; 47H09; 47J05; secondary 49H05.

1. INTRODUCTION

Let H be a real Hilbert space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self-mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [7] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Recall that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitzian semigroup on C (see, e. g., [14]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

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(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

In 2006, Nadezhkina and Takahashi [11], present a hybrid extragradient method for finding a common solution of fixed point problem and variational inequality problems in a real Hilbert space. In 2015, Thuy [13], present a hybrid extragradient method for equilibrium, variational inequality and fixed point problem of a nonexpansive semigroup in Hilbert spaces. In 2017, Dinh et al. [18] present two new extragradient proximal point algorithms for solving split equilibrium problem and fixed point problems of nonexpansive mappings in real Hilbert spaces.

In 2019, I. Inchan [8], give some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup and modified two hybrid projection algorithm to prove the strongly convergence of a sequence $\{x_n\}$ generated by the hybrid projection algorithm of two asymptotically nonexpansive semigroups.

In 2020, P. Kumam et al. [2], modified extragradient method for computing a common solution to the split equilibrium problem and fixed point problem of a nonexpansive semigroup in real Hilbert spaces.

Algorithm 1.1. *let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty subsets of real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying condition $(B1) - (B5)$ and $(A1) - (A4)$, respectively. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be two nonexpansive semigroups. Let $\Gamma = \Big\{x^* \in C : x^* \in EP(C, F) \cap Fix(\mathcal{T})$ and $Ax^* \in EP(Q, G) \cap Fix(\mathcal{S})\Big\} \neq \emptyset$, then*

$$\begin{aligned} x_1 &\in C_1 = C, \\ y_n &= \textit{arg min}\big\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\big\} \\ z_n &= \textit{arg min}\big\{\lambda_n F(y_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\big\} \\ v_n &= (1 - \alpha_n)z_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ u_n &= T_{r_n}^G(Av_n), \\ x_{n+1} &= P_C(v_n + \xi A^*(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n)). \end{aligned}$$

Algorithm 1.2. *Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty subsets of real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying condition $(B1) - (B5)$ and $(A1) - (A4)$, respectively. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be two nonexpansive semigroups. Let $\Gamma = \Big\{x^* \in C : x^* \in EP(C, F) \cap Fix(\mathcal{T})$ and $Ax^* \in EP(Q, G) \cap Fix(\mathcal{S})\Big\} \neq \emptyset$, then*

$$
\begin{aligned}
x_1 &\in C_1 = C,\\
y_n &= \mathit{arg\ min}\big\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\big\}\\
z_n &= \mathit{arg\ min}\big\{\lambda_n F(y_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\big\}\\
v_n &= (1 - \alpha_n)z_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(t)z_n dt,\\
u_n &= T_{r_n}^G(Av_n),\\
w_n &= P_C\Big(v_n + \xi A^*\Big(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big)\Big),\\
C_{n+1} &= \Big\{x \in C_n : \|w_n - x\| \leq \|v_n - x\| \leq \|x_n - x\|\Big\},\\
x_{n+1} &= P_{C_{n+1}} x_1.
\end{aligned}
$$

Then the Algorithm 1.1 converges weakly to solution in Γ and prove that the Algorithm 1.2 converges strongly to to solution in Γ.

Since an asymptotically nonexpansive semigroup is generalized than nonexpansive semigroup. Inspired and motivated of this work, we study and improve a modified extragradient method is defined by Algorithm 1.1 and Algorithm 1.2 and peoved weak and strong convergence for an asymptotically nonexpansive semigroup in real Hilbert spaces.

2. PRELIMINARIES

Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty subsets of real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying condition $(B1) - (B5)$ and $(A1) - (A4)$, respectively. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be two asymptotically nonexpansive semigroups. Recall that a *split equilibrium and fixed point problem (SEFPP)* is to find

$$
x^* \in C \text{ such that } \begin{cases} F_1(x^*, x) \geq 0 \text{ for all } x \in C,\\ x^* \in Fix(\mathcal{T}) \end{cases} \tag{1}
$$

and

$$y^* = Ax^* \in Q \text{ such that } \begin{cases} F_2(y^*, y) \geq 0 \text{ for all } y \in Q, \\ y^* \in Fix(\mathcal{S}). \end{cases} \tag{2}$$

Let $\Gamma = \Big\{x^* \in C : x^* \in EP(C, F_1) \cap Fix(\mathcal{T}) \text{ and } Ax^* \in EP(Q, F_2) \cap Fix(\mathcal{S})\Big\}$. It is remarked that the problem addressed in the inequality (1) represents the classical equilibrium problem and fixed point problem of an asymptotically nonexpansive semigroup. The solution set of an equilibrium problem is denoted as $EP(C, F)$. Recall that a bifunction $F : C \times C \to \mathbb{R}$ is said to be:

1. strongly monotone with constant $\tau > 0$ if

$$F(x, y) + F(y, x) \leq -\tau\|x–y\|^2, \text{ for all } x, y \in C;$$

2. monotone if

$$F(x, y) + F(y, x) \leq 0, \text{ for all } lx, y \in C;$$

3. pseudomonotone if

$$\text{for all } x, y \in C, F(x, y) \geq 0 \Rightarrow F(y, x) \leq 0;$$

4. pseudomonotone with respect to a nonempty subset D of C if

$$\text{for all } x^* \in D \text{ and for all } y \in C, F(x^*, y) \geq 0 \Rightarrow F(y, x^*) \leq 0;$$

5. Lipschitz-type continuous if there exist positive constants c_1 and c_2 such that

$$F(x, y) + F(y, x) \geq F(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \text{ for all } x, y, z \in C.$$

Note that the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are easy to follow. Moreover, if a mapping is Lipschitz continuous on C then for any $\epsilon > 0$ it is Lipschitz-type continuous on C with $c_1 = \frac{L}{2\epsilon}$ and $c_2 = \frac{\epsilon L}{2}$.

