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A Generalization of the Prime Number Theorem

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Abstract

When certain partial sums of $e^{\gamma}\Phi(x) - x(x+1)/2$ do not increase monotonically, almost linear subsequences of abundant numbers result or subsequences of primes that give a staircase approximately equal to a multiple of Gauss' $G(x)$ function result. Using different prime sequences and different constants in Selberg's sigma function lead to variants of the prime number theorem.
Keywords: sum of divisors function, Euler's totient function, abundant numbers, Gauss' $G(x)$ function, staircase of primes, Chebyshev's functions, Selberg's formula.

1. INTRODUCTION

Background material is as follows. If $n \geq 1$ the Euler totient function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n. If $n > 1$, then $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2, \ldots, p_k$ are primes. The Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $a_1 = a_2 = \cdots = a_k = 1$, or $\mu(n) = 0$ otherwise. Let $\Phi(x)$ denote $\sum_{i=1}^{x} \varphi(i)$ and $M(x)$ (Mertens' function) denote $\sum_{i=1}^{x} \mu(i)$. Let $\sigma_x(i)$ denote the sum of positive divisors function ($\sigma_x(i) = \sum_{d|i} d^x$). $\sigma_1(n)$ is commonly denoted by $\sigma(n)$. Let $\pi(x)$ denote the number of primes less than or equal to x. Gauss observed that an approximation of $\pi(x)$ (the staircase of primes) is $x/\log(x)$. Chebyshev's first function $\vartheta(x)$ is defined to be $\sum_{p \leq x} \log p$. The Mangoldt function $\Lambda(n)$ is defined to be $\log p$ if $n = p^m$ for some prime p and some $m \geq 1$, or 0 otherwise. The second Chebyshev function $\psi(x)$ is defined to be $\sum_{n \leq x} \Lambda(n)$. The relations $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$, $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$, and $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$ are logically

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equivalent (see Theorem 4.4 of Apostol's [1] book). Mikolás' [2] Lemma 2 is

Lemma 1 $\sum_{n=1}^{[x]} \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{d=1}^{[x]} f(d) \sum_{\delta=1}^{[x/d]} g(\delta) = \sum_{d=1}^{[x]} g(d) \sum_{\delta=1}^{[x/d]} f(\delta)$

From this, Mikolás determined the following.

Theorem 1 $\Phi(x) = \sum_{n=1}^{[x]} \varphi(n) = \sum_{n=1}^{[x]} nM(\frac{x}{n}) = \frac{1}{2} \sum_{n=1}^{[x]} \mu(n)[\frac{x}{n}]^2 + \frac{1}{2}$

Let γ denote Euler's constant. Gronwall [3] determined the maximal order of the sum of divisors function.

Theorem 2 $\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^\gamma$

2. AN ALGORITHM FOR COMPUTING $\Phi(x) - x(x+1)/2$

The algorithm uses Mikolás' theorem and the following theorem.

Theorem 3 $\sum_{n=1}^{x} M(\frac{x}{n})\sigma(n) = x(x+1)/2$

Proof 1 *Möbius inversion of* $x = \sum_{n\le x} 1$ *gives* $1 = \sum_{d|n} \mu(d)\sigma_0(\frac{n}{d})$. *(See page 95 of Apostol's book). Similarly, Möbius inversion of* $\frac{x(x+1)}{2} = \sum_{n\le x} n$ *gives* $n = \sum_{d|n} \mu(d)\sigma_1(\frac{n}{d})$. *Then* $\frac{x(x+1)}{2} = \sum_{n=1}^{x} \sum_{d|n} \mu(d)\sigma_1(\frac{n}{d})$. *Then by Mikolás' Lemma 2,* $\frac{x(x+1)}{2} = \sum_{d=1}^{x} \sigma_1(d) \sum_{\delta=1}^{[x/d]} \mu(\delta)$*, that is,* $\frac{x(x+1)}{2} = \sum_{d=1}^{x} \sigma_1(d)M(\frac{x}{d})$.

The algorithm uses the following partial sums.

$$L_{x,s} = c_1 \sum_{i=1}^{l-1} M(\frac{x}{i})i - c_2 \sum_{i=1}^{l-1} M(\frac{x}{i})\sigma(i) \quad (1)$$

$$R_{x,s} = c_1 \sum_{i=l}^{x} M(\frac{x}{i})i - c_2 \sum_{i=l}^{x} M(\frac{x}{i})\sigma(i) \quad (2)$$

Setting c_1 and c_2 to 1 gives $\Phi(x) - x(x+1)/2$. The factor of ($c_1 = e^\gamma$, $c_2 = 1$) will be investigated here. Let δ denote a decimation value. The x values are s, $s+\delta$, $s+2\delta$,... and the s values are 1, 2, 3,..., δ. For example, when $\delta = 2$ and $s = 1$, every other $e^\gamma\Phi(x) - x(x+1)/2$ value is computed. Repeating this procedure for $s = 2$ gives the remainder of the $e^\gamma\Phi(x) - x(x+1)/2$ values. The l values start with 1 and are incremented by one for each successive x value. Let $R'_{x,s}$ denote $R_{x,s}/x$ and $L'_{x,s}$ denote $L_{x,s}/x$. See Cox and Ghosh [4] for properties of $L'_{x,s}$ and $R'_{x,s}$ and graphs of staircases of primes.

3. $L'_{x,s}$, $\delta = 2$, $s = 1$

There are 13760 instances where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 1$, and $x \leq 50000$ and only 3 of the x values are abundant numbers. The remaining 13757 values will be referred to as being "sparse" numbers. For the sparse numbers, 3 times 1926 primes are included. An empirical result is

Conjecture 1 *The usual sequence of primes starting with 5 and multiplied by 3 are sparse numbers for* $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, *and* $s = 1$.

