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Canonical Form of Poisson-Nambu Manifold

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Abstract

In this paper, we give the universal property of multi-derivations. We prove the existence and the uniqueness of a canonical form associated with a Nambu-Poisson manifold by using the universal property of multi-derivations. We show that the module of Kähler differentials is a Filippov algebra.

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1. INTRODUCTION

The notion of n-Lie algebra, $n \in \mathbb{N}, n \geq 2$, introduices by Fillipov [2], is a generalization of the concept of Lie algebra. An n-Lie algebra or a Filippov algebra of order n is a vector space L endowed with a skew-symmetric, n-linear bracket,

$$[.,...,.] : L^n = L \times ... \times L \longrightarrow L$$

such that, the following generalized Jacobi identity is satisfied:

$$[x_1, x_2, ..., x_{n-1}, [y_1, y_2, ..., y_n]] = \sum_{i=1}^{n} [y_1, y_2, ..., y_{i-1}, [x_1, x_2, ..., x_{n-1}, y_i] , y_{i+1}, ..., y_n] \quad (1)$$

for all $x_1, x_2, ..., x_{n-1}, y_1, y_2, ..., y_n \in L$ [3]. We shall call such structures Filippov algebras. The identity (1) is called Filippov identity or Fundamental identity. A derivation of an n-Lie algebra $(L, [.,...,.])$ is a linear map $D : L \longrightarrow L$ such that

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$$D\left([x_1, x_2, ..., x_n]\right) = \sum_{i=1}^{n} [x_1, ..., D\left(x_i\right), ..., x_n]$$

for any $x_1, x_2, ..., x_n \in L$. The identity (1) means that the bracket $[x_1, x_2, ...x_{n-1},]$, where the last entry is empty, is a derivation of the Filippov algebra.

In 1973 Nambu [6] proposed a generalization of the standard classical Hamiltonian mechanics based on a three-dimensional 'phase space' spanned by a canonical triplet of dynamical variables and on two 'Hamiltonians'. The notion of Nambu-Poisson structures was introduced in [8] by Takhtajan in order to give a formalism to an idea of Y. Nambu [6].

The purpose of this paper is to show the existence and the uniqueness of a canonical form associated with a Nambu-Poisson manifold by using the universal property of derivations.

The paper is organized as follows. In Section 2, we give the universal property of multi-derivations. In Section 3, we recall the notion of Nambu-Poisson structure and we prove the existence and the uniqueness of a canonical form associated with a Nambu-Poisson manifold by using the universal property of multi-derivations. Finally, in Section 4, we define the Nambu-Poisson form bracket on the module of Kähler differentials and we show that the module of Kähler differentials is a Filippov algebra.

2. UNIVERSAL PROPERTY OF DERIVATIONS

Let M be a smooth manifold and denote by $\Omega_{\mathbb{R}}[C^\infty(M)]$ the module of Kähler differentials of commutative algebra $C^\infty(M)$, that is, the quotient space $\Omega_{\mathbb{R}}[C^\infty(M)] = I/I^2$, where I is the $C^\infty(M)$-submodule of $C^\infty(M) \bigotimes_{\mathbb{R}} C^\infty(M)$ generated by the elements of the form $f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f$ with $f \in C^\infty(M)$ [1], [7]. The linear map $\delta_M : C^\infty(M) \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$ defined by

$$\delta_M\left(f\right) = \overline{f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f}$$

is the canonical derivation which the image of δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$, that is, for $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$, $x = \sum_{i \in I: finite} f_i \cdot \delta_M(g_i)$, with $f_i, g_i \in C^\infty(M)$.

Theorem 1. (Universal property of derivations). The pair $(\Omega_{\mathbb{R}}[C^\infty(M)], \delta_M)$ satisfies the following universal property: for every $C^\infty(M)$-module E and for every derivation $D : C^\infty(M) \longrightarrow E$, there exists a unique $C^\infty(M)$-linear map $\widetilde{D} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow E$ such that $\widetilde{D} \circ \delta_M = D$. Moreover, the linear mapping

$$Hom_{C^\infty(M)}(\Omega_{\mathbb{R}}[C^\infty(M)], E) \longrightarrow Der_{\mathbb{R}}(C^\infty(M), E), \psi \longmapsto \psi \circ \delta_M$$

is an isomorphism of $C^\infty(M)$-modules [1].

