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Some Fixed Point Outcomes in S_b-Metric Spaces using (ϕ, ψ)-Generalized Weakly Contractive Maps in S_b-Metric Spaces

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Abstract

In this result, we define (ψ, ϕ) -generalized weakly contraction map in S_b-metric space. In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)-generalized weakly contractive maps in S-metric spaces and established the existence and uniqueness of fixed point theorem for such maps. By the motivation of B.K.Leta and G.V.R.Babu[3] results in S-metric spaces, we introduced the (ψ, ϕ) - generalized weakly contractive map in S_b-metric spaces and prove a existence and uniqueness of fixed point theorem. We also give an example to support of our result.

Keywords: Fixed point, S-metric space, S_b-metric space, (ψ, ϕ)- generalized weakly contracition map.
2010 MSC: 47H10, 54E50

1. INTRODUCTION

During 1922, Stefan Banach conceived the concept of contraction and established well known Banach contraction theorem. Banach Principle of contraction[9] on metric spaces is the paramount importance cause in the field of fixed points and non linear analysis. Literature's are brought out new outcomes that are related to prove the generalization of metric space and to acquire a refinement about the contractive

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condition. In the year 2006, B Sims and Mustafa[10], established theory on G-metric spaces, that is an extension of metric spaces and established some properties. Later, A.Aliouche, S.Sedghi and N.Shobe [7] initiated S-metric spaces, it is a generalization of G-metric spaces in the year 2012. In 2014, S.Radojevic, N.V.Dung and N.T.Hieu [11] proved by examples shows S-metric spaces are not a generalization of G-metric and controvisely. Recently, N.Mlaiki and N.Souayah[8] introduced the S_b-metric spaces as the generalization of b-metric spaces and S-metric spaces and proved some fixed point results were proved for such spaces in [8]. Very recently Ozur and Tas[5] studied some relations between S_b-metric spaces and some other metric spaces. Fixed points of contractive maps on S-metric spaces were studied in [2,3,7,11-15] and some fixed point results in S_b-metric space were also studied by different authors in [5,6,8].

In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)- generalized weakly contractive maps in S-metric spaces and established the existence and uniqueness of fixed point theorem for such maps. By the motivation of B.K.Leta and G.V.R.Babu[3] results in S-metric spaces, we introduced the (ψ, ϕ) - generalized weakly contractive map in S_b-metric spaces and prove a existence and uniqueness of fixed point theorem. Let us see some basic definitions, Examples and Lemmas for the sake of transparency.

2. PRELIMINARIES

Definition 2.1.[7] Let $X \neq \emptyset$, then a mapping S:$X^3 \rightarrow [0, \infty)$ is said to be an S-metric on X if:
(S1) $S(\xi, \vartheta, w) > 0$ for all $\xi, \vartheta, w \in X$ with $\xi \neq \vartheta \neq w$.
(S2) $S(\xi, \vartheta, w)$ = 0 if $\xi = \vartheta = w$.
(S3) $S(\xi, \vartheta, w) \leq [S(\xi, \xi, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$
for any $\xi, \vartheta, w, a \in X$. Then we call (X, S) is an S-metric space.
Example 2.1.[13] Suppose X=R, Collection of all real numbers and let $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X,S) becomes a S-metric space.
Definition 2.2.[5] Let $X \neq \emptyset$ and s$\geq$1. Then we say that a function
$d : X^2 \rightarrow [0, \infty)$ is a b-metric on X if
(i) $d(\xi, \vartheta) = 0 \iff \xi = \vartheta$.
(ii) $d(\xi, \vartheta) = d(\vartheta, \xi)$ for all $\xi, \vartheta \in X$.
(iii) $d(\xi, \vartheta) \leq s[d(\xi, w) + d(w, \vartheta)]$, for all $\xi, \vartheta, w \in X$.
The pair (X,d) is known as b-metric space with s$\geq$1.
Definition 2.3.[1] Let $X \neq \emptyset$ and s$\geq$1. Then we say a mapping $S_b : X^3 \rightarrow [0, \infty)$ is S_b-metric on X if :
(i) $S_b(\xi, \vartheta, w)$ = 0 if $\xi = \vartheta = w$.