Assumption 2.1. *[10] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

1. *$F(x, x) \geq 0, \forall c \in C$,*
2. *F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$,*
3. *F is upper hemicontinuous, i.e., for each $x, y \in C$, $\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,*
4. *For each $x \in C$ fixed, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous;*

Assumption 2.2. *Let $F_2 : Q \times Q \to \mathbb{R}$ satisfies the following set of standard properties:*

(A1) $F_2(u,u) = 0$ *for all* $u \in Q$*;*

(A2) F_2 *is monotone on* Q*;*

(A3) for each $v \in Q$*, the function* $x \to F_2(u,v)$ *is upper hemicontinuous, that is, for each* $u, w \in Q$*,*

$$\lim_{\lambda \to 0} F_2(\lambda w + (1-\lambda)u, v) \leq F_2(u,v)$$

;

(A4) for each $u \in Q$*, the function* $v \to F_2(u,v)$ *is convex and lower semi-continuous.*

Moreover, the bifunction $F_1 : C \times C \to \mathbb{R}$ *satisfies the following set of standard properties:*

(B1) $F_1(x,x) = 0$ *for all* $x \in C$*;*

(B2) F_1 *is pseudomonotone on* C *with respect to* $EP(C, F_1)$*;*

(B3) F_1 *is weakly continues on* $C \times C$*;*

(B4) for each $x \in C$*, the function* $y \to F_1(x,y)$ *is convex and subdifferentiable;*

(B5) F_1 *is Lipschitz-type continuous on* C*.*

Lemma 2.3. *[3, 6] Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions* $(A1)-(A4)$*. For* $r > 0$ *and* $u \in H$*, there exists* $w \in C$ *such that*

$$F(w,v) + \frac{1}{r}\langle v - w, w - u \rangle \geq 0, \text{ for all } v \in C.$$

Lemma 2.4. *[3, 6] Assume that the bifunctions* $F_1 : C \times C \to \mathbb{R}$ *satisfy Assumption 2.2. For* $r > 0$ *and for all* $x \in H_1$*, define a mapping* $T_r^{F_1} : H_1 \to C$ *as follows:*

$$T_r^{F_1}(x) = \Big\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \Big\}.$$

Then, the following hold:

1. $T_r^{F_1}$ *is single-valued.*
2. $T_r^{F_1}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y \rangle, \forall x, y \in H_1.$$

3. $Fix(T_r^{F_1}) = EP(C, F_1)$*.*
4. $EP(C, F_1)$ *is compact and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \Big\{ d \in Q : F_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \forall e \in Q \Big\}.$$

Then, we easily observe that $T_s^{F_2}$ s is single-valued and firmly nonexpansive, $EP(Q, F_2)$ is compact and convex, and $Fix(T_s^{F_2}) = EP(Q, F_2)$, where $EP(Q, F_2)$ is the solution set of the following equilibrium problem:

Lemma 2.5. *Let H be a real Hilbert space, then the following hold:*

1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H;$
2. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, t \in [0, 1], \forall x, y \in H.$

Lemma 2.6. *[?] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \Big\| \frac{1}{t}\int_0^t T(s)xds - T(u)\Big(\frac{1}{t}\int_0^t T(s)xds\Big)\Big\| = 0.$$

3. WEAK CONVERGENCE THEOREM

In this section, we modifies the algorithm and prove a weak convergence theorem to the common solution of split equilibrium problem and fixed point problems in real Hilbert spaces. First start, we modifies the algorithm as follows:

Algorithm 3.1. *Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty subsets of real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying condition $(B1) - (B5)$ and $(A1) - (A4)$, respectively. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be two asymptotically nonexpansive semigroups. Let $\Gamma = \Big\{x^* \in C : x^* \in EP(C, F_1) \cap Fix(\mathcal{T})$ and $Ax^* \in EP(Q, F_2) \cap Fix(\mathcal{S})\Big\} \neq \emptyset$, we have*

$$\begin{aligned}
x_1 &\in C_1 = C,\\
y_n &= \textit{arg min}\big\{\lambda_n F_1(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\big\}\\
z_n &= \textit{arg min}\big\{\lambda_n F_1(y_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\big\}\\
w_n &= T_{r_n}^{F_1} z_n,\\
v_n &= (1-\alpha_n)w_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt,\\
u_n &= T_{r_n}^{F_2}(Av_n),\\
x_{n+1} &= P_C(v_n + \eta A^*(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n)).
\end{aligned}$$

The following results establishes a crucial relation among the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ for the convergence analysis of Algorithm 3.1.

