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Abundant Numbers and the Riemann Hypothesis

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Abstract

Colossally abundant numbers are likely to result in counterexamples of the Riemann hypothesis. Similarly, abundant numbers keep certain partial sums of $e^{\gamma}\Phi(x)-x(x+1)/2$ from increasing monotonically. Number-theoretic properties of x values where the above expression doesn't increase are investigated. Also, instances where the expression doesn't increase sometimes result in almost linear subsequences of abundant numbers (one such subsequence contains the superabundant numbers). In other instances, subsequences of primes result that give a staircase approximately equal to a multiple of Gauss' $G(x)$ function.

Keywords: sum of divisors function, Euler's totient function, abundant numbers, Riemann hypothesis, Gauss' $G(x)$ function, staircase of primes, second Chebyshev function

1. INTRODUCTION

Background material is as follows. If $n \geq 1$ the Euler totient function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n. If $n > 1$, then $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2, \ldots, p_k$ are primes. The Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $a_1 = a_2 = \cdots = a_k = 1$, or $\mu(n) = 0$ otherwise. Let $\Phi(x)$ denote $\sum_{i=1}^{x} \varphi(i)$ and $M(x)$ (Mertens' function) denote $\sum_{i=1}^{x} \mu(i)$. Mikolás' [1] Lemma 2 is given by equation 1.

$$\sum_{n=1}^{[x]} \sum_{d|n} f(d) g(\frac{n}{d}) = \sum_{d=1}^{[x]} f(d) \sum_{\delta=1}^{[x/d]} g(\delta) = \sum_{d=1}^{[x]} g(d) \sum_{\delta=1}^{[x/d]} f(\delta) \quad (1)$$

From this, Mikolás determined that equation 2 holds.

$$\Phi(x) = \sum_{n=1}^{[x]} \varphi(n) = \sum_{n=1}^{[x]} n M(\frac{x}{n}) = \frac{1}{2} \sum_{n=1}^{[x]} \mu(n) [\frac{x}{n}]^2 + \frac{1}{2} \quad (2)$$

Theorem 3.7 of Apostol's [2] book is

$$\sum_{n\leq x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x) \tag{3}$$

The average order of $\varphi(n)$ is then $3n/\pi^2$.

Let $\sigma_x(i)$ denote the sum of positive divisors function ($\sigma_x(i)=\sum_{d|i}d^x$). Theorem 3.4 of Apostol's book is

$$\sum_{n\leq x}\sigma_1(n)=\frac{1}{2}\zeta(2)x^2+O(x\log x) \tag{4}$$

It can be shown that $\zeta(2)=\pi^2/6$. The average order of $\sigma_1(n)$ is then $\pi^2 n/12$. $\sigma_1(n)$ is commonly denoted by $\sigma(n)$.

Let γ denote Euler's constant. Gronwall [3] determined the maximal order of the sum of divisors function.

$$\limsup_{n\to\infty}\frac{\sigma(n)}{n\log\log n}=e^{\gamma} \tag{5}$$

The corresponding relationship for the totient function is

$$\liminf_{n\to\infty}\frac{\varphi(n)\log\log n}{n}=e^{-\gamma} \tag{6}$$

See 18.3 of Hardy and Wright's [4] book. Ramanujan [5] proved that the Riemann hypothesis implies $\sigma(n)<e^{\gamma}n\log\log n$ for sufficiently large n. Robin [6] proved that this inequality is true for all $n\geq 5041$ if and only if the Riemann hypothesis is true.

Let $\pi(x)$ denote the number of primes less than or equal to x. Gauss observed that an approximation of $\pi(x)$ (the staircase of primes) is $x/\log(x)$. Let $Li(x)$ denote $\int_2^x 1/\log(t)dt$. Usually $\pi(x)$ is less than $Li(x)$ and greater than $x/\log(x)$. A formulation of the Riemann hypothesis is that $\pi(x)$ is approximately equal to $Li(x)$ and that this approximation is essentially square root accurate. The Mangoldt function $\Lambda(n)$ is defined to be $\log(p)$ if $n=p^m$ for some prime p and some $m\geq 1$, or 0 otherwise. The second Chebyshev function $\psi(x)$ is defined to be $\sum_{n\leq x}\Lambda(n)$. The Riemann hypothesis is equivalent to the arithmetic statement $\psi(x)-x=o(x^{1/2+\epsilon})$ for all $\epsilon>0$.