In particular, $(\Omega_{\mathbb{R}}[C^\infty(M)])^* \simeq Der_{\mathbb{R}}[C^\infty(M)]$.

For any $p \in \mathbb{N}$, $\Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$ denotes the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree p from $\Omega_{\mathbb{R}}[C^\infty(M)]$ into $C^\infty(M)$ and $\Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]) =$

$\bigoplus_{p\in\mathbb{N}} \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, the exterior $C^\infty(M)$-algebra of $\Omega_{\mathbb{R}}[C^\infty(M)]$. The set $\Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ is generated by elements of the form $\delta_M\left(f_1\right)\wedge\delta_M\left(f_2\right)\wedge...\wedge\delta_M\left(f_p\right)$, for any $f_1,...,f_p\in C^\infty(M)$.

The $\mathbb{R}$-linear map $\delta^1_M:\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow\Lambda^2\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ defined by

$$\delta^1_M(x)=\sum_{i\in I:finite}\left[\delta_M(f_i)\wedge\delta_M(g_i)\right]$$

satisfies,

$$\delta^1_M\left(f\cdot x\right)=\delta_M\left(f\right)\wedge x+f\cdot\delta^1_M\left(x\right);\quad \delta^1_M\circ\delta_M=0,$$

for any $x=\sum_{i\in I:finite} f_i\cdot\delta_M(g_i)\in\Omega_{\mathbb{R}}[C^\infty(M)]$ and $f\in C^\infty(M)$.

For $D\in Der_{\mathbb{R}}[C^\infty(M)]$, the map $\sigma_D:\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow\Omega_{\mathbb{R}}[C^\infty(M)]$ defined by

$$\sigma_D(x,y)=\widetilde{D}(x)y-\widetilde{D}(y)x$$

is skew-symmetric $C^\infty\left(M\right)$-bilinear, for any $x,y\in\Omega_{\mathbb{R}}[C^\infty(M)]$. Therefore, according to the universal property of exterior algebra [1], there exists a unique $C^\infty(M$-linear map $i_D:\Lambda^2\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)\longrightarrow\Omega_{\mathbb{R}}[C^\infty(M)]$ such that, for any $x,y\in\Omega_{\mathbb{R}}[C^\infty(M)]$,

$$i_D(x\wedge y)=\sigma_D(x,y)=\widetilde{D}(x)y-\widetilde{D}(y)x.$$

We define the Lie derivative with respect to a derivation $D\in Der_{\mathbb{R}}[C^\infty(M)]$ by the $\mathbb{R}$-linear map

$$\mathfrak{L}_D: i_D\circ\delta^1_M+\delta_M\circ\widetilde{D}:\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow\Omega_{\mathbb{R}}[C^\infty(M)].$$

For any $D\in Der_{\mathbb{R}}[C^\infty(M)]$, $x=\sum_{i\in I:finite} f_i\cdot\delta_M(g_i)\in\Omega_{\mathbb{R}}[C^\infty(M)]$ and $f\in C^\infty(M)$, we get

$$\mathfrak{L}_D(x)=\sum_{i\in I:finite}\left[D(f_i)\delta_M(g_i)+f_i\delta_M(D(g_i))\right];\tag{2}$$

$$(\mathfrak{L}_{f\cdot D})(x)=f\cdot\mathfrak{L}_D(x)+\widetilde{D}(x)\delta_M(f);\tag{3}$$