Lemma 3.2. *[1] Suppose that $x^* \in EP(C, F)$, F is pseudomonotone on C and $F(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$, then we have*

1. $\lambda_n\{F(x_n, y) - F(x_n, y_n)\} \geq \langle y_n - x_n, y_n - y\rangle$, *for all* $y \in C$,

2. $\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda c_2)\|z_n - y_n\|^2 - (1 - 2\lambda c_1)\|x_n - y_n\|^2$, *for all* $n \geq 0$

Lemma 3.3. *Let $C \subseteq H_1, Q \subseteq H_2, A, F_1, F_2, \mathcal{T}, \mathcal{S}$ and $\{x_n\}$ be the sequence as in Algorithm 3.1. Assume that the following control conditions are satisfied:*

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$;

(ii) $0 \leq d < e \leq \alpha_n \leq f < 1, \liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} s_n$;

(iii) $0 < \eta < \frac{1}{\|A\|^2}$,

where $\widetilde{t_n} := \left(\frac{1}{t_n}\int_0^{t_n} L_t dt\right) \to 1$ and $\widetilde{s_n} := \left(\frac{1}{s_n}\int_0^{s_n} L_s ds\right) \to 1$ as $n \to \infty$ and $\Gamma \neq \emptyset$. Then $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{z_n\}$, $\{v_n\}$, $\{u_n\}$ are bounded.

Proof Let $x^* \in \Gamma$, from the definition of w_n in Algorithm 3.1, we get

$$\begin{aligned} \|w_n - x^*\|^2 &= \|T_{r_n}^{F_1} z_n - T_{r_n}^{F_1} x^*\|^2 \\ &\leq \langle T_{r_n}^{F_1} z_n - T_{r_n}^{F_1} x^*, z_n - x^*\rangle \\ &= \frac{1}{2}\Big[\|T_{r_n}^{F_1} z_n - T_{r_n}^{F_1} x^*\|^2 + \|z_n - x^*\|^2 - \|T_{r_n}^{F_1} z_n - z_n\|^2\Big], \end{aligned} \tag{3}$$

it follows that

$$\begin{aligned} \|w_n - x^*\|^2 &\leq \|z_n - x^*\|^2 - \|w_n - z_n\|^2 &(4)\\ &\leq \|z_n - x^*\|^2. &(5) \end{aligned}$$

By definition of v_n and (5), we have

$$\begin{aligned}
\|v_n - x^*\| &= \left\|(1-\alpha_n)w_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\right\| \\
&= \left\|(1-\alpha_n)(w_n - x^*) + \alpha_n\left(\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\right)\right\| \\
&\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\right\| \\
&\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\frac{1}{t_n}\int_0^{t_n} \|T(t)w_n - T(t)x^*\| dt \\
&\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\frac{1}{t_n}\int_0^{t_n} L_t\|w_n - x^*\| dt \\
&= (1-\alpha_n)\|w_n - x^*\| + \alpha_n\left(\frac{1}{t_n}\int_0^{t_n} L_t dt\right)\|w_n - x^*\| \\
&= (1-\alpha_n)\|w_n - x^*\| + \alpha_n\widetilde{t_n}\|w_n - x^*\| \\
&= (1-\alpha_n(1-\widetilde{t_n}))\|z_n - x^*\|, \qquad (6)
\end{aligned}$$

where $\widetilde{t_n} := \left(\frac{1}{t_n}\int_0^{t_n} L_t dt\right)$. Moreover, from $(C1)$ and Lemma 3.2(2), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - (1-2\lambda c_2)\|z_n - y_n\|^2 - (1-2\lambda c_1)\|x_n - y_n\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (7)
\end{aligned}$$

From (6) and (7), it follows that

$$\|v_n - x^*\| \leq (1-\alpha_n(1-\widetilde{t_n}))\|x_n - x^*\|. \qquad (8)$$

By the definition of x_{n+1}, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|P_C(v_n + \eta A^*(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n)) - P_C x^*\right\|^2 \\
&\leq \left\|v_n + \eta A^*\Big(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big) - x^*\right\|^2 \\
&= \left\|(v_n - x^*) + \eta A^*\Big(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big)\right\|^2 \\
&= \|v_n - x^*\|^2 + 2\left\langle v_n - x^*, \eta A^*\Big(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big)\right\rangle \\
&+ \left\|\eta A^*\Big(\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big)\right\|^2 \\
&\leq \|v_n - x^*\|^2 + 2\eta\left\langle Av_n - Ax^*, \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\rangle \\
&+ \eta^2\|A^*\|^2\sigma_n^2 \qquad (9)
\end{aligned}$$

where $\sigma_n = \left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\|^2$. On the other hand,

$$\begin{aligned}
&\left\langle Av_n - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
=&\ \left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
&+ \left\langle Av_n - \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
=&\ \left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
&- \left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
=&\ \left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
&- \left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\|^2. \qquad (10)
\end{aligned}$$