2. AN ALGORITHM FOR COMPUTING $\Phi(x)-x(x+1)/2$

The algorithm uses Mikolás' eq.2 and the following theorem.

Theorem 1 $\sum_{n=1}^{x}M(\frac{x}{n})\sigma(n)=x(x+1)/2$

Proof 1 *Möbius inversion of* $x = \sum_{n \le x} 1$ *gives* $1 = \sum_{d|n} \mu(d)\sigma_0(\frac{n}{d})$. *(See page 95 of Apostol's book). Similarly, Möbius inversion of* $\frac{x(x+1)}{2} = \sum_{n \le x} n$ *gives* $n = \sum_{d|n} \mu(d)\sigma_1(\frac{n}{d})$. *Then* $\frac{x(x+1)}{2} = \sum_{n=1}^{x} \sum_{d|n} \mu(d)\sigma_1(\frac{n}{d})$. *Then by Mikolás' Lemma 2,* $\frac{x(x+1)}{2} = \sum_{d=1}^{x} \sigma_1(d) \sum_{\delta=1}^{[x/d]} \mu(\delta)$*, that is,* $\frac{x(x+1)}{2} = \sum_{d=1}^{x} \sigma_1(d) M(\frac{x}{d})$.

The algorithm uses the following partial sums.

$$L_{x,s} = c_1 \sum_{i=1}^{l-1} M(\frac{x}{i})i - c_2 \sum_{i=1}^{l-1} M(\frac{x}{i})\sigma(i) \tag{7}$$

$$R_{x,s} = c_1 \sum_{i=l}^{x} M(\frac{x}{i})i - c_2 \sum_{i=l}^{x} M(\frac{x}{i})\sigma(i) \tag{8}$$

Setting c_1 and c_2 to 1 gives $\Phi(x) - x(x+1)/2$. The factor of ($c_1 = e^\gamma$, $c_2 = 1$) will be investigated here. Let δ denote a decimation value. The x values are s, $s+\delta$, $s+2\delta$,... and the s values are 1, 2, 3,$\ldots, \delta$. For example, when $\delta = 2$ and $s = 1$, every other $e^\gamma \Phi(x) - x(x+1)/2$ value is computed. Repeating this procedure for $s = 2$ gives the remainder of the $e^\gamma \Phi(x) - x(x+1)/2$ values. The l values start with 1 and are incremented by one for each successive x value. A property of the algorithm is that only the $M(1)$, $M(2)$, $M(3)$,...,$M(\delta)$ values are used in the computation of $R_{x,s}$ (in the case $\delta \neq s$ only the values up to $M(\delta - 1)$ are used). This simplifies computations for δ equal to 2 or 3.

Let $R'_{x,s}$ denote $R_{x,s}/x$ and $L'_{x,s}$ denote $L_{x,s}/x$. A plot of $L'_{x,s}$ for $x = 1, 3, 5, \ldots, 999$, $\delta = 2$, and $s = 1$ is

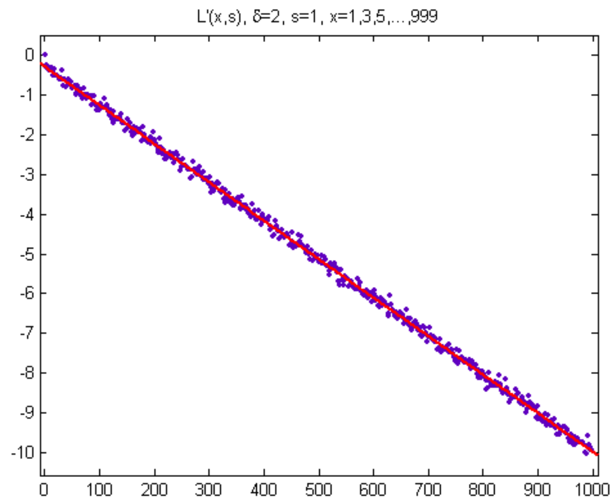

Figure 1: Plot of $L'_{x,s}$, $\delta = 2$, $s = 1$

None of the values are greater than 0. For a linear least-squares fit of the curve, $p_1 = -0.009683$ with a 95% confidence interval of $(-0.009717, -0.00965)$, $p_2 = -0.2931$ with a 95% confidence interval of $(-0.3126, -0.2736)$, SSE=6.136, R-squared=0.9984, and RMSE=0.111.