Sparse numbers also occur for powers of primes times groups of primes. For prime-power factors of 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, and 61, the prime counts are 1926, 941, 630, 353, 295, 159, 141, 95, 64, 56, 39, 32, 16, 7, 4, 2, and 1 respectively. A plot of the reciprocals of 3, 5, 7,. . ., 61 versus the prime counts is

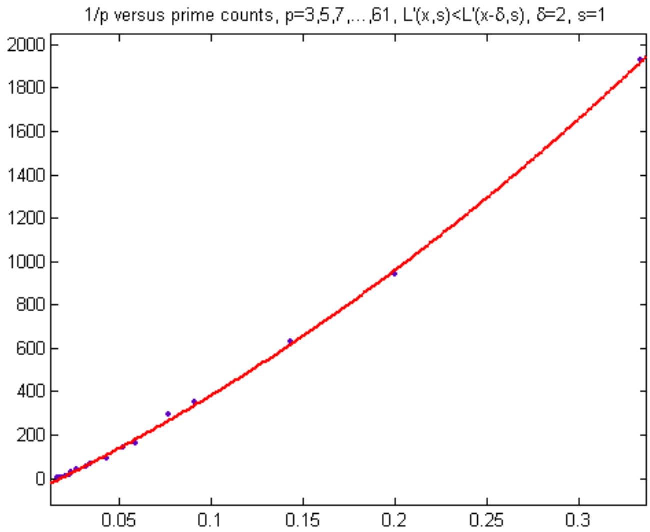

Figure 1: Reciprocal of primes versus prime counts

For a quadratic least-squares fit of the curve, $p_1 = 6025$ with a 95% confidence interval of (5056, 6995), $p_2 = 3983$ with a 95% confidence interval of (3664, 4301), $p_3 = -74.95$ with a 95% confidence of $(-89.21, -60.69)$, SSE=2746, R-squared=0.9993, and RMSE=14.01.

3.1. Another Equivalent Asymptotic Relation and Selberg's Asymptotic Relation

Let p_n denote the nth prime. The asymptotic relation $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$ is logically equivalent to the asymptotic relation $\lim_{x\to\infty} \frac{p_n}{n \log n} = 1$ (see Theorem 4.5 of Apostol's book). A prime-power factor of 5 times the 941 primes 7, 13, 17, 19, 23, 31, 37, 41, 43, 53, 61,. . ., 9973 are sparse numbers. A plot of $\frac{p_n}{n \log n}$ for these primes is

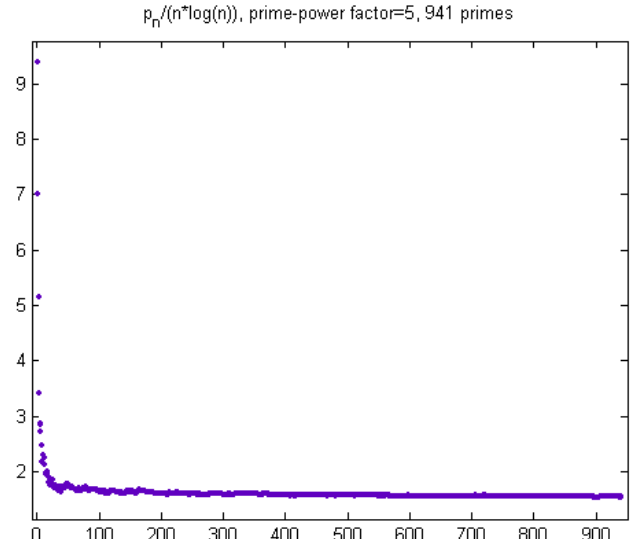

Figure 2: $\frac{p_n}{n \log n}$ values for prime-power factor of 5

Here p_n denotes the nth prime in that group. A plot of the prime-power factors 3, 5, 7,. . ., 43 versus the last $\frac{p_n}{n \log n}$ values is

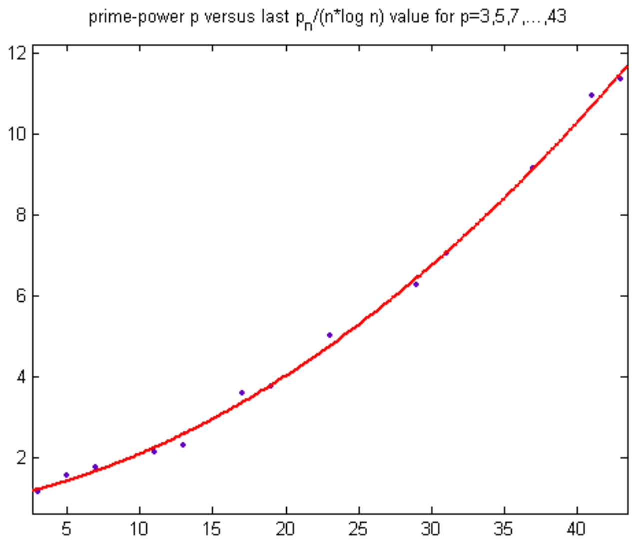

Figure 3: Prime-power factors versus last $\frac{p_n}{n \log n}$ values

For a quadratic least-squares fit of the curve, $p_1 = 0.004008$ with a 95% confidence interval of (0.003175, 0.004841), $p_2 = 0.07318$ with a 95% confidence interval of

(0.03384, 0.1125), $p_3 = 0.9541$ with a 95% confidence interval of (0.5823, 1.325), SSE=0.372, R-squared=0.9976, and RMSE=0.1929.

Selberg's asymptotic formula is $\psi(x)\log x + \sum_{n\leq x}\Lambda(n)\psi(\frac{x}{n}) = 2x\log x + O(x)$ for $x > 0$ (see Theorem 4.18 in Apostol's book). A more convenient form involves the function $\sigma(x) = e^{-x}\psi(e^x) - 1$. Variants of the prime number theorem would use constants other than 1 (as in the above).

3.2. Two More Asymptotic Relations and a Variant of Selberg's Asymptotic Relation

Theorem 3.16 in Apostol's book is

Theorem 4 $\sum_{p\leq x}[\frac{x}{p}]\log p = x\log x + O(x)$

A plot $\sum_{p\leq x}[\frac{x}{p}]\log p$ for the primes associated with a prime-power factor of 5 and $x\log x$ is