(ii)$S_b(\xi, \vartheta, w) \leq s[S_b(\xi, \xi, a) + S_b(\vartheta, \vartheta, a) + S_b(w, w, a)]$
$\forall \xi, \vartheta, w, a \in X$. The pair (X, S_b) is known as S_b-metric space.
Each S-metric space is a S_b-metric space for s=1, but the converse statement is not true. We find an example of S_b-metric, but not an S-metric on X in [5].
Definition 2.4.[1] Consider (X, S_b) be a S_b-metric space for s>1. Then S_b-metric is known as symmetric if $S_b(\xi, \xi, \vartheta) = S_b(\vartheta, \vartheta, \xi), \forall \xi, \vartheta \in X$.
Lemma 2.1.[4] In S_b-metric space, we have
(i) $S_b(\xi, \xi, \vartheta) \leq sS_b(\vartheta, \vartheta, \xi)$ and $S_b(\vartheta, \vartheta, \xi) \leq sS_b(\xi, \xi, \vartheta)$
(ii) $S_b(\xi, \xi, w) \leq 2sS_b(\xi, \xi, \vartheta) + s^2 S_b(\vartheta, \vartheta, w)$.
Definition 2.5.[4] If (X, S_b) is an S_b-metric space and a sequence $\{\xi_n\}$ in X. Then
(i) $\{\xi_n\}$ is called a S_b-Cauchy sequence, if to every $\epsilon > 0$, $\exists\ n_0 \in N$ so that $S_b(\xi_n, \xi_n, \xi_m) \leq \epsilon, \forall n, m > n_0$.
(ii) $\{\xi_n\} \to \xi \iff$ to each $\epsilon > 0$, $\exists\ n_0 \in N$ such that $S_b(\xi_n, \xi_n, \xi) < \epsilon$ and $S_b(\xi, \xi, \xi_n) < \epsilon\ \forall n \geq n_0$, and we write as $\lim_{n\to\infty} \xi_n = \xi$.
Definition 2.6.[4] We say that (X, S_b) is complete if each S_b-Cauchy sequence is S_b-Convergent in X.
Tas and Ozgur[5] proved the following theorems in S_b-metric spaces.
Theorem 2.1.[5] Consider (X, S_b) be a complete S_b-metric space and s ≥ 1. If h is a self map on X satisfying

$$S_b(h\xi, h\xi, h\vartheta) \leq c\, S_b(\xi, \xi, \vartheta)\ \ \forall \xi, \vartheta \in X,\ \ where\ \ 0 < c < \tfrac{1}{s^2}.$$

Then h has a unique fixed point ξ in X.
In this article we indicate:
(i)$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi$ is non decreasing, continuous and ψ(t)=0 $\iff$ t=0.$\}$
(ii) $\Phi = \{\phi : [0, \infty) \to [0, \infty)$: ϕ is continuous, $\phi(t) = 0 \iff t = 0\}$.

In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)- generalized weakly contractive maps in S-metric spaces and proved existence and uniqueness of fixed point theorem for such maps as follows.
Definition 2.7[3] Consider (X,S) be an S-metric space and h be a self map on X. Suppose that $\exists\ \alpha \in (0, 1)$, $\psi \in \Psi$ and $\phi \in \Phi$ so that
$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(P_\alpha(\xi, \vartheta, w)) - \phi(P_\alpha(\xi, \vartheta, w))$ (2.1)
where $P_\alpha(\xi, \vartheta, w)$ = max$\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw)$,
$\alpha S(h\xi, h\xi, \vartheta) + (1 - \alpha)S(h\vartheta, h\vartheta, w)\}, \forall \xi, \vartheta, w \in X$.
Then h is called a (α, ψ, ϕ)- generalized weakly contractive map on X.
Theorem 2.2.[3] Let h be a self map on a complete S-metric space (X,S) and h satisfies (α, ψ, ϕ)- generalized weakly contractive map. Then h have a unique fixed point in X.

Lemma 2.2.[6] Let $\{\xi_n\}$ is S_b-convergent to ξ in S_b-metric space (X, S_b) for s$\geq$1, then we obtain:
(i)$\frac{1}{2s}S_b(\vartheta,\vartheta,\xi) \leq \liminf_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq \limsup_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq 2sS_b(\vartheta,\vartheta,\xi)$ and
(ii)$\frac{1}{s^2}S_b(\xi,\xi,\vartheta) \leq liminf_{n\to\infty}S_b(\xi_n,\xi_n,\vartheta) \leq \limsup_{n\to\infty} S_b(\xi_n,\xi_n,\vartheta) \leq s^2S_b(\xi,\xi,\vartheta)$.
Lemma 2.3.[2] Let $\{\xi_n\}$ be a sequence in S_b-metric space (X,S_b) so that $\lim_{n\to\infty} S_b(\xi_n,\xi_n,\xi_{n+1})$ = 0.
If sequence $\{\xi_n\}$ is not Cauchy, then we find an $\epsilon > 0$ and $\{m_k\}$ and $\{n_k\}$ are sequences of natural numbers with $n_k > m_k > k$ so that $S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) \geq \epsilon$, $S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) < \epsilon$ and
(i) $\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) = \epsilon$. (ii) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k-1}) = \epsilon$. (ii) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}) = \epsilon$.
In this article, we define (α,ψ,ϕ)-almost generalized weakly contractive maps in S_b-metric spaces and establish the existence and uniqueness of fixed point of maps. Also, we draw some corollaries and provide an example in support of our results.