$$\mathfrak{L}_D(f\cdot x)=[D(f)]\,x+f\cdot\mathfrak{L}_D(x);\tag{4}$$

$$\mathfrak{L}_D\left[\delta_M(f)\right]=\delta_M\left[D(f)\right].\tag{5}$$

For any integer $p\geq 1$, recall that a skew-symmetric $\mathbb{R}$-multilinear map $D:[C^\infty(M)]^p\longrightarrow E$ is a skew-symmetric p-derivation if for any $f_1,f_2,...,f_p\in C^\infty(M)$, the map

$$D^i=D\left(f_1,...,\widehat{f_i},...,f_p\right):C^\infty(M)\longrightarrow E, f_i\longmapsto D\left(f_1,f_2,...,f_{i-1},f_i,f_{i+1},...,f_p\right)$$

is a derivation, for any $i=1,2,...,p$ [7]. We denote by $Der^p_{sks}\left[C^\infty(M),E\right]$ the $C^\infty(M)$-module of the skew-symmetric p-derivations on $C^\infty(M)$ with coefficients in E. Let

$$\delta^{(p)}_M:[C^\infty(M)]^p\longrightarrow(\Omega_{\mathbb{R}}[C^\infty(M)])^p$$

be the map such that $f_1, ..., f_p$ in $C^\infty(M)$,

$$\delta_M^{(p)}(f_1, ..., f_p) = (\delta_M(f_1), \delta_M(f_2), ..., \delta_M(f_p)) .$$

Theorem 2. For any $D \in \mathcal{D}er^p_{sks}[C^\infty(M), E]$, there exists a unique skew-symmetric $C^\infty(M)$-multilinear map of degree p,
$\tilde{D} : [\Omega_{\mathbb{R}}[C^\infty(M)]]^p \longrightarrow E$ such that

$$\tilde{D}\left(\delta_M\left(f_1\right), ..., \delta_M\left(f_p\right)\right) = D\left(f_1, ..., f_p\right) \tag{6}$$

and there exists a unique $C^\infty(M)$-linear map
$\overline{D} : \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right) \longrightarrow E$ such that

$$\overline{D}\left(\delta_M\left(f_1\right) \wedge \delta_M\left(f_2\right) \wedge ... \wedge \delta_M\left(f_p\right)\right) = D\left(f_1, f_2, ..., f_p\right), \tag{7}$$

for any $f_1, ..., f_p \in C^\infty(M)$.
Proof. By definition of skew-symmetric p-derivation D, the map

$$D^i : C^\infty(M) \longrightarrow E, f_i \longmapsto D^i(f_i) = D\left(f_1, f_2, ..., f_{i-1}, f_i, f_{i+1}, ..., f_p\right)$$

is a derivation, for any $i = 1, 2, ..., p$. Then, using Theorem 1, there is a unique $C^\infty(M)$-lineair map $\widetilde{D^i} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow E$ such that $\widetilde{D^i} \circ \delta_M = D^i$ that is,

$$\widetilde{D^i}\left[\delta_M(f_i)\right] = D^i(f_i) = D\left(f_1, f_2, ..., f_{i-1}, f_i, f_{i+1}, ..., f_p\right)$$

for any $f_1, ..., f_p \in C^\infty(M)$. We deduce the exitence and the uniqueness of skew-symmetric $C^\infty(M)$-multilinear map $\widetilde{D} : (\Omega_{\mathbb{R}}[C^\infty(M)])^p \longrightarrow E$ of degree p such that $\widetilde{D} \circ \delta_M^{(p)} = D$, that is, for any $f_1, ..., f_p \in C^\infty(M)$,

$$\widetilde{D}\left(\delta_M\left(f_1\right), ..., \delta_M\left(f_p\right)\right) = D\left(f_1, ..., f_p\right).$$

According to the universal property of exterior produit [1], there exists a unique skew-symmetric $C^\infty(M)$-multilinear map $\widetilde{D}$ of degree p such that $\widetilde{D} \circ \delta_M^{(p)} = D$ induces a unique $C^\infty(M)$-linear map

$$\overline{D} : \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right) \longrightarrow E$$

such that, for any $f_1, ..., f_p \in C^\infty(M)$,

$$\begin{aligned}\overline{D}\left(\delta_M\left(f_1\right) \wedge \delta_M\left(f_2\right) \wedge ... \wedge \delta_M\left(f_p\right)\right) &= \widetilde{D}\left(\delta_M\left(f_1\right), \delta_M\left(f_2\right), ..., \delta_M\left(f_p\right)\right) \\ &= D\left(f_1, f_2, ..., f_p\right).\end{aligned}$$