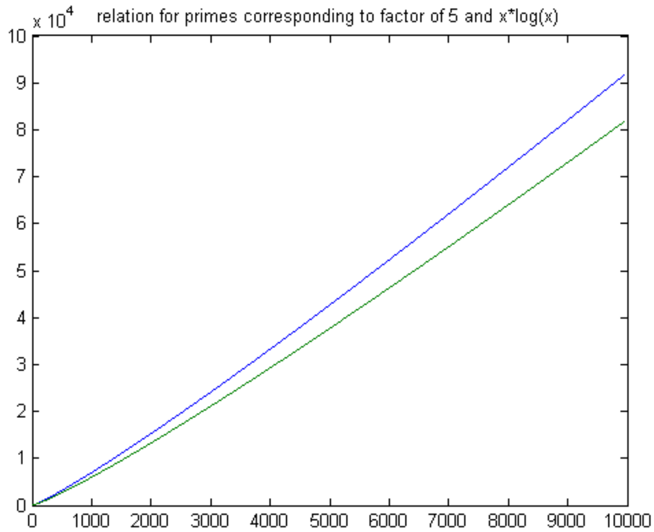

Figure 4: $\sum_{p\leq x}[\frac{x}{p}]\log p$ for prime-power factor of 5 and $x\log x$

The greater curve is $x\log x$. Theorem 4.10 in Apostol's book is

Theorem 5 *For all* $x \geq 1$ *we have* $\sum_{p\leq x}\frac{\log p}{p} = \log x + O(1)$. *Also, there exist positive constants* c_1 *and* c_2 *such that* $\vartheta(x) \leq c_1 x$ *for all* $x \geq 1$ *and* $\vartheta(x) \geq c_2 x$ *for all sufficiently large* x.

A plot of $\sum_{p \le x} \frac{\log p}{p}$ for the primes associated with a prime-power factor of 5 and $\log x$ is

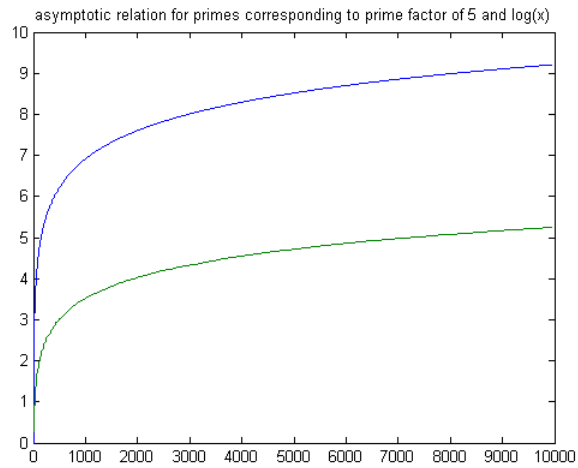

Figure 5: $\sum_{p \le x} \frac{\log p}{p}$ for prime-power factor of 5 and $\log x$

The greater curve is $\log x$. Selberg's formula is equivalent to the following

$$\vartheta(x) \log x + \sum_{p \le x} \vartheta(\frac{x}{p}) \log p = 2x \log x + O(x) \tag{3}$$

A plot of $2x \log x$ and $\vartheta(x) \log x + \sum_{p \le x} \vartheta(\frac{x}{p}) \log p$ for the different $\vartheta(x)$ values for prime-power factors of $p = 3, 5, 7, 11, 13, 17,$ and 19 and $x \le 2598$ is

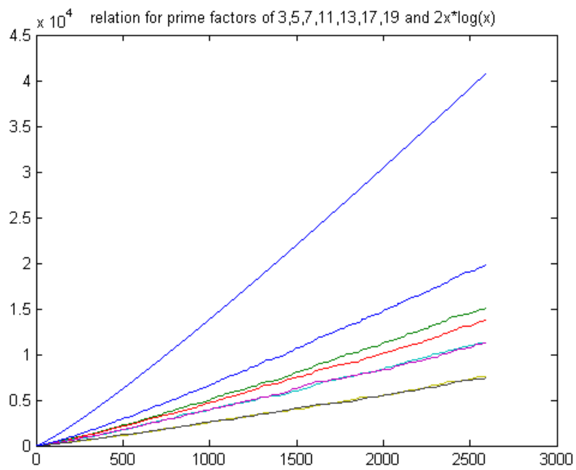

Figure 6: Asymptotic relations for prime-power factors of 3, 5, 7, 11, 13, 17, and 19.

The greater curve is $2x \log x$. The successively smaller curves correspond to the increasing prime-power factors. The curves almost overlap for the prime-pair factors (11, 13) and (17, 19).

3.3. Variants of Chebyshev's First Function and Analytic Proofs of the Prime Number Theorem

A plot of the sums of the $\vartheta(x)$ values for the primes associated with a prime-power factor of 7 is

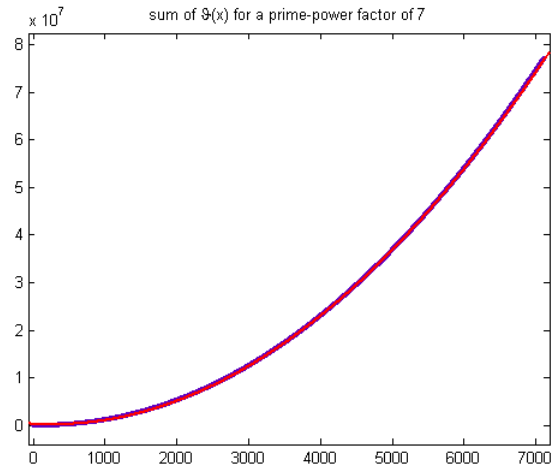

Figure 7: Sums of $\vartheta(x)$ values for a prime-power factor of 7.

For a quadratic least-squares fit of the curve, $p_1 = 1.629$ with a 95% confidence interval of (1.628, 1.629), $p_2 = -844.8$ with a 95% confidence interval of (-848.5, -840.33), $p_3 = 3.309 \cdot 10^5$ with a 95% confidence interval of ($3.246 \cdot 10^5$, $3.372 \cdot 10^5$), SSE=$5.85 \cdot 10^{13}$, R-squared=1, and RMSE=$9.067 \cdot 10^4$. These parameters are likely to change for larger x values.

The first step in an analytic proof of the prime number theorem is to take the integral of $\psi(x)$, that is, $\psi_1(x) = \int_1^x \psi(t)dt$. It is then shown that $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$ implies $\psi(x) \sim x$ as $x \to \infty$. This is the significance of the above quadratic curve.