3. MAIN RESULTS

Definition 3.1. Let (X, S_b) be an S_b-metric space for s$\geq$1. Let h be a self map of X. Then we say h be a (ψ,ϕ)-generalized weakly contractive map if $\exists$ L$\geq$0, $\psi \in \Psi$ and $\phi \in \Phi$ such that
$\psi(4s^4S_b(h\xi,h\vartheta,hw)) \leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w)$ (3.1.)
where P(ξ,ϑ,w) = max$\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}$
and Q(ξ,ϑ,w) = min$\{S_b(hw,\xi,\xi), S_b(h\xi,\vartheta,\vartheta), S_b(h\xi,w,w), S_b(h\xi,\vartheta,w)\}$
$\forall\xi,\vartheta,w \in X$.
Example 3.1. Consider (X, S_b) be a complete S_b-metric space for s=4, where $X = [0,\frac{7}{3}]$ and $S_b : X^3 \to$ R is defined by
$S_b(\xi,\vartheta,w) = \frac{1}{16}[|\xi-\vartheta| + |\vartheta-w| + |w-\xi|]^2$, $\forall\xi,\vartheta,w \in X$.
We define a self map h on X by

$$h\xi = \begin{cases} \frac{1}{8} & \text{if } \xi \in [0,2] \\ \frac{\xi}{16} - \frac{1}{32} & \text{if } \xi \in (2,\frac{7}{3}] \end{cases}.$$

Also, Consider $\phi,\psi : [0,\infty) \to [0,\infty)$ be two functions defined by ψ(t) = t and $\phi(t) = \frac{t}{4}$ for all t$\in [0,\infty)$.
Now, we verify the inequality (3.1.)
case(i) when $\xi,\vartheta,w \in$ [0,2], we have $\psi(4s^4S_b(h\xi,h\vartheta,hw))$ = 0.
Then inequality (3.1.) holds good.

case(ii) Let $\xi,\vartheta,w \in (2,\frac{7}{3}]$. Suppose that $\xi > \vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16}-\frac{\vartheta}{16}|+|\frac{\vartheta}{16}-\frac{w}{16}|+|\frac{w}{16}-\frac{\xi}{16}|]^2\\
&\leq \frac{4^5}{16}[3|\frac{\xi}{16}-\frac{w}{16}|]^2\\
&\leq \frac{9}{4}|\xi-w|^2=\frac{1}{4}\\
&\leq \frac{15123}{16384}=\frac{5041}{4096}-\frac{5041}{16384}\\
&= \frac{3}{4}S_b(\xi,\xi,h\xi)\leq \frac{3}{4}P(\xi,\vartheta,w)\\
&= P(\xi,\vartheta,w)-\frac{1}{4}P(\xi,\vartheta,w)\\
&= \psi(P(\xi,\vartheta,w))-\phi(P(\xi,\vartheta,w)).
\end{aligned}$$

case(iii) When $\xi,\vartheta \in$[0,2] and $\in (2,\frac{7}{3}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5S_b(\frac{1}{8},\frac{1}{8},\frac{w}{16}-\frac{1}{32})\\
&= \frac{4^5}{16}[|\frac{1}{8}-(\frac{w}{16}-\frac{1}{32})|+|\frac{w}{16}-\frac{1}{32}-\frac{1}{8}|]\\
&= \frac{4^5}{16}[2|\frac{1}{8}-\frac{w}{16}+\frac{1}{32}|]^2\\
&= \frac{1}{4}[5-2w]^2\\
&\leq \frac{1}{4}\leq \frac{15123}{16384}=\frac{5041}{4096}-\frac{5041}{16384}\\
&= S_b(w,w,hw)-\frac{1}{4}S_b(w,w,hw)\\
&= P(\xi,\vartheta,w)-\frac{1}{4}P(\xi,\vartheta,w)\\
&= \psi(P(\xi,\vartheta,w))-\phi(P(\xi,\vartheta,w)).
\end{aligned}$$

case(iv) When $\vartheta, w \in [0,2]$ and $\xi \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{1}{8}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{\xi}{16} - \frac{1}{32} - \frac{1}{8}| + |0| + |\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= 4^4[\frac{5-2\xi}{32}]^2 = \frac{1}{4}[5-2\xi]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= \frac{3}{4}S_b(\xi, \xi, h\xi) \leq \frac{3}{4}P(\xi, \vartheta, w) \\
&= P(\xi, \vartheta, w) - \frac{1}{4}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(v) When w$\in [0,2]$ and $\xi, \vartheta \in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[\frac{\xi - \vartheta}{16} + \frac{2\vartheta - 5}{32} + \frac{5 - 2\xi}{32}]^2 \\
&= \frac{4^5}{16}[\frac{10 - 4\vartheta}{32}]^2 \\
&\leq \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4}S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vi) When ξ ∈[0,2] and $\vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[\frac{5-2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5-2w}{32}]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vii) When ξ, w ∈[0,2] and $\vartheta \in (2, \frac{7}{3}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2\vartheta)]^2 \\
&= \frac{1}{4}[5-2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(viii) When ξ ∈[0,2] and $\vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2 \\
&= \frac{1}{4}[5-2w]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w,w,hw) - \frac{1}{4}S_b(w,w,hw) \\
&= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w) \\
&= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)).
\end{aligned}$$

Therefore h satisfies (ψ, ϕ) - generalized weakly contractive map.