A plot of $R'_{x,s}$ for $x = 1, 3, 5, \ldots, 999$, $\delta = 2$, and $s = 1$ is

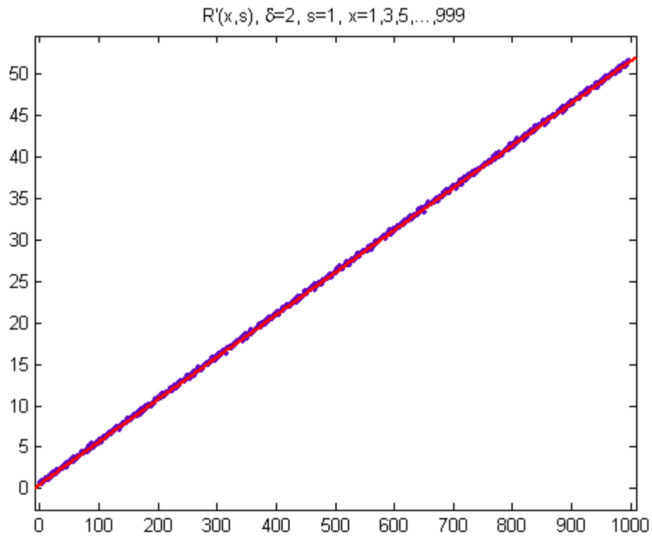

Figure 2: Plot of $R'_{x,s}$, $\delta = 2$, $s = 1$

None of the values are less than 0. For a linear least-squares fit of the curve, $p_1 = 0.05105$ with a 95% confidence interval of (0.051, 0.5541), $p_2 = 0.5271$ with a 95% confidence interval of (0.5, 0.5541), SSE=11.81, R-squared=0.9999, and RMSE=11.81.

Similar results are obtained for other δ and s values, so $R'_{x,s}$ appears to be an upper bound of $(e^\gamma \Phi(x) - x(x+1)/2)/x$. More generally,

Conjecture 1 $R'_{x,s} > (e^\gamma \Phi(x) - x(x+1)/2)/x > R'_{x,s} \cdot \frac{\delta-1}{\delta}$ *for sufficiently large* x.

For $\delta = 2$, $R'_{x,s} > (e^\gamma \Phi(x) - x(x+1)/2)/x > R'_{x,s} \cdot \frac{\delta-1}{\delta}$ without exception.

A plot of $R'_{x,s}$, $(e^\gamma \Phi(x) - x(x+1)/2)/x$, and $R'_{x,s} \cdot \frac{\delta-1}{\delta}$ for $\delta = 3$, $s = 1$, and $x = 1, 4, 7, \ldots, 198$, is

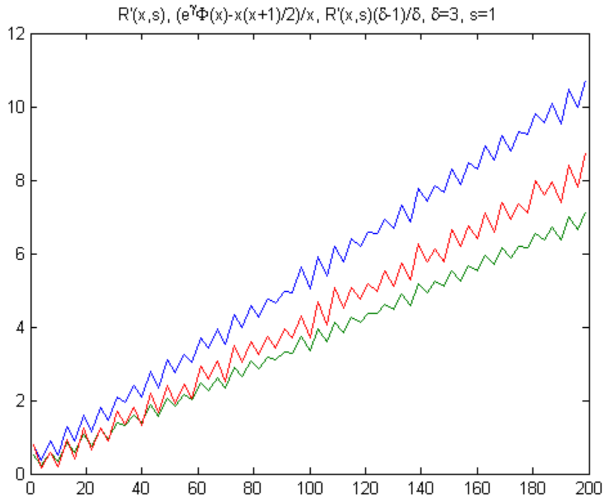

Figure 3: Plot of $R'_{x,s}$, $(e^{\gamma}\Phi(x) - x(x+1)/2)/x$, $R'_{x,s} \cdot \frac{\delta-1}{\delta}$, $\delta = 3$, $s = 1$

$R'_{x,s} \cdot \frac{\delta-1}{\delta}$ is greater than $(e^{\gamma}\Phi(x) - x(x+1)/2)/x$ for x=4, 7, 10, 16, 22, 28, and 40. In general, there are fewer exceptions for larger δ values.