Consider,

$$\begin{aligned}
\|Av_n - Ax^*\|^2 &= \left\| \left(\frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^* \right) - \left(\frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right) \right\|^2 \\
&= \left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^* \right\|^2 \\
&\quad -2 \left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
&\quad + \left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left\langle \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^*, \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \\
&= \frac{1}{2} \Big[\left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Ax^* \right\|^2 \qquad (11) \\
&+ \left\| \frac{1}{s_n} \int_0^{s_n} S(s)u_n ds - Av_n \right\|^2 - \|Av_n - Ax^*\|^2 \Big].
\end{aligned}$$

From (10) and (11), we have

$$\begin{aligned}&\left\langle Av_n - Ax^*, \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\rangle\\ &\le \frac{1}{2}\Big[\Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Ax^*\Big\|^2 + \Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big\|^2\\ &\qquad -\|Av_n - Ax^*\|^2\Big] - \Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big\|^2\\ &= \frac{1}{2}\Big[\Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Ax^*\Big\|^2 - \Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\Big\|^2\\ &\qquad -\|Av_n - Ax^*\|^2\Big].\end{aligned}$$

Now we consider,

$$\begin{aligned}\Big\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Ax^*\Big\|^2 &\le \frac{1}{s_n}\int_0^{s_n} \|S(s)u_n - S(s)Ax^*\|^2 ds\\ &\le \frac{1}{s_n}\int_0^{s_n} L_s\|u_n - Ax^*\|^2 ds\\ &= \left(\frac{1}{s_n}\int_0^{s_n} L_s ds\right)\|u_n - Ax^*\|^2\\ &= \widetilde{s_n}\|u_n - Ax^*\|^2, \qquad (12)\end{aligned}$$

where $\widetilde{s_n} := \left(\frac{1}{s_n}\int_0^{s_n} L_s ds\right)$. From (12) and (12), it implies that

$$\begin{aligned}&\left\langle Av_n - Ax^*, \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\rangle\\ &\qquad\le \frac{1}{2}\Big(\widetilde{s_n}\|u_n - Ax^*\|^2 - \|Av_n - Ax^*\|^2\Big) - \frac{1}{2}\sigma_n^2. \qquad (13)\end{aligned}$$

Note that

$$\begin{aligned}\|u_n - Ax^*\|^2 &= \|T_{r_n}^{F_2}Av_n - T_{r_n}^{F_2}Ax^*\|^2\\ &\le \langle T_{r_n}^{F_2}Av_n - T_{r_n}^{F_2}Ax^*, Av_n - Ax^*\rangle\\ &= \frac{1}{2}\Big(\|T_{r_n}^{F_2}Av_n - T_{r_n}^{F_2}Ax^*\|^2 + \|Av_n - Ax^*\|^2 - \|T_{r_n}^{F_2}Av_n - Av_n\|^2\Big)\\ &= \|Av_n - Ax^*\|^2 - \|u_n - Av_n\|^2. \qquad (14)\end{aligned}$$

Replace (14) in (13) implies that

$$\left\langle Av_n - Ax^*, \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n \right\rangle \leq \frac{-1}{2}\Big(\|u_n - Av_n\|^2 + (1-\widetilde{s_n})\|Av_n - Ax^*\|^2 + \sigma_n^2\Big). \quad (15)$$

It follows from (9) and (15)

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|v_n - x^*\|^2 - \eta\Big(1 - \eta\|A\|^2\Big)\sigma_n^2 - \eta\Big(\|u_n - Av_n\|^2 \\ &\quad +(1-\widetilde{s_n})\|Av_n - Ax^*\|^2\Big) \\ &\leq (1-\alpha_n(1-\widetilde{t_n}))^2\|x_n - x^*\|^2 - \eta\Big(1-\eta\|A\|^2\Big)\sigma_n^2 - \eta\Big(\|u_n - Av_n\|^2 \\ &\quad +(1-\widetilde{s_n})\|Av_n - Ax^*\|^2\Big)\end{aligned} \quad (16)$$

From (iii), (8) and (16) that

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n(1-\widetilde{t_n}))^2\|x_n - x^*\|^2 - \eta(1-\widetilde{s_n})\|Av_n - Ax^*\|^2. \quad (17)$$

Taking $n \to \infty$ in (17), we have $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, from (5), (7) and (8), it follows that

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|v_n - x^*\| = \lim_{n\to\infty}\|z_n - x^*\| = \lim_{n\to\infty}\|w_n - x^*\|.$$

Hence, $\{x_n\}$. Consequently, all other sequences in Algorithm 3.1 are bounded.

Next, we prove a weak convergence theorem of Algorithm 3.1 to a point in Γ.