Theorem 3.1. Suppose h be a self map on a complete symmetric S_b-metric space (X, S_b) for s≥1. If h be a (ψ, ϕ) - generalized weakly contraction map, then h has a unique fixed point in X.

Proof. Let $\xi_0 \in X$ and define a sequence $\{\xi_n\}$ in X by $\xi_n = h\xi_{n-1}$, for n =1,2,3...
Suppose $\xi_{n-1} = \xi_n$ to some n, then h has a fixed point ξ_{n-1}.
Now, we suppose that $\xi_{n-1} \neq \xi_n$, ∀ $n \in$ N.
By choosing $\xi = \vartheta = \xi_{n-2}, w = \xi_{n-1}$ in (3.1.), we obtain

$$\psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)) \leq \psi(4s^4 S_b(h\xi_{n-2}, h\xi_{n-2}, h\xi_{n-1}))$$
$$\leq \psi(P(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})) - \phi(P(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})) + L.Q(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}) \quad (3.2.)$$

where

$$\begin{aligned}
&P(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}) = \\
&\max\{S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), S_b(\xi_{n-2}, \xi_{n-2}, h\xi_{n-2}), S_b(\xi_{n-2}, \xi_{n-2}, h\xi_{n-2}), \\
&\quad S_b(\xi_{n-1}, \xi_{n-1}, h\xi_{n-1}), \tfrac{1}{4s^2}[S_b(h\xi_{n-2}, h\xi_{n-2}, h\xi_{n-1}) + \\
&\quad S_b(h\xi_{n-2}, h\xi_{n-2}, \xi_{n-2}) S_b(h\xi_{n-2}, h\xi_{n-2}, \xi_{n-1}) S_b(h\xi_{n-1}, h\xi_{n-1}, \xi_{n-2})]\}) \\
&= \max\{S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), \\
&\quad S_b(\xi_{n-1}, \xi_{n-1}, \xi_n), \tfrac{1}{4s^2}[S_b(\xi_{n-1}, \xi_{n-1}, \xi_n) + \\
&\quad S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}) S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-1}) S_b(\xi_n, \xi_n, \xi_{n-2})]\}) \\
&= \max\{S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), S_b(\xi_{n-1}, \xi_{n-1}, \xi_n), \tfrac{1}{4s^2} S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)\} \\
&= max\{S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)\} \qquad (3.3.)
\end{aligned}$$

and

$Q(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})$= min$\{S_b(h\xi_{n-1}, \xi_{n-2}, \xi_{n-2}), S_b(h\xi_{n-2}, \xi_{n-2}, \xi_{n-2}),$

$$S_b(h\xi_{n-2},\xi_{n-1},\xi_{n-1}),S_b(h\xi_{n-2},\xi_{n-2},\xi_{n-1})\}$$
$$=\min\{S_b(\xi_n,\xi_{n-2},\xi_{n-2}),S_b(\xi_{n-1},\xi_{n-2},\xi_{n-2}),$$
$$S_b(\xi_{n-1},\xi_{n-1},\xi_{n-1}),S_b(\xi_{n-1},\xi_{n-2},\xi_{n-1})\}$$
$$=0. \qquad (3.4.)$$

If $S_b(\xi_{n-1},\xi_{n-1},\xi_n)$ is the maximum in (3.3.) and using (3.4.) and (3.2.), we get
$\psi(S_b(\xi_{n-1},\xi_{n-1},\xi_n))\leq\psi(S_b(\xi_{n-1},\xi_{n-1},\xi_n))-\phi(S_b(\xi_{n-1},\xi_{n-1},\xi_n))$.
This implies $\phi(S_b(\xi_{n-1},\xi_{n-1},\xi_n))$ = 0. Therefore, $\xi_{n-1}=\xi_n$, is a contradiction to our assumption. Thus,

$$\psi(S_b(\xi_{n-1},\xi_{n-1},\xi_n))\leq$$
$$\psi(S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}))-\phi(S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1})). \qquad (3.5.)$$
$$<\psi(S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1})).$$