3. ABUNDANT NUMBERS

Perfect numbers are considered to be abundant numbers. Only odd abundant numbers divisible by 3 are considered. An empirical result is

Conjecture 2 $R'_{x,s} < R'_{x-\delta,s}$, $2|x$, *only if* x *is an abundant number or 6 does not divide* x.

If $R'_{x,s} < R'_{x-\delta,s}$, $2|x$, $6|\delta$, and $6|s$, the only possibility is that x is an abundant number.

3.1. $\delta = 2$

Conjecture 3 *If* x *is an odd abundant number,* $\delta = 2$*, and* $s = 1$ *then* $R'_{x,s} < R'_{x-\delta,s}$.

The odd abundant numbers less than 10000 are 945, 1575, 2205, 2835, 3465, 4095, 4725, 5355, 5775, 5985, 6435, 6615, 6825, 7245, 7425, 7875, 8085, 8415, 8505, 8925, 9135, 9555, and 9765. To satisfy the first premise of the article, the odd abundant numbers must increase almost linearly. A plot of the odd abundant numbers less than 100000 is

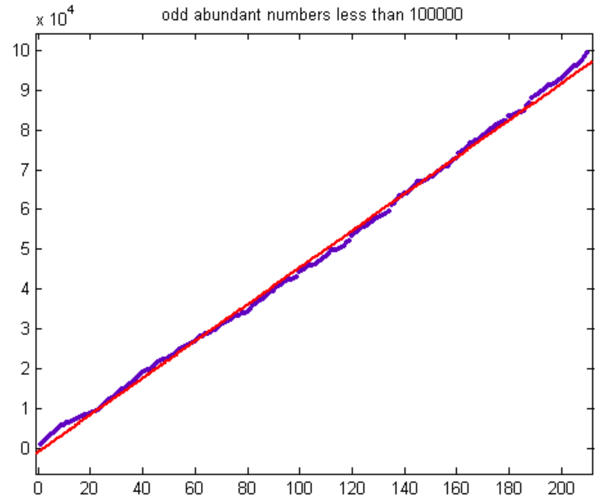

Figure 4: Odd abundant numbers less than 100000.

For a linear least-squares fit of the values, $p_1 = 461.8$ with a 95% confidence interval of (459.1, 464.5), $p_2 = -704.6$ with a 95% confidence interval of $(-1033, -376.1)$, SSE=$3.01 \cdot 10^8$, R-squared=0.9982, and RMSE=1203.

$R'_{x,s} < R'_{x-\delta,s}$, $\delta = 2$, $s = 1$, for $x \leq 50000$ in 9002 instances, but only 114 of these x values are odd abundant numbers. The remaining 8888 x values will be referred to as being "sparse" numbers. The 930 prime sparse numbers are 3, 31, 61, 127, 151, 181, 199, 211, 241,. . ., 33769. A plot of these values is

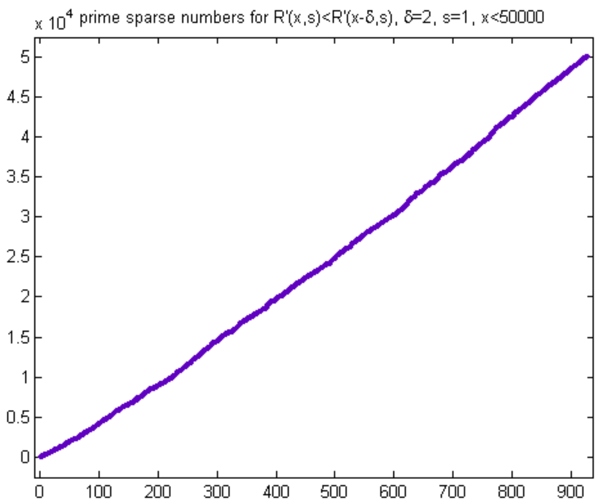

Figure 5: Prime sparse numbers.