3.4. Variants of Chebyshev's Second Function and a Generalization of the Mangoldt Function

The analogues of $\psi(x)$ increase almost linearly. The fundamental theorem of arithmetic is not applicable to the analogue of the Mangoldt function since subsets of the primes 2, 3, 5,... are used. Also, unexpected n values are "prime powers". For example, for the primes corresponding to a prime-power factor of 3, $2 \cdot 5$ is a "prime power". A detailed example of how this method works is given in the next subsection. For the primes corresponding to a prime-power factor of 3 and $x \leq 50000$, there are 3949 instances where an n value is even (and possibly divisible by 3) and $\Lambda(n)$ is not zero. There are 1250 instances where an n value is divisible by 3 but not by 2 and $\Lambda(n)$ is not zero. Otherwise, there are 1966 instances where $\Lambda(n)$ is not zero. A plot of the analogue of $\psi(x)$ (henceforth denoted by $\psi'(x,p)$) and its components is

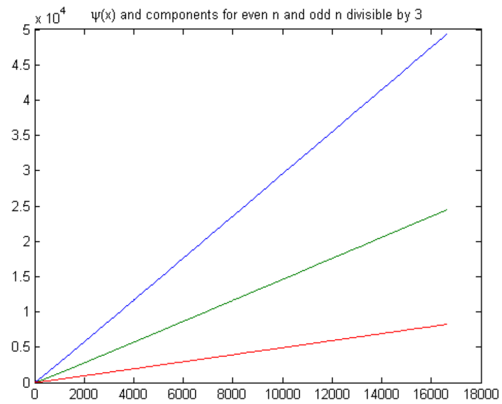

Figure 8: $\psi(x)$ for a prime-power factor of 3 and components.

The two components account for about 2/3 (0.6669 for the last $\psi'(x,3)$ value) of an $\psi'(x,3)$ value. The remaining component of a $\psi'(x,3)$ value is the sum of the logarithms of the usual prime powers (with the exception of $\log 2$ and $\log 3$). $\psi'(x,3)$ is then about 3 times larger than $\psi(x)$.

A plot of the sums of the $\psi'(x,3)$ values is

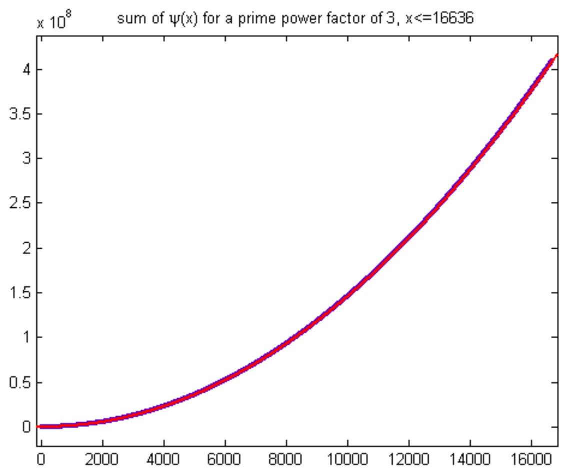

Figure 9: Sums of $\psi(x)$ values for a prime-power factor of 3.

For a quadratic least-squares fit of the curve, $p_1 = 1.49$ with a 95% confidence interval of (1.49, 1.49), $p_2 = -303.8$ with a 95% confidence interval of (-304.3, -303.3), $p_3 = 1.785 \cdot 10^5$ with a 95% confidence interval of ($1.766 \cdot 10^5$, $1.803 \cdot 10^5$), SSE=$2.813 \cdot 10^{13}$, R-squared=1, and RMSE=$4.11 \cdot 10^4$. These parameters are likely to change for larger x values. The sum of these almost linear curves is an almost quadratic curve.

For the primes corresponding to a prime-power factor of 5 and $x \leq 50000$, there are 3033 instances where an n value is even (and possibly divisible by 3 or 5) and $\Lambda(n)$ is not zero. There are 1066 instances where an n value is divisible by 3 (and possibly by 5) but not by 2 and $\Lambda(n)$ is not zero. There are 406 instances where an n value is divisible by 5 but not by 2 or 3 and $\Lambda(n)$ is not zero. Otherwise, there are 1433 instances where $\Lambda(n)$ is not zero. A plot of $\psi'(x, 5)$ and its components is

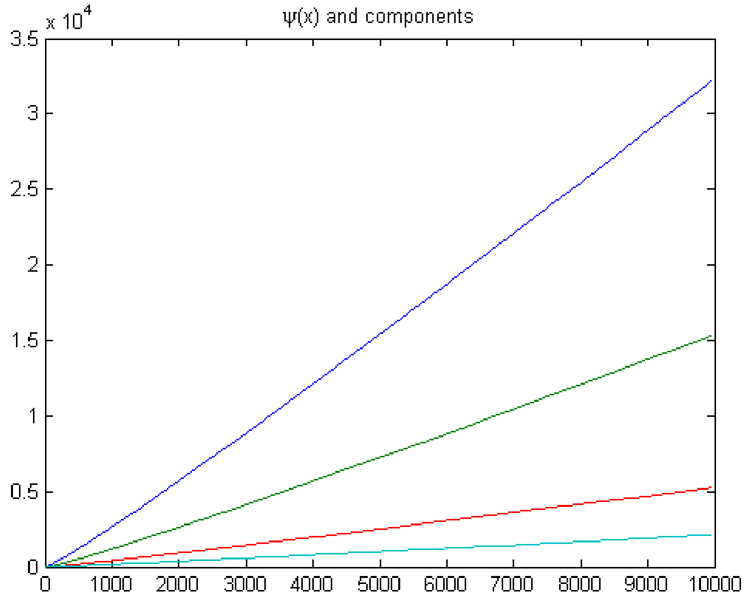

Figure 10: $\psi(x)$ for a prime-power factor of 5 and components.

The three components account for about 7/10 (0.7044 for the last $\psi'(x, 5)$ value) of an $\psi'(x, 5)$ value.

A plot of the sums of the $\psi'(x, 5)$ values is

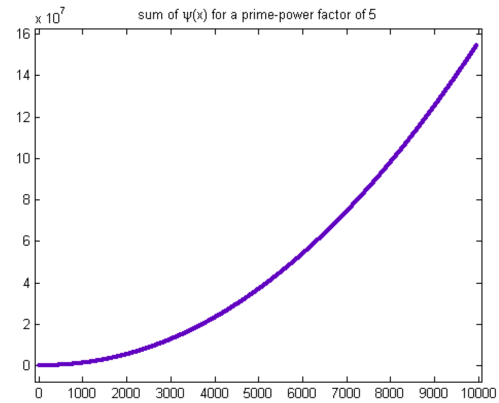

Figure 11: Sums of $\psi(x)$ values for a prime-power factor of 5.