Theorem 3.4. *Let $C \subseteq H_1, Q \subseteq H_2, A, F_1, F_2, \mathcal{T}, \mathcal{S}$ and $\{x_n\}$ be the sequence as in Algorithm 3.1, and satisfying the control conditions in Lemma 3.3. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by Algorithm 3.1 converge weakly to a point in Γ.*

Proof From (16), we have

$$\begin{aligned}\eta\|u_n - Av_n\|^2 &\leq (1-\alpha_n(1-\widetilde{t_n}))^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad -\eta(1-\widetilde{s_n})\|Av_n - Ax^*\|^2,\end{aligned} \quad (18)$$

and

$$\begin{aligned}&\eta\Big(1-\eta\|A\|^2\Big)\sigma_n^2 \\ &\leq (1-\alpha_n(1-\widetilde{t_n}))^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 - \eta(1-\widetilde{s_n})\|Av_n - Ax^*\|^2\end{aligned} \quad (19)$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists and $\sigma_n = \left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\|^2$, it implies that

$$\lim_{n\to\infty}\|u_n - Av_n\| = 0 = \lim_{n\to\infty}\left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\|. \tag{20}$$

Consider,

$$\left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - u_n\right\| \leq \left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - Av_n\right\| + \|Av_n - u_n\|. \tag{21}$$

From (20) and (21), we have

$$\lim_{n\to\infty}\left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - u_n\right\| = 0. \tag{22}$$

Observe that

$$\begin{aligned}
&\|u_n - S(t)u_n\| \\
&\leq \left\|u_n - \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - S(t)\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\| \\
&\quad + \left\|S(t)\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - S(t)u_n\right\| \\
&\leq \left\|u_n - \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - S(t)\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\| \\
&\quad + L_t^S\left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - u_n\right\| \\
&= (1 + L_t^S)\left\|u_n - \frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds - S(t)\frac{1}{s_n}\int_0^{s_n} S(s)u_n ds\right\|.
\end{aligned}$$

From (22) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|u_n - S(t)u_n\| = 0. \tag{23}$$

From (10), we have

$$(1 - 2\lambda c_2)\|z_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2, \tag{24}$$

and

$$(1 - 2\lambda c_1)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2. \tag{25}$$

Since $\{\lambda_n\} \subset [a, b]$ for some $a, b \in \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, (24), (25) and $\lim_{n\to\infty}\|x_n - x^*\|$, $\lim_{n\to\infty}\|z_n - x^*\|$ are exist, it follows that

$$\lim_{n\to\infty}\|z_n - y_n\| = 0 = \lim_{n\to\infty}\|x_n - y_n\|. \tag{26}$$

Note that $\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\|$ and from (26), therefore

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{27}$$

By definition of v_n and Lemma 2.5, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &= \Big\|(1-\alpha_n)w_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\Big\|^2 \\
&= \Big\|(1-\alpha_n)(w_n - x^*) + \alpha_n\Big(\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\Big)\Big\|^2 \\
&= (1-\alpha_n)\|w_n - x^*\|^2 + \alpha_n\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - x^*\Big\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|^2 \\
&\leq \|w_n - x^*\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|^2,
\end{aligned} \tag{28}$$

it follows that

$$\alpha_n(1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|^2 \leq \|w_n - x^*\|^2 - \|v_n - x^*\|^2. \tag{29}$$

Since, $0 \leq d < e \leq \alpha_n \leq f < 1$, $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} s_n$ and (29), we have

$$e(1-f)\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|^2 \leq \|w_n - x^*\|^2 - \|v_n - x^*\|^2. \tag{30}$$

From Lemma 3.3 and (30), we have

$$\lim_{n\to\infty} \Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|^2 = 0. \tag{31}$$

Since $\|v_n - w_n\| = \alpha_n\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\|$, it follows that

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

Similarly, we observe that

$$\begin{aligned}\|w_n - T(s)w_n\| \leq &\\ & \Big\|w_n - \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\| + \Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - T(s)\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\| \\ & + \Big\|T(s)\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - T(s)w_n\Big\| \\ \leq\ & \Big\|w_n - \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\| + \Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - T(s)\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\| \\ & + L_s^T\Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - w_n\Big\| \\ =\ & (1+L_s^T)\Big\|w_n - \frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\| + \Big\|\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt - T(s)\frac{1}{t_n}\int_0^{t_n} T(t)w_n dt\Big\|.\end{aligned}$$

From (31) and Lemma 2.6, we have

$$\lim_{n\to\infty} \|w_n - T(s)w_n\| = 0. \tag{32}$$

From (4), we have

$$\|w_n - z_n\|^2 \leq \|z_n - x^*\|^2 - \|w_n - x^*\|^2. \tag{33}$$

By Lemma 3.3, that $\lim_{n\to\infty} \|z_n - x^*\|$, $\lim_{n\to\infty} \|w_n - x^*\|$ are exist, then

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{34}$$

Since $\|w_n - x_n\| \leq \|w_n - z_n\| + \|z_n - x_n\|$, from (27) and (34), we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{35}$$

Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} \rightharpoonup w$ for some $w \in C$.