This curve resembles the usual curve for all the primes. See Mazur and Stein's [7] book for graphs of the staircase of primes. A plot of the staircase obtained from the 930 primes (the "count") and Gauss' $G(x)$ curve ($x/\log x$) is

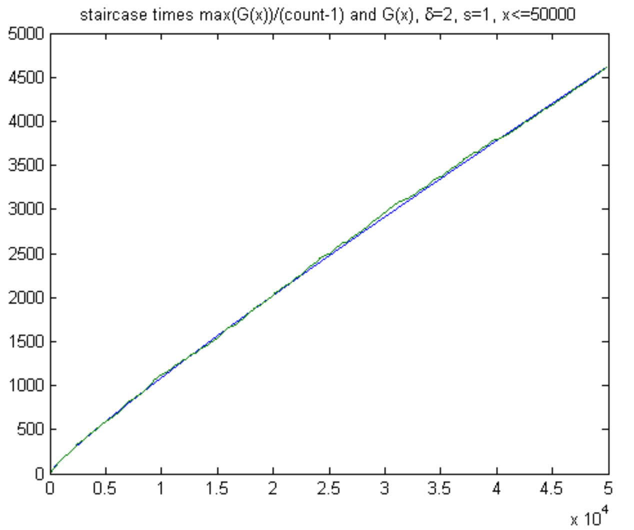

Figure 6: Scaled staircase of primes and $G(x)$

The portion of the staircase consisting of zeros (the first two elements) is not included in the graph and the scaling factor is 4616.0/929 (about 4.9688). The curves roughly coincide - this is the second premise of the article. This phenomenon also occurs for a fixed prime-power multiple of primes. In the above, 3 times the 345 primes 5, 7, 17, 37, 47, 61, 67, 97, 103, 127, 137, 167,. . .,16657 are sparse numbers. A plot of the staircase obtained from the 345 primes and Gauss' $G(x)$ curve is

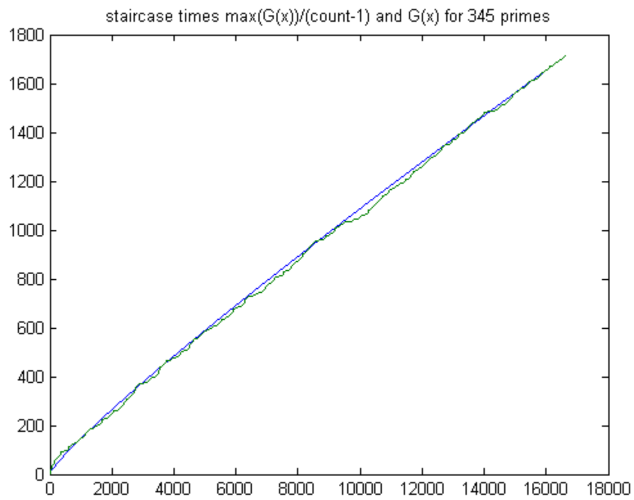

Figure 7: Scaled staircase of primes and $G(x)$

The portion of the staircase consisting of zeros (the first four elements) is not included in the graph and the scaling factor is 1713.5/344 (about 4.9520). In the above 8888 sparse numbers, the numbers of primes for prime factors of 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, and 41 are 345, 422, 276, 156, 128, 98, 80, 70, 56, 48, 42, and 34 respectively. The numbers of primes for prime-power factors of 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, and 29^2 are 290, 148, 49, 15, 13, 8, 5, 3, and 2 respectively. The numbers of primes for prime-power factors of 3^3, 5^3, 7^3, and 11^3 are 137, 38, 9, and 2 respectively.