3. CANONICAL FORM OF POISSON-NAMBU MANIFOLD

Let M be a smooth manifold and $r \in \mathbb{N}$, $2 \leq r \leq dimM$. Recall that a Nambu-Poisson structure on M of order r is a skew-symmetric r-linear

$$\{, ..., \} : [C^\infty(M)]^r = C^\infty(M) \times ... \times C^\infty(M) \longrightarrow C^\infty(M)$$

which satifies the Leibniz rule:

$$\{f_1, f_2, , f_{r-1}, gh\} = \{f_1, f_2, , f_{r-1}, g\}\, h + g\, \{f_1, f_2, , f_{r-1}, h\} \tag{8}$$

and the Fundamental identity:

$$\{f_1, f_2, ..., f_{r-1}, \{g_1, g_2, ..., g_r\}\} = \sum_{i=1}^{r} \{g_1, g_2, ..., g_{i-1}, \{f_1, f_2, ..., f_{r-1}, g_i\}, g_{i+1}, ..., g_r\}$$

for any $f_1, f_2, ..., f_{r-1}, g, h, g_1, g_2, ..., g_r$ in $C^\infty(M)$. The pair $(M, \{, ..., \})$ is called a Nambu-Poisson manifold of order r. The bracket $\{, ..., \}$ is called a Nambu-Poisson bracket of order r. If $r = 2$, we rediscover Poisson structures [4], [5] and [9]. Thus, Nambu-Poisson structures can be seen as a kind of generalization of Poisson structures.

When $(M, \{, ..., \})$ is a Nambu-Poisson manifold of order r, the map

$$ad\,(f_1, f_2, ..., f_{r-1}) : C^\infty(M) \longrightarrow C^\infty(M)\,, g \longmapsto \{f_1, f_2, ..., f_{r-1}, g\}$$

is a derivation and the map

$$ad : [C^\infty(M)]^{r-1} \longrightarrow Der_{\mathbb{R}}\,[C^\infty(M)]\,, (f_1, f_2, ..., f_{r-1}) \longmapsto ad\,(f_1, f_2, ..., f_{r-1})$$

is a skew-symmetric $(r-1)$-derivation. By the Theorem 2 , there exists a unique $C^\infty(M)$-linear map

$$\overline{ad} : \Lambda^{r-1}\left(\Omega_{\mathbb{R}}\,[C^\infty(M)]\right) \longrightarrow Der_{\mathbb{R}}\,[C^\infty(M)]$$

such that, for any $f_1, f_2, ..., f_{r-1} \in C^\infty(M)$,

$$\overline{ad}\,(\delta_M(f_1) \wedge \delta_M(f_2) \wedge ... \wedge \delta_M(f_{r-1})) = ad\,(f_1, f_2, ..., f_{r-1})\,. \tag{9}$$

That is, for any $f \in C^\infty(M)$,

$$\left[\overline{ad}\,(\delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}))\right](f) = \{f_1, f_2, ..., f_{r-1}, f\}\,. \tag{10}$$

Given any $(r-1)$ functions $f_1, f_2, ..., f_{r-1} \in C^\infty(M)$, the Hamiltonian vector field $X_{f_1 \dots f_{r-1}}$ associated to these functions is defined by

$$X_{f_1 \dots f_{r-1}} = \overline{ad}\,(\delta_M(f_1) \wedge \delta_M(f_2) \wedge ... \wedge \delta_M(f_{r-1}))\,.$$

That is,

$$\{f_1, f_2, ..., f_{r-1}, g\} = X_{f_1 \dots f_{r-1}}(g)$$

$f_1, f_2, ..., f_{r-1}, g \in C^\infty(M)$.