By the definition of ψ, we have
$S_b(\xi_{n-1},\xi_{n-1},\xi_n)<S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1})$.
Thus, $\{S_b(\xi_{n-1},\xi_{n-1},\xi_n)\}$ be a positive real of strictly decreasing sequence.
Then we find a r$\geq$0 so that $\lim_{n\to\infty}S_b(\xi_{n-1},\xi_{n-1},\xi_n)$ = r.
Taking n$\to\infty$ in (3.5.), we obtain
$\psi(r)\leq\psi(r)-\phi(r)$. This implies $\phi(r)=0$. Hence r = 0. Thus,

$$\lim_{n\to\infty}S_b(\xi_{n-1},\xi_{n-1},\xi_n)=0. \qquad (3.6.)$$

By choosing $\xi=\vartheta=\xi_{n-1},w=\xi_{n-2}$ in (3.1.), we get

$$\psi(S_b(\xi_n,\xi_n,\xi_{n-1}))\leq\psi(4s^4S_b(h\xi_{n-1},h\xi_{n-1},h\xi_{n-2}))$$
$$\leq\psi(P(\xi_{n-1},\xi_{n-1},\xi_{n-2}))-\phi(P(\xi_{n-1},\xi_{n-1},\xi_{n-2}))+L.Q(\xi_{n-1},\xi_{n-1},\xi_{n-2}) \qquad (3.7.)$$

where

$$P(\xi_{n-1},\xi_{n-1},\xi_{n-2})$$
$$=\max\{S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2}),S_b(\xi_{n-1},\xi_{n-1},h\xi_{n-1}),S_b(\xi_{n-1},\xi_{n-1},h\xi_{n-1}),$$
$$S_b(\xi_{n-2},\xi_{n-2},h\xi_{n-2}),\tfrac{1}{4s^2}[S_b(h\xi_{n-1},h\xi_{n-1},h\xi_{n-2})+$$
$$S_b(h\xi_{n-1},h\xi_{n-1},\xi_{n-1})S_b(h\xi_{n-1},h\xi_{n-1},\xi_{n-2})S_b(h\xi_{n-2},h\xi_{n-2},\xi_{n-1})]\}$$
$$=\max\{S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2}),S_b(\xi_{n-1},\xi_{n-1},\xi_n),S_b(\xi_{n-1},\xi_{n-1},\xi_n),$$
$$S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}),\tfrac{1}{4s^2}[S_b(\xi_n,\xi_n,\xi_{n-1})+$$
$$S_b(\xi_n,\xi_n,\xi_{n-1})S_b(\xi_n,\xi_n,\xi_{n-2})S_b(\xi_{n-1},\xi_{n-1},\xi_{n-1})]\}$$
$$=\max\{S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2}),S_b(\xi_n,\xi_n,\xi_{n-1}),\tfrac{1}{4s^2}S_b(\xi_n,\xi_n,\xi_{n-1})\}$$
$$=max\{S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2}),S_b(\xi_n,\xi_n,\xi_{n-1})\} \qquad (3.8.)$$

and

$$Q(\xi_{n-1},\xi_{n-1},\xi_{n-2})=\min\{S_b(h\xi_{n-2},\xi_{n-1},\xi_{n-1}),S_b(h\xi_{n-1},\xi_{n-1},\xi_{n-1}),$$
$$S_b(h\xi_{n-1},\xi_{n-2},\xi_{n-2}),S_b(h\xi_{n-1},\xi_{n-1},\xi_{n-2})\}$$
$$=\min\{S_b(\xi_{n-1},\xi_{n-1},\xi_{n-1}),S_b(\xi_n,\xi_{n-1},\xi_{n-1}),$$
$$S_b(\xi_n,\xi_{n-2},\xi_{n-2}),S_b(\xi_n,\xi_{n-1},\xi_{n-2})\}$$
$$=0. \qquad (3.9.)$$

If $S_b(\xi_n,\xi_n,\xi_{n-1})$ is maximum in (3.8.) and using (3.7.) and (3.9.), we get

$\psi(S_b(\xi_n, \xi_n, \xi_{n-1})) \leq \psi(S_b(\xi_n, \xi_n, \xi_{n-1})) - \phi(S_b(\xi_n, \xi_n, \xi_{n-1}))$+L.0
This implies $\phi(S_b(\xi_n, \xi_n, \xi_{n-1}))$ = 0. Hence, $\xi_n = \xi_{n-1}$, is a contradiction to our assumption.
Thus

$$\psi(S_b(\xi_n, \xi_n, \xi_{n-1})) \leq \psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})) - \phi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})) \quad (3.10.)$$
$$\leq \psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}))$$

From the definition of ψ, we obtain
$S_b(\xi_n, \xi_n, \xi_{n-1}) < S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})$.
Thus, $\{S_b(\xi_n, \xi_n, \xi_{n-1})\}$ be a positive reals of strictly decreasing sequence.
Hence, we can find r ≥ 0 so that
$\lim_{n\to\infty} S_b(\xi_n, \xi_n, \xi_{n-1})$ = r.
Taking $n \to \infty$ in (3.10.), we obtain
$\psi(r) \leq \psi(r) - \phi(r)$. This implies $\phi(r)$ = 0. Therefore r = 0. Thus,
$\lim_{n\to\infty} S_b(\xi_n, \xi_n, \xi_{n-1})$ = 0.
Now we verify that $\{\xi_n\}$ is a S_b-cauchy sequence in X.
Suppose that sequence $\{\xi_n\}$ is not a S_b- cauchy, $\exists\ \epsilon > 0$ and monotone increasing sequence of real numbers $m(\kappa)$ and $n(\kappa)$ with $n(\kappa) > m(\kappa) > \kappa$ so that $S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}) \geq \epsilon$ and $S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-2}) < \epsilon$. (3.11.)
Now from (3.1.),(3.7) and (3.11.), we have