For $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 2$, $s = 1$, and $x <= 5000$, there are 1391 sparse numbers. For a linear least-squares fit of these values, $p_1 = 3.581$ with a 95% confidence interval of (3.58, 3.582), $p_2 = 14.36$ with a 95% confidence interval of (13.68, 15.05), SSE=$5.855 \cdot 10^4$, R-squared=1, and RMSE=6.493. A plot of $\psi(x)$ for $x \leq 1391$ is

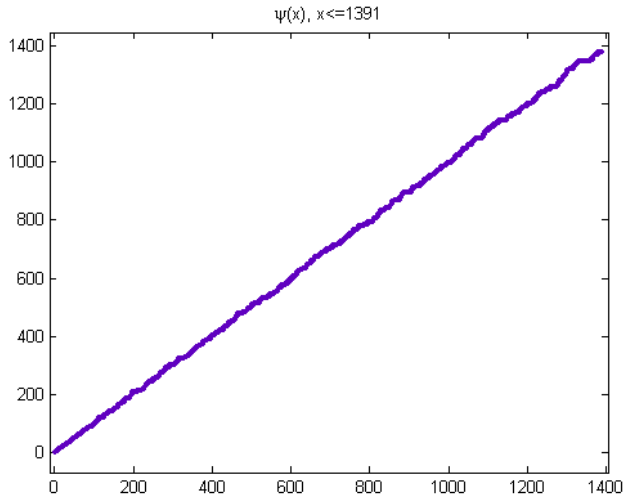

Figure 8: Second Chebyshev function

For a linear least-squares fit of the curve, $p_1 = 1.001$ with a 95% confidence interval of (0.9998, 1.001), $p_2 = -1.529$ with a 95% confidence interval of (-2.096, -0.9626), SSE=$4.027 \cdot 10^4$, R-squared=0.9995, and RMSE=5.384. The approximation error is somewhat smaller than that for the sparse numbers (except for R-squared) but the largest sparse number is about 3.626 times larger than the largest $\psi(x)$ value. More empirical results are

Conjecture 4 *If $\delta = 2$, $s = 2$, $R'_{x,s} < R'_{x-\delta,s}$, and $R'_{x,s} < R'_{x-2\delta,s}$, then x is an abundant number.*

Conjecture 5 *If $\delta = 2$, $s = 2$, $R'_{x,s} < R'_{x-\delta,s}$, and $3|x$, then x is an abundant number.*

$R'_{x,s} < R'_{x-\delta,s}$, $\delta = 2$, $s = 2$, for $x <= 10000$ in 1794 instances and 1201 of these x values are abundant numbers. A plot of the abundant numbers is

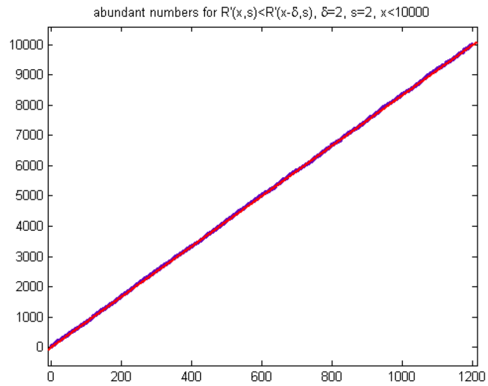

Figure 9: Abundant numbers

For a linear least-squares fit of the curve, $p_1 = 8.323$ with a 95% confidence interval of (8.321, 8.326), $p_2 = -3.422$ with a 95% confidence interval of $(-5.114, -1.73)$, SSE=$2.674 \cdot 10^5$, R-squared=1, and RMSE=14.93. For a linear least-squares fit of all the 2491 abundant numbers less than or equal to 10000, $p_1 = 4.002$ with a 95% confidence interval of (4.001, 4.002), $p_2 = 17.25$ with a 95% confidence interval of (16.69, 17.84), SSE=$1.395 \cdot 10^5$, R-squared=1, and RMSE=7.485. The ratio of the number of abundant numbers (1201/2491) times the ratio of the slopes (8.323/4.002) is 1.0027, approximately equal to 1.