In terms of Hamiltonian vector fields, the fundamental identity can be expressed as

$$\left[X_{f_1 \dots f_{r-1}}, X_{g_1 \dots g_{r-1}}\right] = \sum_{i=1}^{r-1} X_{g_1 \dots \{f_1, f_2, \dots, f_{r-1}, g_i\} \dots g_{r-1}}$$

for $f_1, f_2, ..., f_{r-1}, g_1, g_2, ..., g_{r-1} \in C^\infty(M)$, where the bracket on the left-hand side is the Lie bracket of vector fields.

Proposition 1. If $(M, \{., ..., .\})$ is a Nambu-Poisson manifold of order r, then for any $f_1, f_2, ..., f_r \in C^\infty(M)$,

$$\mathfrak{L}_{\overline{ad}\left(\delta_M(f_1)\wedge\delta_M(f_2)\wedge...\wedge\widehat{\delta_M(f_i)}\wedge...\wedge\delta_M(f_r)\right)}\delta_M(f_i) = (-1)^{r-i}\,\delta_M\{f_1, f_2, ..., f_r\}. \tag{11}$$

Proof. For any $f_1, f_2, ..., f_r \in C^\infty(M)$, From (5),

$$\begin{aligned}
&\mathfrak{L}_{\overline{ad}\left(\delta_M(f_1)\wedge\delta_M(f_2)\wedge...\wedge\widehat{\delta_M(f_i)}\wedge...\wedge\delta_M(f_r)\right)}\delta_M(f_i)\\
&= \delta_M\left[\overline{ad}\left(\delta_M(f_1)\wedge\delta_M(f_2)\wedge...\wedge\widehat{\delta_M(f_i)}\wedge...\wedge\delta_M(f_r)\right)(f_i)\right]\\
&= \delta_M\left\{f_1, f_2, ..., \widehat{f_i}, ..., f_r, f_i\right\}\\
&= (-1)^{r-i}\,\delta_M\{f_1, f_2, ..., f_r\}.
\end{aligned}$$

Theorem 3. Let $(M, \{., ..., .\})$ be a Nambu-Poisson manifold of order r. Then there exists a unique skew symmetric r-form $\pi : (\Omega_{\mathbb{R}}[C^\infty(M)])^r \longrightarrow C^\infty(M)$ such that, for any $f_1, f_2, ..., f_r \in C^\infty(M)$,

$$\{f_1, f_2, ..., f_r\} = \pi(\delta_M(f_1), \delta_M(f_2), ..., \delta_M(f_r)) \tag{12}$$

defines an r-Lie algebra structure on $C^\infty(M)$. Moreover, π induces a unique $C^\infty(M)$-linear map

$$\omega_M : \Lambda^r(\Omega_{\mathbb{R}}[C^\infty(M)]) \longrightarrow C^\infty(M)$$

such that, for any $f_1, f_2, ..., f_r \in C^\infty(M)$,

$$\omega_M(\delta_M(f_1)\wedge...\wedge\delta_M(f_{r-1})\wedge\delta_M(f_r)) = \{f_1, ..., f_{r-1}, f_r\}. \tag{13}$$

Proof. When $(M, \{., ..., .\})$ is a Nambu-Poisson manifold of order r, the map

$$\{., ..., .\} : [C^\infty(M)]^r \longrightarrow C^\infty(M), (f_1, f_2, ..., f_r) \longmapsto \{f_1, f_2, ..., f_r\}$$

is a skew-symmetric r-derivation. By the Theorem 2, there exists a unique skew-symmetric $C^\infty(M)$-multilinear map $\pi : (\Omega_{\mathbb{R}}[C^\infty(M)])^r \longrightarrow C^\infty(M)$ such that, for any $f_1, f_2, ..., f_r \in C^\infty(M)$,

$$\{f_1, f_2, ..., f_r\} = \pi(\delta_M(f_1), \delta_M(f_2), ..., \delta_M(f_r))$$

and π induces a unique $C^\infty(M)$-linear map

$$\omega_M : \Lambda^r(\Omega_{\mathbb{R}}[C^\infty(M)]) \longrightarrow C^\infty(M)$$

such that,

$$\omega_M(\delta_M(f_1)\wedge...\wedge\delta_M(f_{r-1})\wedge\delta_M(f_r)) = \{f_1, ..., f_{r-1}, f_r\}.$$

The $C^\infty(M)$-linear map ω_M is called canonical form of Nambu-Poisson manifold $(M, \{., ..., .\})$.