Now, we prove that $w \in \Gamma$. First, we show that $w \in Fix(\mathcal{T})$. Assume that $w \notin Fix(\mathcal{T})$. Since, $w_{n_i} \rightharpoonup w$ and $T(s)w \neq w$, from opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty} \|w_{n_i} - w\| &< \liminf_{i\to\infty} \|w_{n_i} - T(s)w\| \\ &\leq \liminf_{i\to\infty} \Big(\|w_{n_i} - T(s)w_{n_i}\| + \|T(s)w_{n_i} - T(s)w\|\Big) \\ &\leq L_s^T \liminf_{i\to\infty} \|w_{n_i} - w\|.\end{aligned}$$

Taking $s \to \infty$, we have

$$\liminf_{i\to\infty} \|w_{n_i} - w\| \quad < \quad \liminf_{i\to\infty} \|w_{n_i} - w\|.$$

A contradiction. Then we obtain $w \in Fix(\mathcal{T})$. Since, $w_n = T_{r_n}^{F_1} z_n$, we have

$$F_1(w_n, x) + \frac{1}{r_n}\langle x - w_n, w_n - z_n\rangle \geq 0, \forall x \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_n}\langle x - w_n, w_n - z_n\rangle \quad \geq \quad F_1(x, w_n),$$

and hence,

$$\left\langle x - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{r_{n_i}}\right\rangle \quad \geq \quad F_1(x, w_{n_i}).$$

Since, $\|w_n - z_n\| \to 0$, we get $w_{n_i} \rightharpoonup w$ and $\frac{w_{n_i} - z_{n_i}}{r_{n_i}} \to 0$, it follows that

$$F_1(x, w) \leq 0, \forall x \in C. \tag{36}$$

For, $0 < \lambda < 1$ and $x, w \in C$, let $x_\lambda = \lambda x + (1 - \lambda)w \in C$ and from convex function of F_1, we have

$$\begin{aligned} 0 &\leq F_1(x_\lambda, x_\lambda) \\ &\leq \lambda F_1(x_\lambda, x) + (1 - \lambda)F_1(x_\lambda, w) \\ &\leq F_1(x_\lambda, x). \end{aligned}$$

By Assumption 2.2 and (36), we get $F_1(w, x) \geq 0$. This implies that $w \in EP(C, F_1)$. Next, we show that $Aw \in Fix(\mathcal{S})$. Assume that $Aw \notin Fix(\mathcal{S})$. Since, $\|w_n - v_n\| \to 0$ and

$$v_n - w = (v_n - w_n) + (w_n - w) \rightharpoonup 0,$$

it follows that $v_n \rightharpoonup w$. Since A is bounded linear operator, so $Av_n \rightharpoonup Aw$. Since, $\|u_n - Av_n\| \to 0$ and

$$u_n - Aw = (u_n - Av_n) + (Av_n - Aw) \rightharpoonup 0,$$

it follows that $u_n \rightharpoonup Aw$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup Aw$, and Assume that $S(t)Aw \neq Aw$, from opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty}\|u_{n_j}-Aw\| &< \liminf_{j\to\infty}\|u_{n_j}-S(t)Aw\|\\ &\le \liminf_{j\to\infty}\Big(\|u_{n_j}-S(t)u_{n_j}\|+\|S(t)u_{n_j}-S(t)Aw\|\Big)\\ &\le L_t^S\liminf_{j\to\infty}\|u_{n_j}-Aw\|.\end{aligned}$$

Taking $t\to\infty$, we have

$$\liminf_{j\to\infty}\|u_{n_j}-Aw\| \quad < \quad \liminf_{j\to\infty}\|u_{n_j}-Aw\|.$$

A contradiction. Then we obtain $Aw\in Fix(\mathcal{S})$. Since, $u_n=T_{r_n}^{F_2}Av_n$, we have

$$F_2(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-Av_n\rangle\ge 0,\forall y\in Q.$$

It follows from the monotonicity of F_2 that

$$\frac{1}{r_n}\langle y-u_n,u_n-Av_n\rangle \quad \ge \quad F_2(y,u_n),$$

and hence,

$$\left\langle y-u_{n_i},\frac{u_{n_i}-Av_{n_i}}{r_{n_i}}\right\rangle \quad \ge \quad F_2(y,u_{n_i}).$$

Since, $\|u_n-Av_n\|\to 0$, we get $u_{n_i}\rightharpoonup Aw$ and $\frac{u_{n_i}-Av_{n_i}}{r_{n_i}}\to 0$, it follows that

$$F_2(y,Aw)\le 0,\forall y\in Q. \tag{37}$$

For, $0<\rho<1$ and $y,Aw\in Q$, let $y_\rho=\rho y+(1-\rho)Aw\in Q$ and from convex function of F_2, we have

$$\begin{aligned}0 &\le F_2(y_\rho,y_\rho)\\ &\le \rho F_2(y_\rho,y)+(1-\rho)F_2(y_\rho,Aw)\\ &\le F_2(Aw,y).\end{aligned}$$

By Assumption 2.2 and (37), we get $F_2(Aw,y)\ge 0$. This implies that $Aw\in EP(Q,F_2)$. This proves $w\in\Gamma$.

Finally, we show that $\{x_n\}$ converges weakly to w and $\{u_n\}$ converges weakly to Aw. Assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q$ as $i\to\infty$

where $q \in \Gamma$ such that $q \neq w$. By Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - q\| &< \liminf_{i\to\infty} \|x_{n_i} - w\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - w\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - q\| \\
&= \liminf_{i\to\infty} \|x_{n_i} - q\|.
\end{aligned}$$

A contradiction. This implies that $x_n \rightharpoonup w$ as $n \to \infty$. Since, $Av_n \rightharpoonup Aw$ as $n \to \infty$, therefore from $\|u_n - Av_n\| \to 0$ as $n \to \infty$ we conclude that $u_n \rightharpoonup Aw$. This completes the proof.