In the above 593 sparse numbers, 2 times the 118 primes 5, 11, 23, 53, 59, 83, 113, 137,. . .,4973 are sparse numbers. A plot of the staircase obtained from the 118 primes and Gauss's $G(x)$ curve is

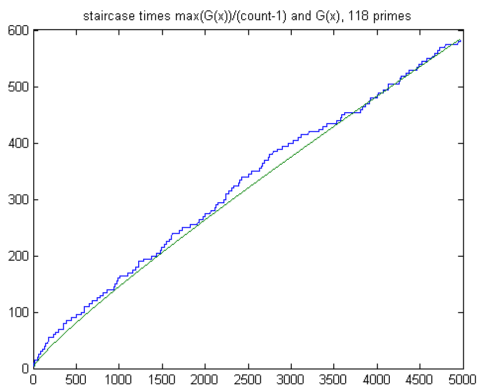

Figure 10: Scaled staircase of primes and $G(x)$

The portion of the staircase consisting of zeros (the first four elements) is not included in the graph and the scaling factor is 584.1455/117 (about 4.9927). The numbers of primes corresponding to prime-power factors of 2, 2^2, 2^3, 2^4, 2^5, and 2^6 are 118, 66, 37, 18, 7, and 1 respectively. The number approximately halves for each increase in the exponent. A plot of the 593 sparse numbers is

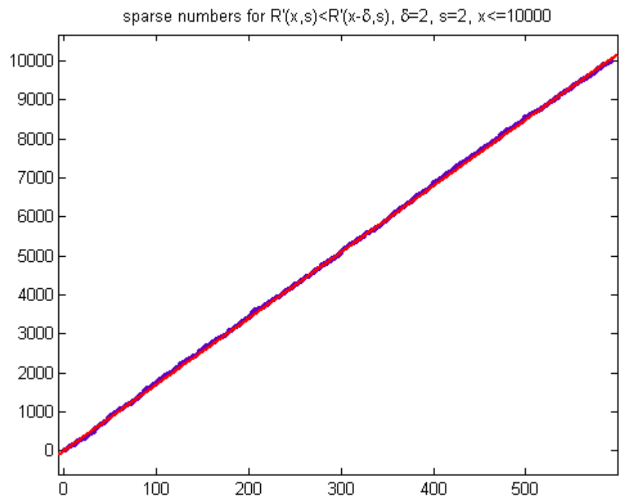

Figure 11: Sparse numbers.

For a linear least-squares fit of the curve, $p_1 = 16.97$ with a 95% confidence interval of (16.96, 16.99), $p_2 = -5.153$ with a 95% confidence interval of $(-10.08, -0.2211)$, SSE=$5.51 \cdot 10^5$, R-squared=0.9999, and RMSE=30.53. In general, the sparse numbers increase almost linearly.

3.2. $\delta = 3$

For $\delta = 3$, $s = 1$, and $x \leq 50000$, there are 7846 instances where $R'_{x,s} < R'_{x-\delta,s}$ and only 1959 of these x values are abundant numbers. None of the sparse numbers are primes of the form $6k + 1$. The 5887 sparse numbers include 2 times the 1219 primes 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83,. . ., 24989 (primes of the form $6k - 1$). There are 1389 primes of the form $6k - 1$ less than or equal to 24989, so 1219/1389 (about 87.76%) of them are included. The sparse numbers include 2^2 times the 650 primes 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103,. . ., 12487 (primes of the form $6k + 1$). There are 736 primes of the form $6k + 1$ less than or equal to 12487, so 650/736 (about 88.32%) of them are included. For prime-power factors of 2, 2^2, 2^3, 2^4, 2^5, 2^6, and 2^7, the numbers of primes are 1219, 650, 356, 187, 106, 47, and 12 respectively, the smallest primes are 5, 13, 19, 37, 71, 139, and 257 respectively, and the largest primes are 24989, 12487, 6203, 3121, 1559, 769, and 389 respectively. For a linear least-squares fit of the logarithms of the numbers of primes, the slope is -0.726, the intercept is 8.002, and R-squared=0.9755 (excluding the last count of 12, the slope is -0.6389, the y-intercept is 7.77, and R-squared=0.9977). A plot of the logarithms of the largest primes is

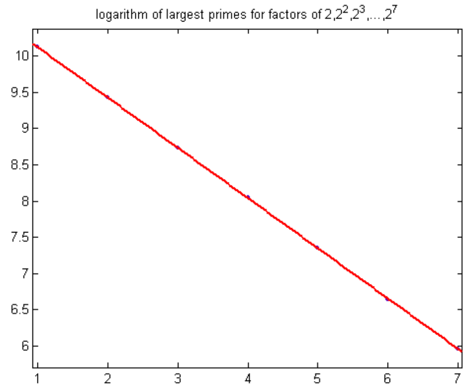

Figure 12: Logarithms of largest primes

For a linear least-squares fit of the curve, the slope is 0.9644 with a 95% confidence interval of $(-0.6969, -0.6919)$, the y-intercept is 10.82 with a 95% confidence interval of (10.81, 10.83), SSE=0.0001362, R-squared=1.0, and RMSE=0.005219. This is not entirely unexpected since $\log(2) = 0.69$. For a linear least-squares fit of the logarithms of the smallest primes, the slope is 0.6424, the y-intercept is 1.054, and R-squared=0.993.