Proposition 2. For any $u \in \Lambda^{r-1}(\Omega_{\mathbb{R}}[C^\infty(M)])$ and for any $f \in C^\infty(M)$

$$\left[\overline{ad}(u)\right](f) = \omega_M(u \wedge \delta_M(f)). \tag{14}$$

Proof. For any $u \in \Lambda^{r-1}(\Omega_{\mathbb{R}}[C^\infty(M)])$, putting $u = g \cdot \delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1})$. Since $\overline{ad}$ is a $C^\infty(M)$-linear map and by (10), we get

$$\left[\overline{ad}(g \cdot \delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}))\right](f) = g \cdot \{f_1, f_2, ..., f_{r-1}, f\}.$$

On the other hand, by (13), we get

$$\omega_M(g \cdot \delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}) \wedge \delta_M(f)) = g \cdot \{f_1, f_2, ..., f_r\}.$$

Thus,

$$\left[\overline{ad}(u)\right](f) = \omega_M(u \wedge \delta_M(f)).$$

Proposition 3. For any $X = x_1 \wedge x_2 \wedge ... \wedge x_r \in \Lambda^r(\Omega_{\mathbb{R}}[C^\infty(M)])$,

$$\omega_M(x_1 \wedge x_2 \wedge ... \wedge x_r) = i_{\overline{ad}(x_1 \wedge ... \wedge x_{r-1})}(x_r). \tag{15}$$

Proof. Since for $f \in C^\infty(M)$,

$$\left[\overline{ad}(x_1 \wedge x_2 \wedge ... \wedge x_{r-1})\right](f) = \omega_M(x_1 \wedge x_2 \wedge ... \wedge x_{r-1} \wedge \delta_M(f))$$

and

$$i_{\overline{ad}(x_1 \wedge x_2 \wedge ... \wedge x_{r-1})}(\delta_M(f)) = \left[\overline{ad}(x_1 \wedge x_2 \wedge ... \wedge x_{r-1})\right](f), \tag{16}$$

then

$$\omega_M(x_1 \wedge x_2 \wedge ... \wedge x_{r-1} \wedge \delta_M(f)) = i_{\overline{ad}(x_1 \wedge x_2 \wedge ... \wedge x_{r-1})}(\delta_M(f)).$$

On the other hand, if $x_r = \sum_{i \in I: finite} f_i \cdot \delta_M(g_i)$, we have

$$\omega_M(x_1 \wedge ... \wedge x_r) = \sum_{i \in I: finite} f_i \cdot \omega_M(x_1 \wedge ... \wedge x_{r-1} \wedge \delta_M(g_i))$$

By (16), we get

$$\begin{aligned} \omega_M(x_1 \wedge ... \wedge x_r) &= i_{\overline{ad}(x_1 \wedge ... \wedge x_{r-1})}\left(\sum_{i \in I: finite} f_i \cdot \delta_M(g_i)\right) \\ &= i_{\overline{ad}(x_1 \wedge ... \wedge x_{r-1})}(x_r). \end{aligned}$$