For a quadratic least-squares fit of the curve, $p_1 = 1.645$ with a 95% confidence interval of (1.644, 1.645), $p_2 = -965.3$ with a 95% confidence interval of (-969.3, -961.3), $p_3 = 5.339 \cdot 10^5$ with a 95% confidence interval of ($5.252 \cdot 10^5$, $5.427 \cdot 10^5$), SSE=$2.197 \cdot 10^{14}$, R-squared=1, and RMSE=$1.485 \cdot 10^5$. These parameters are likely to change for larger x values.

3.5. More on Components of Variants of Chebyshev's Second Function

For $x \leq 70000$, 17 times the 215 primes 19, 31, 37, 43, 61, 67, 79, 97, 109, 127, 139, 151,. . ., 4111 are sparse numbers. The sizes (numbers of elements) of the components of $\psi'(x, 17)$ corresponding to 2, 3, 5, 7, 11, 13, and 17 are 689, 237, 99, 58, 32, 23, and 20 respectively. A plot of the logarithms of these values is

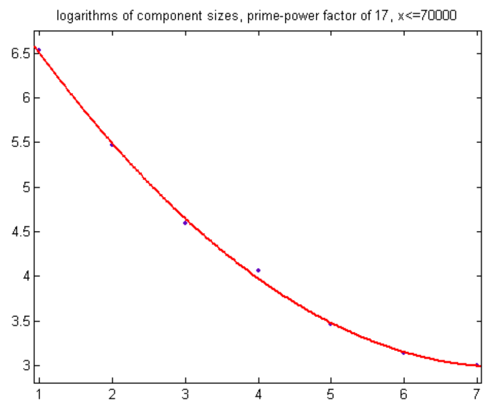

Figure 12: Logarithms of sizes of components of $\psi(x)$ analogue for a prime-power factor of 17.

For a quadratic least-squares fit of the curve, $p_1 = 0.08608$ with a 95% confidence interval of (0.06985, 0.1023), $p_2 = -1.275$ with a 95% confidence interval of (-1.408, -1.142), $p_3 = 7.7$ with a 95% confidence interval of (7.468, 7.932), SSE=0.01148, R-squared=0.9989, and RMSE=0.05357. A plot of the components is

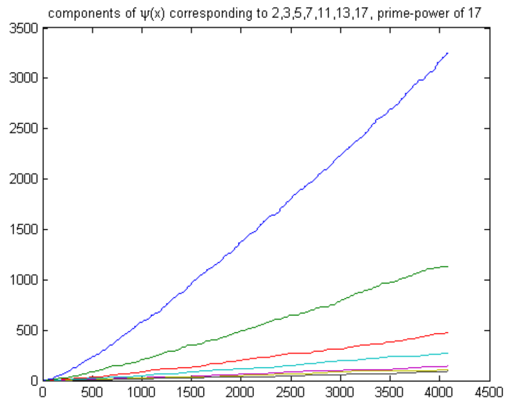

Figure 13: Components of $\psi(x)$ analogue for a prime-power factor of 17.

Let $g(n)$ denote n divided by the largest power of 3 that divides n. Let $f_1(x) = \sum_{n=2j+1, n\neq 3^k}^{[x/2^i]} \Lambda(g(n))$, $i = 1, 2, 3, \ldots$, $j = 2, 3, 4, \ldots$, and $k = 2, 3, 4, \ldots$. The sum of $f_1(x)$ for $i = 1, 2, 3, \ldots$ is the first component of $\psi'(x, 3)$. Let $f_2(x) = \sum_{n=2j+1, n\neq 3^k}^{[x/3^i]} \Lambda(g(n))$, $i = 1, 2, 3, \ldots$, $j = 2, 3, 4, \ldots$, and $k = 2, 3, 4, \ldots$. When 3^l divides n, i must be incremented by l for the next iteration of the sum. This caveat is necessary to avoid Mangoldt values from being counted more than once. (In practice, the algorithm is not implemented this way.) The sum of $f_2(x)$ for $i = 1, 2, 3, \ldots$ is the second component of $\psi'(x, 3)$. This insures that no n value is just the product of powers of 2 and powers of 3. A plot of the sums of the first components of the $\psi'(x, 3)$ values for $x \leq 70000$ is

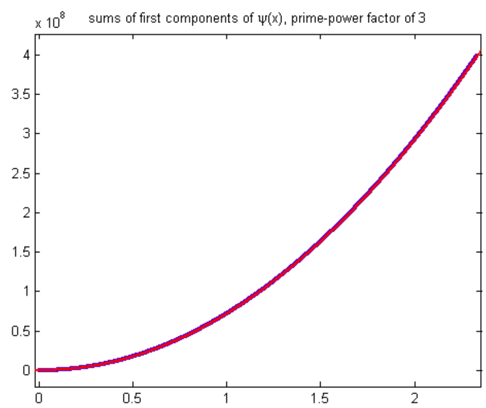

Figure 14: Sums of first components of $\psi(x)$ analogue for a prime-power factor of 3.

For a quadratic least-squares fit of the curve, $p_1 = 0.7434$ with a 95% confidence interval of (0.7434, 0.7434), $p_2 = -273.9$ with a 95% confidence interval of (-274.3, -273.5), $p_3 = 2.264 \cdot 10^5$ with a 95% confidence interval of ($2.244 \cdot 10^5$, $2.284 \cdot 10^5$), SSE=$6.27 \cdot 10^{13}$, R-squared=1, and RMSE=$5.185 \cdot 10^4$.