$$\psi(4s^4\epsilon) \leq \psi(4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) =$$
$$\psi(4s^4 S_b(h\xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2}, h\xi_{n(\kappa)-2}))$$
$$\leq \psi(P(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})) - \phi(P(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}))$$
$$+ L.Q(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})$$

where

$$P(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})$$
$$= \max\{S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}), S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2}),$$
$$S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2}), S_b(\xi_{n(\kappa)-2}, \xi_{n(\kappa)-2}, h\xi_{n(\kappa)-2}),$$
$$\tfrac{1}{4s^2}[S_b(h\xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2}, h\xi_{n(\kappa)-2})$$
$$+ S_b(h\xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}) S_b(h\xi_{m(\kappa)-2}, h\xi_{m(\kappa)-2},$$
$$\xi_{n(\kappa)-2}) S_b(h\xi_{n(\kappa)-2}, h\xi_{n(\kappa)-2}, \xi_{m(\kappa)-2})]\})$$
$$= \max\{S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}), S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-1}),$$
$$S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-1}), S_b(\xi_{n(\kappa)-2}, \xi_{n(\kappa)-2}, \xi_{n(\kappa)-1}), \tfrac{1}{4s^2}[S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})$$
$$+ S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{m(\kappa)-2}) S_b(\xi_{m(\kappa)-1},$$
$$\xi_{m(\kappa)-1}, \xi_{n(\kappa)-2}) S_b(\xi_{n(\kappa)-1}, \xi_{n(\kappa)-1}, \xi_{m(\kappa)-2})]\})$$

As $\kappa \to \infty$

$$lim_{n\to\infty} A(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}) = \max\{S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}),$$
$$\tfrac{1}{4s^2} S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})\}.$$

and
$Q(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})$ =
$\min\{S_b(h\xi_{n(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}), S_b(h\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}),$
$S_b(h\xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}, \xi_{n(\kappa)-2}), S_b(h\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})\}.$
= 0.
If $\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})$ is maximum,
$\psi(4s^4S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) \leq$
$\psi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) - \phi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}))$
This implies
$\psi(4s^4S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) < \psi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}))$
From the property of ψ, we have
$4s^4S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}) < \frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})$
This gives rise to
$4s^4 < \frac{1}{4s^2} \Rightarrow 16s^6 < 1$, a contradiction as s ≥ 1.
Therefore, we have
$\psi(4s^4S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) \leq$
$\psi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})) - \phi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}))$
$< \psi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}))$
Now using lemma(2.1), we have
$4s^4S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}) \leq$
$2sS_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-1}) + s^2S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-2}).$
Letting $\kappa \to \infty$, we get
$4s^4\epsilon \leq s^2\epsilon$, a contradiction as s$\geq$1.
Hence $\{\xi_n\}$ be a S_b-Cauchy sequence of complete space X, $\exists \tau \in X$ so that $\lim_{n\to\infty} \xi_n = \tau$.
Now we show that $h\tau = \tau$. Suppose that $h\tau \neq \tau$. Then by lemma (2.2.), we have
$\frac{1}{2s}S_b(f\tau, f\tau, \tau) \leq \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
This implies
$\frac{4s^4}{2s}S_b(f\tau, f\tau, \tau) \leq 4s^4 \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
$\leq 4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
Thus
$2s^3S_b(h\tau, h\tau, \tau) \leq 4s^4 \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
$\leq 4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
From the property of ψ, we have
$\psi(2s^3S_b(h\tau, h\tau, \tau)) \leq \psi(4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n))$
$\leq$
$\psi(\limsup_{n\to\infty} P(\tau, \tau, \xi_n)) - \phi(\limsup_{n\to\infty} P(\tau, \tau, \xi_n)) + L(\limsup_{n\to\infty} Q(\tau, \tau, \xi_n))$
Now