4. NAMBO-POISSON FORM BRACKET ON THE MODULE OF KÄHLER DIFFERENTIALS

Let $(M, \{., ..., .\})$ be a Nambu-Poisson manifold of order r equipped with canonical form ω_M. We define the Nambu-Poisson form bracket on $\Omega_{\mathbb{R}}[C^\infty(M)]$ as the map

$$[.,...,.]_{\omega_M} : (\Omega_{\mathbb{R}}[C^\infty(M)])^r \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$$

such that

$$\begin{aligned}[x_1, x_2, ..., x_r]_{\omega_M} &= \sum_{i=1}^{r} (-1)^{r+i} \mathfrak{L}_{\overline{ad}(x_1\wedge x_2\wedge...\wedge\widehat{x_i}\wedge...\wedge x_r)} x_i \\ &\quad - (r-1)\,\delta_M[\omega_M(x_1\wedge x_2\wedge...\wedge x_r)]\end{aligned} \tag{17}$$

for any $x_1, x_2, ..., x_r \in \Omega_{\mathbb{R}}[C^\infty(M)]$.

For $r = 2$, π is the Poisson 2-form of a Poisson algebra $C^\infty(M)$. For any $x, y \in \Omega_{\mathbb{R}}[C^\infty(M)]$, $\pi(x,y) = \omega_M(x\wedge y)$ and

$$[x,y]_\pi = \mathfrak{L}_{\overline{ad}(x)} y - \mathfrak{L}_{\overline{ad}(y)} x - \delta_M(\pi(x,y)).$$

Proposition 4. For any $f_1, f_2, ..., f_r \in C^\infty(M)$,

$$\delta_M\{f_1, f_2, ..., f_r\} = [\delta_M(f_1), \delta_M(f_2), ..., \delta_M(f_r)]_{\omega_M}. \tag{18}$$

and for any $f \in C^\infty(M)$, we have

$$[x_1, ..., x_{r-1}, f\cdot x_r]_{\omega_M} = f\cdot[x_1, ..., x_r]_{\omega_M} + \left[\overline{ad}(x_1\wedge...\wedge x_{r-1})\right](f)\cdot x_r. \tag{19}$$

Proof. From (11) and (17), for any $f_1, f_2, ..., f_{2r} \in C^\infty(M)$,

$$\begin{aligned}&[\delta_M(f_1), \delta_M(f_2), ..., \delta_M(f_r)]_{\omega_M} \\ =\ & \sum_{i=1}^{r} (-1)^{r+i}(-1)^{r-i} \mathfrak{L}_{\overline{ad}\left(\delta_M(f_1),\delta_M(f_2),...,\widehat{\delta_M(f_i)},...,\delta_M(f_r)\right)} \delta_M(f_i) \\ & - (r-1)\,\delta_M[\omega_M(\delta_M(f_1)\wedge\delta_M(f_2)\wedge...\wedge\delta_M(f_r))] \\ =\ & \delta_M(\{f_1, f_2, ..., f_r\}).\end{aligned}$$

From (17), for any $f \in C^\infty(M)$, we have

$$\begin{aligned}&[x_1, x_2, ..., x_{r-1}, f\cdot x_r]_{\omega_M} \\ =\ & \sum_{i=1}^{r} (-1)^{r+i} \mathfrak{L}_{\overline{ad}(x_1\wedge x_2\wedge...\wedge\widehat{x_i}\wedge...\wedge f\cdot x_r)} x_i - (r-1)\,\delta_M[\omega_M(x_1\wedge x_2\wedge...\wedge f\cdot x_r)] \\ =\ & \mathfrak{L}_{\overline{ad}\left(x_1\wedge x_2\wedge...\wedge x_{r-1}\wedge\widehat{f\cdot x_r}\right)}(f\cdot x_r) + \sum_{i=1}^{r-1} (-1)^{r+i} \mathfrak{L}_{\overline{ad}(x_1\wedge x_2\wedge...\wedge\widehat{x_i}\wedge...\wedge x_{r-1}\wedge f\cdot x_r)} x_i \\ & - (r-1)\,(\delta_M(f)\cdot\omega_M(x_1\wedge...\wedge x_r) + f\cdot\delta_M[\omega_M(x_1\wedge...\wedge x_r)])\end{aligned}$$