3.6. The Generalized Mangoldt Function and Legendre's Identity

Theorem 3.12 of Apostol's book is

Theorem 6 *For $x \geq 1$ we have $\sum_{n \leq x} \mu(n)[\frac{x}{n}] = 1$ and $\sum_{n \leq x} \Lambda(n)[\frac{x}{n}] = \log[x]!$.*

Legendre's identity (Theorem 3.14 in Apostol's book) is

Theorem 7 *For every $x \geq 1$ we have $[x]! = \prod_{p \leq x} p^{\alpha(p)}$ where the product is extended over all primes $\leq x$, and $\alpha(p) = \sum_{m=1}{}^{\infty}[\frac{x}{p^m}]$.*

Using Legendre's identity instead of $\log[x]!$ allows the generalized Mangoldt function (henceforth denoted by $\Lambda'(x, p)$) to be used. Let $L(x, p)$ denote the variant of Legendre's identity computed using the primes corresponding to a prime-power factor of p. A plot of $\log(L(x, 5))$ versus $\log(\sum_{n \leq x} \Lambda'(n)[\frac{x}{n}])$ for $x = 7$ to 941 is

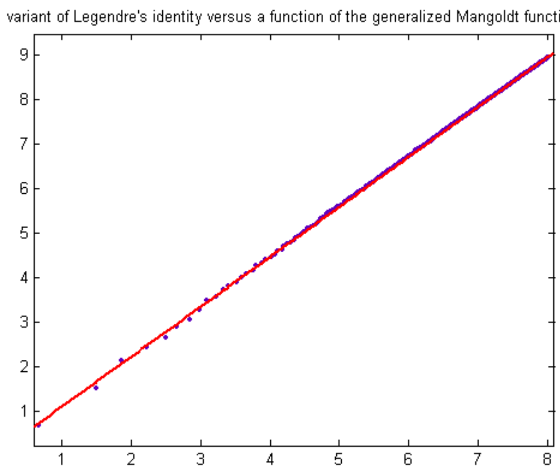

Figure 15: Variant of Legendre's identity versus function of generalized Mangoldt function.

For a linear least-squares fit of the curve, $p_1 = 1.121$ with a 95% confidence interval of (1.12, 1.122), $p_2 = -0.0206$ with a 95% confidence of (-0.02695, -0.01421), SSE=0.3196, R-squared=0.9998, and RMSE=0.01851. The slopes for prime-power factors of 3, 5, 7, and 11 are 1.076, 1.121, 1.149, and 1.173. The values increase almost quadratically. Apostol's Theorem 3.15 is

Theorem 8 *If* $x \geq 2$ *we have* $\log[x]! = x\log x - x + O(\log x)$ *and hence* $\sum_{n\leq x}\Lambda(n)[\frac{x}{n}] = x\log x - x + O(\log x)$.

This in turn is used to prove the above Theorem 3.16.

3.7. The Generalized Mangoldt Function and the Möbius Function

Theorem 2.11 in Apostol's book is

Theorem 9 *If* $n \geq 1$ *we have* $\Lambda(n) = \sum_{d|n}\mu(d)\log\frac{n}{d} = -\sum_{d|n}\mu(d)\log d$.

The generalized Möbius function would still have possible values of -1, 1, or 0. In addition to a value of 0 when d is not square-free, $\mu(d)$ would be 0 when d has no prime factors in the primes corresponding to a prime-power factor of p. Let $\mu'(d,p)$ denote this generalized Möbius function. Then $\Lambda'(n,p) = -\sum_{d|n}\mu'(d,p)\log d$. Theorem 2.11 is used to prove the following (Theorem 4.14 in Apostol's book)

Theorem 10 *The prime number theorem implies* $\lim_{x\to\infty}\frac{M(x)}{x} = 0$.

A plot of the generalized Mertens function for $\mu'(x,5)$ is

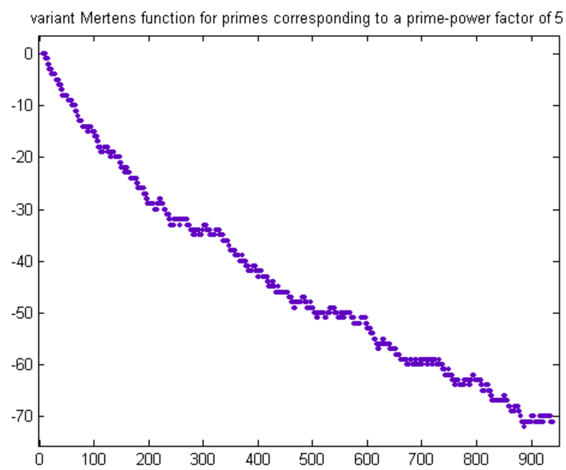

Figure 16: Generalized Mertens function for primes corresponding to a prime-power factor of 5.

Similarly to the Mertens function, the first few values are negative.

3.8. More on the Analytic Proof of the Prime Number Theorem

Apostol's Theorem 13.1 is

Theorem 11 $\psi_1(x) = \sum_{n \leq x}(x-n)\Lambda(n)$. *Also, the asymptotic relation* $\psi_1(x) \sim x^2/2$ *implies* $\psi(x) \sim x$ *as* $x \to \infty$.

A lemma used in the proof of this is

Lemma 2 *Let* $A(x) = \sum_{n \leq x} \alpha(n)$ *and let* $A_1(x) = \int_1^x A(t)dt$. *Assume also that* $\alpha(n) \geq 0$ *for all* n. *If we have the asymptotic formula* $A_1(x) \sim Lx^c$ *as* $x \to \infty$ *for some* $c > 0$ *and* $L > 0$, *then we also have* $A(x) \sim cLx^{c-1}$ *as* $x \to \infty$.

Let $\zeta(s)$ denote the Riemann zeta function. Apostol's Theorem 13.2 is

Theorem 12 *If* $c > 1$ *and* $x \geq 1$ *we have* $\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds$.

To be applicable to $\psi'(x,p)$, the starting x values must be adjusted. For the primes corresponding to a prime-power factor of 3 (the primes greater than 3), the L value is increased from 0.5 to about 1.448 (for $x = 5$ to 1926). For the primes corresponding to prime-power factors of 5, 7, 11, 13, 17, 19, 23, and 29, the respective L values are about 1.353, 1.243, 0.9731, 0.872, 0.4045, 0.3392, 0.2307, and 0.1347 respectively. A plot of the prime-power factors (including $p = 3$) versus the L values is

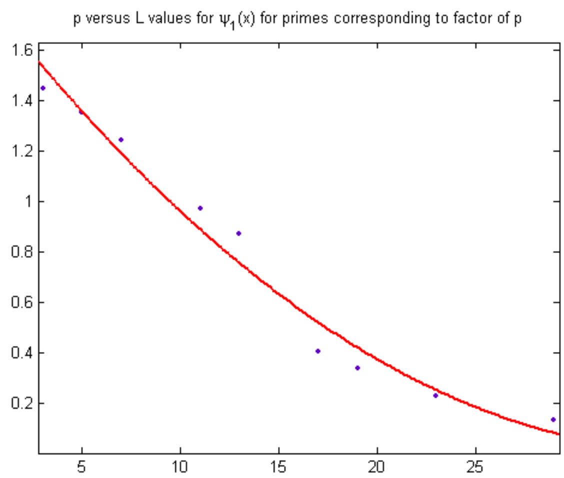

Figure 17: Prime-power factors versus L values.