$P(\tau,\tau,\xi_n)$ = max$\{S_b(\tau,\tau,\xi_n), S_b(\tau,\tau,h\tau), S_b(\tau,\tau,h\tau), S_b(\xi_n,\xi_n,h\xi_n),$
$\frac{1}{4s^2}[S_b(h\tau,h\tau,h\xi_n) + S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,\xi_n)S_b(h\xi_n,h\xi_n,\tau)]\}$
= max $\{S_b(\tau,\tau,h\tau), \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)\}$
$Q(\tau,\tau,\xi_n) = min\{S_b(h\xi_n,\tau,\tau), S_b(h\tau,\tau,\tau), S_b(h\tau,\xi_n,\xi_n), S_b(h\tau,\tau,\xi_n)\}$
= 0
If $\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)$ is maximum
$\psi(2s^3S_b(h\tau,h\tau,\tau)) \leq \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) - \phi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) + L.0$
$< \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau))$
From the property of ψ, we have
$2s^3S_b(h\tau,h\tau,\tau) < \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)$
this implies
$8s^5 < 1$, a contradiction. Therefore
$\psi(2s^3S_b(h\tau,h\tau,\tau)) \leq \psi(S_b(\tau,\tau,h\tau)) - \phi(S_b(\tau,\tau,h\tau)) + L.0$
$\Rightarrow \psi(2s^3S_b(h\tau,h\tau,\tau)) < \psi(S_b(\tau,\tau,h\tau)).$ (3.12.)
If $\tau \neq h\tau$, in (3.12.), we have
$2s^3S_b(h\tau,h\tau,\tau) < S_b(\tau,\tau,h\tau) \leq sS_b(h\tau,h\tau,\tau)$
which implies
$2s^2 < 1$, is a contradiction. Therefore, $h\tau = \tau$.
Now, we show that τ is unique.
Let τ and j be two distinct fixed points of h.
Now, consider
$\psi(S_b(\tau,\tau,j)) = \psi(S_b(h\tau,h\tau,hj))$
$\leq \psi(4s^4S_b(h\tau,h\tau,hj))$ (3.13.)
$\leq \psi(P(\tau,j,j)) - \phi(P(\tau,j,j)) + L.Q(\tau,j,j)$
where
$P(\tau,j,j)$ = max$\{S_b(\tau,j,j), S_b(\tau,\tau,h\tau), S_b(j,j,hj), S_b(j,j,hj),$
$\frac{1}{4s^4}[S_b(h\tau,hj,hj) + S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,j)S_b(hj,hj,j)]\}$
$= \{S_b(\tau,j,j), \frac{1}{4s}S_b(\tau,j,j)\} = S_b(\tau,j,j)$ (3.14.)
and $Q(\tau,j,j) = min\{S_b(fj,\tau,\tau), S_b(f\tau,j,j), S_b(f\tau,f\tau,j), S_b(fj,fj,j)\}$
$= 0$ (3.15.)
From (3.13.),(3.14.) and (3.15.) we get
$\psi(\frac{1}{4s^4}S_b(\tau,j,j)) \leq \psi(S_b(\tau,j,j)) - \phi(S_b(\tau,j,j)) + L.0$
$< \psi(S_b(\tau,j,j)).$
From the property of ψ, we have $4s^4 < 1$, a contradiction.
There fore, we get $S_b(\tau,j,j) = 0$
Hence τ=j. Hence τ is the unique fixed point of h.

In the Theorem (3.1.), if we substitue L=0, we get the following.

Corollary 3.1. Let h be a self map of X and here X is an S_b-metric space. Suppose $\exists$ $\phi \in \Phi$ and $\psi \in \Psi$ so that $\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w))$
where $P(\xi, \vartheta, w) = \max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$
$\frac{1}{4s^2}[S_b(h\xi, h\vartheta, hw) + S_b(h\xi, h\xi, \xi)S_b(h\xi, h\xi, w)S_b(hw, hw, \vartheta)]\}$.
$\forall \xi, \vartheta, w \in X$. Then h contains unique fixed point in X.

If ψ is the identity map in the Corollary (3.1.), we get a Corollary as follows.
Corollary 3.2. Let h be a self map of X and here X is an S_b-metric space. Suppose there exists $\phi \in \Phi$ so that $4s^4 S_b(h\xi, h\vartheta, hw) \leq P(\xi, \vartheta, w) - \phi(P(\xi, \vartheta, w))$
where $P(\xi, \vartheta, w) = \max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$
$\frac{1}{4s^2}[S_b(h\xi, h\vartheta, hw) + S_b(h\xi, h\xi, \xi)S_b(h\xi, h\xi, w)S_b(hw, hw, \vartheta)]\}$.
$\forall\ \xi, \vartheta, w \in X$. Then h contains unique fixed point in X.

If we substitute P(ξ, ϑ, w) = P*(ξ, ϑ, w) in the Theorem (3.1.), we obtain the following.
Corollary 3.3. Let h be a self map of X and here X is an S-metric space. Suppose $\exists$ $\phi \in \Phi$ and $\psi \in \Psi$ so that $\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P^*(\xi, \vartheta, w)) - \phi(P^*(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w)$
where P*(ξ, ϑ, w) = $\max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$
$\frac{S_b(\xi,\xi,h\xi)S_b(\vartheta,\vartheta,h\vartheta)}{1+S_b(\xi,\xi,h\xi)+S_b(\xi,\vartheta,w)}, \frac{S_b(\xi,\xi,h\xi)S_b(w,w,hw)}{1+S_b(w,w,hw)+S_b(\xi,\vartheta,w)},$
$\frac{1}{4s^2}[S_b(h\xi, h\vartheta, hw) + S_b(h\xi, h\xi, \xi)S_b(h\xi, h\xi, w)S_b(hw, hw, \vartheta)]\}$.
and Q(ξ, ϑ, w) = $\min\{S_b(hw, \xi, \xi), S_b(h\xi, \vartheta, \vartheta), S_b(h\xi, w, w), S_b(h\xi, \vartheta, w)\}$
$\forall \xi, \vartheta, w \in X$. Then h contains unique fixed point in X.