By (3) and (4), we get

$$\begin{aligned}&[x_1, x_2, ..., x_{r-1}, f\cdot x_r]_{\omega_M} \\ =\ & f\cdot[x_1, ..., x_r]_{\omega_M} + \left[\overline{ad}(x_1\wedge x_2\wedge...\wedge x_{r-1})\right](f)\cdot x_r \\ & +\delta_M(f)\cdot\left[-(r-1)\,\omega_M(x_1\wedge...\wedge x_r) + \sum_{i=1}^{r-1}(-1)^{r+i}(-1)^{r-i}\, i_{\overline{ad}(x_1\wedge x_2...\wedge x_{r-1})} x_r\right].\end{aligned}$$

That is,

$$\begin{aligned}&[x_1, x_2, ..., x_{r-1}, f \cdot x_r]_{\omega_M} \\ =\ & f \cdot [x_1, ..., x_r]_{\omega_M} + \left[\overline{ad}\left(x_1 \wedge x_2 \wedge ... \wedge x_{r-1}\right)\right](f) \cdot x_r \\ & +\delta_M(f) \cdot \left[-(r-1)\,\omega_M(x_1 \wedge ... \wedge x_r) + (r-1)\, i_{\overline{ad}(x_1 \wedge x_2 ... \wedge x_{r-1})} x_r\right].\end{aligned}$$

Therefore, by (15), we have

$$[x_1, ..., x_{r-1}, f \cdot x_r]_{\omega_M} = f \cdot [x_1, ..., x_r]_{\omega_M} + \left[\overline{ad}\left(x_1 \wedge ... \wedge x_{r-1}\right)\right](f) \cdot x_r.$$

Theorem 4. Let $(M, \{., ..., .\})$ be a Nambu-Poisson manifold of order r equipped with canonical form ω_M. Then the bracket $[., ..., .]_{\omega_M}$ defines an r-Lie algebra structure on $\Omega_{\mathbb{R}}[C^\infty(M)]$.

Proof. The Nambu-Poisson form-bracket is skew-symmetric. For any $x_1, ..., , x_r, x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f_1, ..., f_{r-1} \in C^\infty(M)$,

$$[\delta_M(f_1), ..., \delta_M(f_{r-1}), x]_{\omega_M} = \mathfrak{L}_{\overline{ad}(\delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}))}(x) \tag{20}$$

and

$$\mathfrak{L}_{\overline{ad}(\delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}))}\left([x_1, ..., x_r]_{\omega_M}\right) = \sum_{i=1}^{r} \left[x_1, ..., \mathfrak{L}_{\overline{ad}(\delta_M(f_1) \wedge ... \wedge \delta_M(f_{r-1}))} x_i, ..., x_r\right]_{\omega_M} \tag{21}$$

From (20 and (21, for any $x_1, ..., x_{r-1}, x_r \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f_1, ..., f_{r-1} \in C^\infty(M)$, the following identity holds:

$$\begin{aligned}&\left[\delta_M(f_1), ..., \delta_M(f_{r-1}), [x_1, ..., x_r]_{\omega_M}\right]_{\omega_M} \\ =\ & \sum_{i=1}^{r} \left[x_1, ..., x_{i-1}, [\delta_M(f_1), ..., \delta_M(f_{r-1}), x_i]_{\omega_M}, ..., x_r\right]_{\omega_M}\end{aligned}$$

Since δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$, for any $x_1, x_2, ..., x_{r-1}, y_1, y_2, ..., y_r \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have

$$\begin{aligned}&\left[x_1, x_2, ..., x_{r-1}, [y_1, y_2, ..., y_r]_{\omega_M}\right]_{\omega_M} \\ &= \sum_{i=1}^{r} \left[y_1, y_2, ..., y_{i-1}, [x_1, x_2, ..., x_{r-1}, y_i]_{\omega_M}, y_{i+1}, ..., y_r\right]_{\omega_M}.\end{aligned}$$

That is the bracket the $[., ..., .]_{\omega_M}$ satisfies the fundamental identity. Thus the pair $\left(\Omega_{\mathbb{R}}[C^\infty(M)], [., ..., .]_{\omega_M}\right)$ is an r-Lie algebra.

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