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Traces of the Idempotent of K[G], when *char* K is not equal to 0

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Abstract

As we know that the trace *e* of idempotent element of group ring K[G] lies in open interval]0, 1[when the characteristic of field K is zero. But in this research paper we have extended the thought of group ring K[G] as the characteristic of field K as positive. Because by doing so we do not keep restricted our self as the characteristic of field K is zero. When we assume so means group ring structure K[G] with field as characteristic positive, then trace *e* of the idempotent element of K[G] lies in the prime subfield and so produces trace *e* in]0, 1[. Thus here we choose only algebraic polynomials which give trace *e* in real range]0, 1[. But when we pick a transcendental polynomial on this mapping φ to complex group ring structure C[G] then trace *e* of idempotent *e* as trace *e* lies outside the real range of field K. We have taken the help from very important theorem of Zalesskii and commutator algebra for a ring to get our result in this research paper.

Keywords: Group ring K[G], idempotent element, trace *e* of idempotent element, Zalesskii's theorem, prime subfield of field K, commutator algebra in ring theory.

1. Introduction:

If we consider a mapping φ from K[x] to C[G] in which *char* K=0 then the idempotent of K[G] lies strictly between 0 and 1. Similarly if this mapping φ from rational field Q[x] to C[G] then it also produces trace *e* in]0, 1[. Thus we choose only algebraic polynomials which give trace e in real range]0, 1[. But when we pick a transcendental polynomial on this mapping $\emptyset$ to complex group ring structure C[G] then trace e of idempotent e as trace e lies outside the real range.

2. Zalesskii's Theorem:

If e be the idempotent element of group ring K[G], then trace e lies in the prime subfield of K.

Remark: If we think upon the commutative algebra of characteristic $p>0$ as prime positive then we have, $(a+b)^p = a^p + b^p$. As we know that in commutator subspace [A, A], an algebra A becomes the linear span of commutators xy-yx in which x, y ∈ A. Here we take A = K[G] as a group ring over a field K and group G, then trace e (xy-yx) = trace e (xy)- trace e(yx) =0. Thus we can say that [A, A] means [K[G], K[G]] is contained in the kernel of trace e.

3. Lemma:

If $a_1, a_2, \ldots\ldots, a_k$ are the elements of an algebra A over a field K, with characteristic $p>0$, then if we choose $q=p^n$, as n is a positive integer, we have

$(a_1 + a_2 + a_3 \ldots\ldots\ldots\ldots\ldots\ldots. + a_k)^q = a_1^q + a_2^q \ldots\ldots\ldots. + a_k^q$ mod [A, A].

Now, we rewrite the Zalesskii's theorem as follow in fields of positive characteristic.

4. Theorem:

If e be the idempotent of K[G], in which field K has characteristic $p>0$. Then the trace e will be contained in the prime subfield of K.

Proof: Let us suppose that e be an idempotent of K[G] and we can write idempotent e as follow,

$e = \sum_{x\in G} a_x . x \in$ K[G]. Again we can rewrite idempotent $e = t + \sum_{x\in P} a_x$.x, here P⊂G and set P has elements of order as power of p.

We choose $q = p^n$ for some integer $n>0$ then from the previous lemma $e = e^q = (t + \sum_{x\in P} a_x . x)^q$ mod [K[G],K[G]].

Now we take n large enough in $q=p^n$ then we have $x^q = 1$ ∀ x∈ G.

Again we have $t = \sum_{x\notin P} a_x x$ but no any element of group G in the support of b whose order is the power of p. So, we can write $t^q = \sum_{x\notin P} a_x^q x^q$ mod [K[G], K[G]] and no $x^q = 1_G$ as well as trace e $(b^q) = 0$. It is also clear that trace e [K[G], K[G]] = {0}. Now we have, trace e = trace $e^q = e^q = \sum_{x\notin P} a_x^q = (\sum_{x\in P} a_x)^q$ in the field of characteristic $p>0$. We have chosen n sufficiently large and we take $q' = qp$ so that we get trace e= trace $e^{qp} = \sum_{x\in P} a_x^{qp} = (\sum_{x\in P} a_x)^{qp} = \{(\sum_{x\in P} a_x)^q\}^P$ =(trace $e)^{P.}$ Thus we have obtained, trace e = (trace $e)^p$ and we have found an equation just as, trace e – (trace e $)^p = 0$ or, $x - x^p = 0$ in field K.

As it is seen that the solution of this equation lies in the prime subfield of K. Because the multiplication identity of this field K satisfies the given above equation. So we can write here trace $e = 1$ as well as trace $e = 0$. Therefore trace e is contained in the prime subfield of K.

5. Lemma:

Let us suppose that K be a field with *char* K = 0 and $x_0, x_1, x_2, \ldots\ldots, x_n$ be elements of K in which $x_0 \notin Q$. Then there will be a prime number p and a valuation R of field K which contains $x_0, x_1, x_2, \ldots, x_n$ and a ring homomorphism ∅ from R to $GF_{(P)}$ such that $\emptyset(x_0) \in GF_{(P)}$. The kernel of ∅ is the maximum ideal M of R.
Now we use this above lemma in the following theorem to complete the proof.

6. Zalesskii Theorem:

If $e \in K[G]$ be an idempotent, then trace e is contained in the prime subfield of field K. Here K[G] is group ring over field K.

Proof: Let us suppose that $x_0, x_1, x_2, \ldots, x_n \in K$ be the coefficient of the support of the idempotent of K[G]. Here we choose $e \in K[G]$ be the idempotent and x_0= trace e. If $x_0 \notin Q$ (rational field) then from the previous lemma, 5 we have R, ∅, p such that the coefficients of idempotent e lie in valuation ring R. Thus we get $e \in R[G] \subseteq K[G]$. Now we suppose M as kernel of mapping ∅ and also let a residue field F = R/M. Then we have natural mapping ∅ : → F includes group ring structure homomorphism ∅ : R[G] → F[G] provided that the characteristic of residue field F is p > 0. According to the rule of a ring homomorphism, we know that any homomorphic map relates the idempotent of R[G] to the idempotent of F[G]. But we have assume that the image of $\emptyset\,(x_0) = \emptyset($ trace $e)$ does not lie in the prime subfield of F. The assumption contradicts the result of idempotent in group ring F[G] over field F with characteristic $p > 0$.
Hence, we can say that the image of trace e lies in prime subfield of F and so trace $e \in$ Q (rational field).

7. Conclusions:

We have found that the trace e of idempotent of group ring K[G] lies in prime subfield of K, when the characteristic of field K is $p > 0$. We also presented the Zalesskii's theorem in this paper to show the trace e of K[G] contained in the prime subfield of K, regardless the characteristic of field K.

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