For a quadratic least-squares fit of the curve, R-squared=0.9653. The respective ranges of the x values are (5, 1296), (7, 941), (11, 630), (13, 353), (17, 295), (19, 159), (23, 141), (31, 95), and (31, 64). The other lemma used in the proof of Theorem 13.1 is

Lemma 3 *For any arithmetical function* $\alpha(n)$ *let* $A(x) = \sum_{n\leq x} \alpha(n)$*, where* $A(x) = 0$ *if* $x < 1$*. Then* $\sum_{n\leq x}(x-n)\alpha(n) = \int_1^x A(t)dt$*.*

These lemmas are applicable to the variant $\psi(x)$ values. Apostol's Theorem 13.2 is

Theorem 13 *If* $c > 1$ *and* $x \geq 1$ *we have* $\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds$*.*

This gives $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$ where $\zeta'(s) = -\sum_{n=2}^{\infty} \frac{\log n}{n^s}$. Let $r(s,p,i)$ denote $\zeta(s)\sum_{n=2}^{i} \frac{\Lambda'(n,p)}{n^s} / \sum_{n=2}^{i} \frac{\log n}{n^s}$. For $s = 2$, the primes corresponding to a prime-power factor of 3, and $i = 2599$ (for $x \leq 70000$), this gives a value of about 0.5305. The i values for the primes corresponding to the prime-power factors of 5, 7, 11, 13, 17, 19, and 23 are 1261, 845, 475, 386, 215, 181, and 122 respectively. The respective $r(2,p,i)$ values are about 0.2877, 0.1885, 0.1593, 0.1228, 0.05637, 0.0375, and 0.02056. A plot of the logarithms of these values is

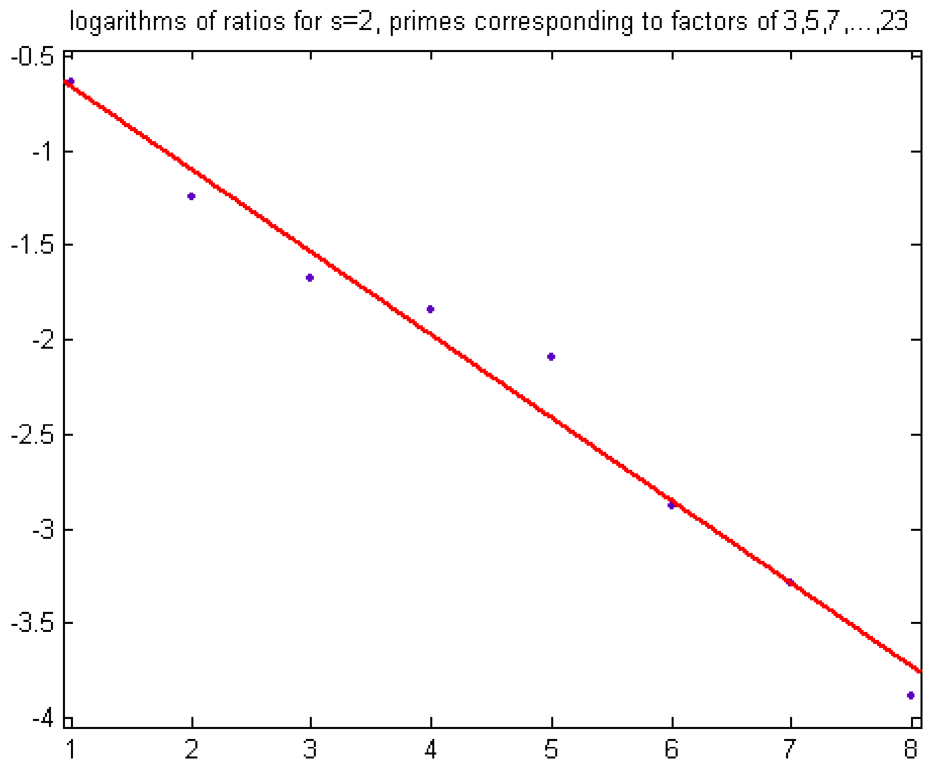

Figure 18: Logarithms of ratios for $s = 2$ and primes corresponding to prime-power factors of 3,5,7,. . .,23.

For a linear least-squares fit of the curve, the slope is about -0.438 and the y-intercept is about -0.218. The $r(4, p, i)$ values are about 0.06322, 0.01662, 0.004117, 0.002723, 0.001356, 0.0005114, 0.0002413, and 0.00009139. The slope of the linear least-squares fit of the logarithms of these values is about -0.880 and the y-intercept is about -2.30. The $r(6, p, i)$ values are about 0.009796, 0.001392, 0.0001259, 0.00005827, 0.00001709, 0.000005525, 0.000001817, and 0.0000003699. The slope of the linear least-squares fit of the logarithms of these values is about -1.37 and the y-intercept is about -3.97. The $r(8, p, i)$ values are about 0.001154, 0.0001186, 0.000004104, 0.000001298, 0.0000002192, 0.0000006211, 0.00000001415, and 0.000000001521. The slope of the linear least-squares fit of the logarithms of these values is about -1.86 and the y-intercept is about -5.59. These slopes and intercepts decrease almost linearly. A plot of the slopes for $s = 2, 3, 4, \ldots, 9$ is

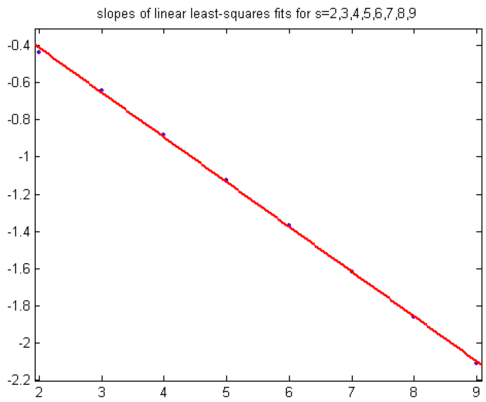

Figure 19: Slopes of linear least-squares fits for $s = 2, 3, 4, \ldots, 9$

For a linear least-squares fit of the curve, R-squared=0.9995. A plot of the y-intercepts for $s = 2, 3, 4, \ldots, 9$ is