Corollary 3.5. *[2] Let $C \subseteq H_1, Q \subseteq H_2, A, F, G, \mathcal{T}, \mathcal{S}$ and $\{x_n\}$ be the sequence as in Algorithm 1.1. Assume that the following set of control conditions are satisfied:*

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$;

(ii) $0 \leq d < e \leq \alpha_n \leq f < 1, \liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} s_n$;

(iii) $0 < \xi < \frac{1}{\|A\|^2}$.

If $\Gamma \neq \emptyset$ then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by Algorithm 1.1 converge weakly to a point in Γ.

4. STRONG CONVERGENCE THEOREM

In this section, we modified the iterative method together with the classical shrinking projection algorithm to establish the strong convergence results. Our algorithm reads as follows.

Algorithm 4.1. *Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty subsets of real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying condition $(B1) - (B5)$ and $(A1) - (A4)$, respectively. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be two asymptotically nonexpansive semigroups. Let $\Gamma = \Big\{x^* \in C : x^* \in EP(C, F_1) \cap Fix(\mathcal{T})$ and $Ax^* \in EP(Q, F_2) \cap Fix(\mathcal{S})\Big\} \neq \emptyset$,* *we have*

$$\begin{aligned} x_1 &\in C_1 = C, \\ y_n &= \textit{arg min}\Big\{\lambda_n F_1(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\Big\} \\ z_n &= \textit{arg min}\Big\{\lambda_n F_1(y_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\Big\} \\ w_n &= T_{r_n}^{F_1} z_n, \\ v_n &= (1 - \alpha_n) w_n + \alpha_n \frac{1}{t_n} \int_0^{t_n} T(t) w_n dt, \\ u_n &= T_{r_n}^{F_2}(Av_n), \\ p_n &= P_C\Big(v_n + \eta A^*\Big(\frac{1}{s_n} \int_0^{s_n} S(s) u_n ds - Av_n\Big)\Big), \\ C_{n+1} &= \Big\{x \in C_n : \|p_n - x\| \le \|v_n - x\| \le \|x_n - x\|\Big\}, \\ x_{n+1} &= P_{C_{n+1}} x_1. \end{aligned}$$

Now, we prove a strong convergence theorem of the Algorithm 4.1 to common solution of Γ.

Theorem 4.2. *Let $C \subseteq H_1, Q \subseteq H_2, A, F_1, F_2, H_1, H_2, \mathcal{T}, \mathcal{S}$ and $\{x_n\}$ be the sequence as in Algorithm 4.1. Assume that the following control conditions are satisfied:*

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$*;*

(ii) $0 \le d < e \le \alpha_n \le f < 1$, $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} s_n$*;*

(iii) $0 < \eta < \frac{1}{\|A\|^2}$.

If $\Gamma \neq \emptyset$, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by Algorithm 4.1 converge strongly to a point in Γ.

Proof First, show that C_n is nonempty colsed and convex for all $n \ge 1$. Since

$$\{x \in C_n : \|p_n - x\|^2 \le \|x_n - x\|^2\} = \{x \in C_n : \|p_n\|^2 - \|x_n\|^2 \le 2\langle p_n - x_n, x\rangle\},$$

the set C_{n+1} is closed and convex.

Let $x^* \in \Gamma$, it follows from (16) and (8) we have

$$\begin{aligned} \|p_n - x^*\|^2 &\le \|v_n - x^*\|^2 - \eta\Big(1 - \eta\|A\|^2\Big)\sigma_n^2 \\ &\quad -\eta\Big(\|u_n - Av_n\|^2 + (1 - \widetilde{s_n})\|Av_n - Ax^*\|^2\Big) \\ &\le \|v_n - x^*\|^2 \\ &\le (1 - \alpha_n(1 - \widetilde{t_n}))^2\|x_n - x^*\|^2. \end{aligned} \tag{38}$$

From (38), taking $t \to \infty$ we have

$$\|p_n - x^*\| \leq \|v_n - x^*\| \leq \|x_n - x^*\|. \tag{39}$$

From (39), it follows that $\Gamma \subseteq C_n$ for $n \geq 1$. It implies that the Algorithm 4.1 is well-defined.

Next, we prove that, $\{x_n\}$ and $\{p_n\}$ are bounded and $x_n \to p$ as $n \to \infty$ for some $p \in \Gamma$. Since, $x_{n+1} \in P_{C_{n+1}} x_1$, therefore $\|x_{n+1} - x_1\| \leq \|x_2 - x_1\|$ for all $x_1 \in C$. Inparticular, we have $\|x_{n+1} - x_1\| \leq \|P_\Gamma x_1 - x_1\|$. Hence, $\{x_n\}$ and $\{p_n\}$ are bounded. Since $x_n = P_{C_n} x_1$, it follows that

$$\langle x_1 - x_n, x_n - y \rangle \geq 0$$

for all $y \in \Gamma$ and $n \in \mathbb{N}$.