In Theorem (3.1.), if we put s=1, we get the following.
Corollary 3.4. Let h be a self map of X and here X is an S-metric space. Suppose that $\exists$ L$\geq 0, \phi \in \Phi$ and $\psi \in \Psi$ so that
$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w)$
where $P(\xi, \vartheta, w) = \max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw),$
$\frac{1}{2}[S(h\xi, h\vartheta, hw) + S(h\xi, h\xi, \xi)S(h\xi, h\xi, w)S(hw, hw, \vartheta)]\}$
and $Q(\xi, \vartheta, w) = \min\{S(hw, \xi, \xi), S(h\xi, \vartheta, \vartheta), S(h\xi, w, w), S(h\xi, \vartheta, w)\}$
$\forall \xi, \vartheta, w \in X$. Then h contains unique fixed point in X.

Example 3.2. Consider $X = [0, \frac{12}{5}]$ and define $S : X^3 \to$ R by
$S_b(\xi, \vartheta, w) = \frac{1}{16}[|\xi - \vartheta| + |\vartheta - w| + |w - \xi|]^2, \forall \xi, \vartheta, w \in X$. Then (X, S_b) is a complete S_b-metric space for s=4.
We define a self map h on X by

hξ = $\begin{cases} \frac{1}{8} & \text{if } \xi \in [0,2] \\ \frac{\xi}{16} - \frac{1}{32} & \text{if } \xi \in (2, \frac{12}{5}] \end{cases}$.

Also, Consider $\phi, \psi : [0,\infty) \to [0,\infty)$ be two functions defined by ψ(t) = t and $\phi(t) = \frac{t}{3}$, for any t $\in [0,\infty)$.

Now, we validate the inequality (3.1.).

case(i) when $\xi, \vartheta, w \in$ [0,2], we have $\psi(4s^4 S_b(h\xi, h\vartheta, hw))$ = 0.

Then inequality (3.1.) holds good.

case(ii) Let $\xi, \vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16} - \frac{\vartheta}{16}| + |\frac{\vartheta}{16} - \frac{w}{16}| + |\frac{w}{16} - \frac{\xi}{16}|]^2 \\
&\le \frac{4^5}{16}[3|\frac{\xi}{16} - \frac{w}{16}|]^2 \\
&\le \frac{9}{4}|\xi - w|^2 = \frac{9}{25} \\
&\le \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(\xi, \xi, h\xi) - \frac{1}{3} S_b(\xi, \xi, h\xi) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(iii) When $\xi, \vartheta \in$[0,2] and w$\in (2, \frac{12}{5}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{w}{16} - \frac{1}{32})| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - \frac{w}{16} + \frac{1}{32}|]^2 \\
&= \frac{1}{4}[5 - 2w]^2 \\
&\le \frac{1}{4} \le \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(iv) When $\vartheta, w \in [0,2]$ and $\xi \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{1}{8}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{\xi}{16} - \frac{1}{32} - \frac{1}{8}| + |0| + |\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= 4^4[\frac{5-2\xi}{32}]^2 = \frac{1}{4}[5-2\xi]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= \frac{2}{3} S_b(\xi, \xi, h\xi) \leq \frac{2}{3} P(\xi, \vartheta, w) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(v) When w$\in [0,2]$ and $\xi, \vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[\frac{\xi - \vartheta}{16} + \frac{2\vartheta - 5}{32} + \frac{5 - 2\xi}{32}]^2 \\
&= \frac{4^5}{16}[\frac{10 - 4\vartheta}{32}]^2 \\
&\leq \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vi) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[\frac{5 - 2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5 - 2w}{32}]^2 \\
&= \frac{4^5}{16} \cdot \frac{1}{32^2}[2(5 - 2w)]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vii) When $\xi, w \in$[0,2] and $\vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2 \\
&= \frac{4^5}{16} \cdot \frac{1}{32^2}[2(5 - 2\vartheta)]^2 \\
&= \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(viii) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2 \\
&= \frac{1}{4}[5-2w]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w,w,hw) - \frac{1}{3}S_b(w,w,hw) \\
&= P(\xi,\vartheta,w) - \frac{1}{3}P(\xi,\vartheta,w) \\
&= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)).
\end{aligned}$$

Hence the conditions of Theorem (3.1.) are satisfied by h and also $\frac{1}{8}$ is the unique fixed point of h.

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