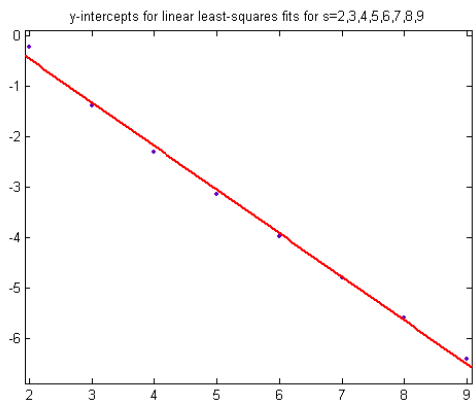

Figure 20: y-intercepts of linear least-squares fits for $s = 2, 3, 4, \ldots, 9$

For a linear least-squares fit of the curve, R-squared=0.9969.

Theorem 13.3 in Apostol's book is

Theorem 14 *If* $c > 1$ *and* $x \geq 1$ *we have* $\frac{\psi_1(x)}{x^2} - \frac{1}{2}(1 - \frac{1}{x})^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds$ *where* $h(s) = \frac{1}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1})$.

The right hand side of the equation in Theorem 13.3 tends to 0 as $x \to \infty$. A plot of $\frac{\psi_1'(x,p)}{x^2} - 1.243(1 - \frac{1}{x})^2$ for $p = 7$ and $x = 11$ to 845 is

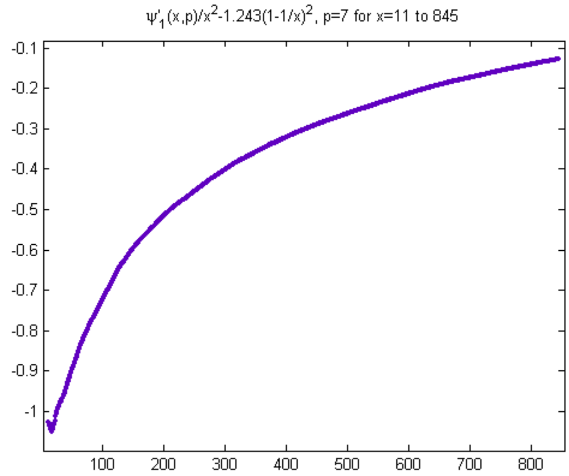

Figure 21: $\frac{\psi_1'(x,p)}{x^2} - 1.243(1 - \frac{1}{x})^2$ for $p = 7$

The values appear to be approaching 0.

4. CONCLUSION

Except for 2 and 3, the usual prime sequence 2, 3, 5, 7, 11,... is generated by a sparse number sequence. Subsequences of primes in sparse numbers give staircases approximately equal to a multiple of Gauss' $G(x)$ function. Much of the theoretical machinery of the proof of the prime number theorem does not depend on what the asymptotic limit of $\frac{\pi(x)\log x}{x}$ is. Broadening the definition of what a prime sequence is (as in the above) leads to variants of the prime number theorem.

5. METHODS

C code for computing $L_{x,s}$ and $R_{x,s}$ and finding abundant and sparse numbers when the values do not increase monotonically is as follows.

```
#include <math.h>
#include <stdio.h>
extern char *malloc();
// compute sum of divisors
unsigned int numdiv(unsigned int a) {
    unsigned int i,sum;
    sum=0;
    for (i=1; i<=a; i++) {
        if (a==(a/i)*i)
            sum=sum+i;
        }
    return sum;
    }
unsigned int max=20000; // maximum x value
unsigned int start=1; // beginning x value
unsigned int delta=2; // increment
unsigned int swap=1; // if set, output L(x,s) results instead of R(x,s)
unsigned int out=2; // set to 1 for abundant numbers
                    // set to 2 for sparse numbers
void main() {
    int sum,t,*m;
    unsigned int i,j,k,index,count;
    double temp,tempsum,tempsum1,oldsum;
    int newsum,tmpsum,newsum1,tmpsum1;
    FILE *Outfp;
    Outfp = fopen("out7fd.dat","w");
    m=(int*) malloc(400004);
    if (m==NULL) {
        printf("not enough memory");
        return;
        }
    if (max>20000) {
        printf("x value too large");
        return;
        }
    // compute Mertens function
```

```
m[0]=1;
for (index=2; index<=max; index++) {
    sum=0;
    for (i=2; i<=(index/3); i++)
        sum=sum+m[index/i-1];
    sum=sum+(index+1)/2;
    t=1-sum;
    m[index-1]=t;
    }
count=0;
temp=exp(.57721566490153286060); // e^γ
oldsum=0.0;
j=1;
for (index=start; index<=max; index+=delta) {
    newsum=0;
    tmpsum=0;
    newsum1=0;
    tmpsum1=0;
    for (i=1; i<j; i++) {
        newsum1=newsum1+m[index/i-1]*i;
        tmpsum1=tmpsum1+m[index/i-1]*(int)numdiv(i);
        }
    tempsum1=(double)newsum1*temp;
    tempsum1=tempsum1-(double)tmpsum1;
    for (i=j; i<=index; i++) {
        newsum=newsum+m[index/i-1]*i;
        tmpsum=tmpsum+m[index/i-1]*(int)numdiv(i);
        }
    tempsum=(double)newsum*temp;
    tempsum=tempsum-(double)tmpsum;
    j=j+1;
    if (swap==0)
        tempsum=tempsum/(double)index;
    else
        tempsum=tempsum1/(double)index;
    printf(" %d %d %e \n ",j-1,index,tempsum);
    if (tempsum<oldsum) {
        k=numdiv(index);
```

```
        if (k<(index*2)) { // not an abundant number
            if (out==2) {
                fprintf(Outfp," %d, \n",index);
                count=count+1;
                }
            }
        else {
            if (out==1) {
                fprintf(Outfp," %d,\n",index);
                count=count+1;
                }
            }
        }
    oldsum=tempsum;
    }
printf("count=%d",count);
fclose(Outfp);
return;
}
```

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