Since, $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_1 - x_n, x_n - x_{n+1} \rangle \geq 0 \tag{40}$$

So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle = -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - x_{n+1} \rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_1 - x_n\|^2 \leq \|x_1 - x_n\| \|x_1 - x_{n+1}\|,$$

and hence

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_1 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Next, we claim that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From (40), we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) + (x_1 - x_{n+1})\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\langle x_1 - x_n, x_1 - x_n \rangle - 2\langle x_1 - x_n, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_n - x_1\|^2 \\ &- 2\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2 \\ &= -\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2. \end{aligned}$$

Since, $\lim_{n\to\infty} \|x_n - x_1\|$ exists, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{41}$$

Consider, for $m \geq n$ we have

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\|. \quad (42)$$

From (41) and (42), we have $\{x_n\}$ is Cauchy sequence. By completeness of H_1 we have $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Claim that, $p \in \Gamma$. Since $x_{n+1} \in C_{n+1}$ thus

$$\|p_n - x_{n+1}\| \leq \|v_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \quad (43)$$

Consider,

$$\|p_n - x_n\| \leq \|p_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|,$$

similarly,

$$\|v_n - x_n\| \leq \|v_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|.$$

From (39), it follows that

$$\lim_{n\to\infty} \|p_n - x_n\| = 0 = \lim_{n\to\infty} \|v_n - x_n\|. \quad (44)$$

Consider,

$$\begin{aligned} \|z_n - T(s)z_n\| &\leq \|z_n - w_n\| + \|w_n - T(s)w_n\| + \|T(s)w_n - T(s)z_n\| \\ &\leq \|z_n - w_n\| + \|w_n - T(s)w_n\| + L_s\|w_n - z_n\| \\ &= (1 + L_s)\|w_n - z_n\| + \|w_n - T(s)w_n\| \end{aligned}$$

From (32) and (34) it implies that

$$\lim_{n\to\infty} \|z_n - T(s)z_n\| = 0. \quad (45)$$

Since, $x_n \to p$ as $n \to \infty$. From (26), (27) and (44) it follows that $z_n \to p$, $y_n \to p$ and $v_n \to p$ as $n \to \infty$. Consider,

$$\begin{aligned} \|T(s)p - p\| &\leq \|T(s)p - T(s)z_n\| + \|T(s)z_n - z_n\| + \|z_n - p\| \\ &\leq L_s\|p - z_n\| + \|T(s)z_n - z_n\| + \|z_n - p\| \\ &= (1 + L_s)\|z_n - p\| + \|T(s)z_n - z_n\|. \quad (46) \end{aligned}$$

From (45) and $z_n \to p$ as $n \to \infty$, it follows that $p \in Fix(\mathcal{T})$. From Lemma 3.2(1), we have

$$\lambda_n\{F_1(x_n, x) - F_1(x_n, y_n)\} \geq \langle y_n - x_n, y_n - x\rangle, \text{ for all } x \in C. \quad (47)$$

From (26), we have $F(p, x) \geq 0$, it follows that $p \in EP(C, F_1)$. Hence, $p \in$

$EP(C, F_1) \cap Fix(\mathcal{T})$.

Finally, we prove that $Ap \in EP(Q, F_2) \cap Fix(\mathcal{S})$. Since, $v_n \to p$ as $n \to \infty$, we have $Av_n \to Ap$ as $n \to \infty$. Consider,

$$\|u_n - Ap\| \le \|u_n - Av_n\| + \|Av_n - Ap\|, \tag{48}$$

from (20) and (48), it implies that

$$\lim_{n\to\infty} \|u_n - Ap\| = 0. \tag{49}$$

Now observe that

$$\begin{aligned} \|S(t)Ap - Ap\| &\le \|S(t)Ap - S(t)u_n\| + \|S(t)u_n - u_n\| + \|u_n - Ap\| \\ &\le L_t\|Ap - u_n\| + \|S(t)u_n - u_n\| + \|u_n - Ap\| \\ &= (1 + L_t)\|u_n - Ap\| + \|S(t)u_n - u_n\|. \end{aligned}$$

From (23) and (49) it follows that $Ap \in Fix(\mathcal{S})$. Similarly, from Lemma 3.2(1) again, we have

$$\lambda_n\{F_2(u_n, y) - F_2(u_n, Av_n)\} \ge \langle Av_n - u_n, Av_n - y\rangle, \text{ for all } y \in Q. \tag{50}$$

From (i), (20) and (49), we have $F_2(Ap, y) \ge 0$, it follows that $Ap \in EP(Q, F_2)$. Hence, $Ap \in EP(Q, F_2) \cap Fix(\mathcal{S})$. Therefor, $p \in \Gamma$. This completes the proof.

Corollary 4.3. *[2] Let $C \subseteq H_1, Q \subseteq H_2, A, F, G, H_1, H_2, \mathcal{T}, \mathcal{S}$ and $\{x_n\}$ be the sequence as in Algorithm 1.2. Assume that the following set of control conditions are satisfied:*

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$;

(ii) $0 \le d < e \le \alpha_n \le f < 1, \liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} s_n$;

(iii) $0 < \xi < \frac{1}{\|A\|^2}$.

If $\Gamma \neq \emptyset$, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by Algorithm 1.2 converge strongly to a point in Γ.

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