For the 5887 sparse numbers, $2 \cdot 5$ times 329 primes are included. The minimum prime is 13 and the maximum prime is 4999. For $2 \cdot 5$, $2 \cdot 7$, $2 \cdot 11$,. . ., $2 \cdot 127$ times primes, the prime counts are 329, 251, 164, 146, 105, 102, 77, 65, 60, 53, 44, 40, 35, 33, 28, 28, 24, 23, 23, 22, 15, 15, 15, 14, 13, 10, 10, 8, and 6 respectively, the minimum primes are 13, 11, 13, 17, 19, 23, 31, 37, 41, 41, 43, 47, 61, 61, 61, 71, 71, 73, 83, 83, 97, 97, 101, 103, 107, 109, 113, 127, and 127 respectively, and the maximum primes are 4999, 3557, 2269, 1913, 1459, 1307, 1069, 859, 797, 659, 607, 569, 523, 463, 431, 401, 359, 349, 317, 311, 283, 277, 257, 241, 239, 229, 227, 211, and 179 respectively. A plot of 1/5, 1/7, 1/11,. . ., 1/127 versus the prime counts is

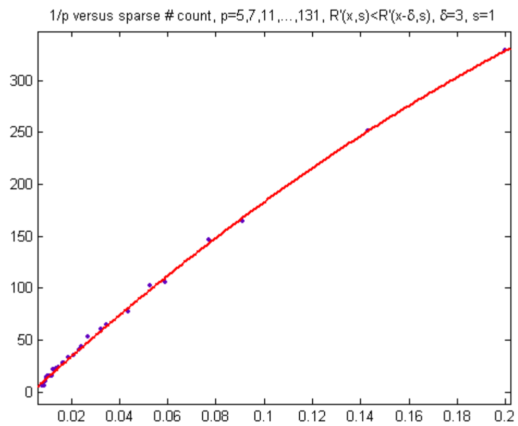

Figure 13: Reciprocals of primes versus prime counts

For a quadratic least-squares fit of the curve, $p_1 = -2263$ with a 95% confidence interval of $(-2670, -1856)$, $p_2 = 2135$ with a 95% confidence interval of (2059, 2211), $p_3 = -7.847$ with a 95% confidence interval of $(-9.666, -6.027)$, SSE=173.2, R-squared=0.9989, and RMSE=2.581. A plot of prime counts versus the maximum primes is

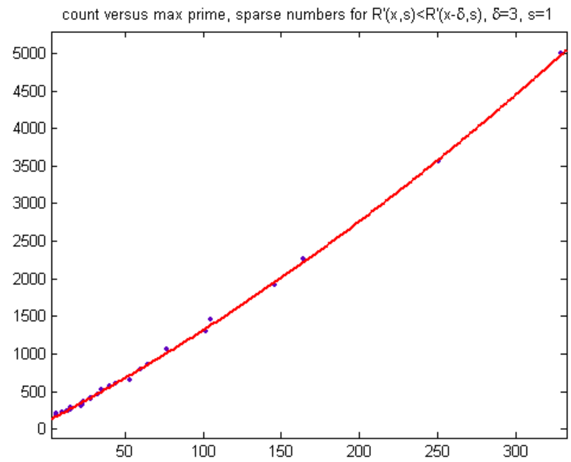

Figure 14: Prime counts versus maximum primes

For a quadratic least-squares fit of the curve, $p_1 = 0.01168$ with a 95% confidence interval of (0.00992, 0.01343), $p_2 = 11.01$ with a 95% confidence interval of (10.48, 11.53), $p_3 = 99.76$ with a 95% confidence interval of (78.48, 121), SSE=$2.713 \cdot 10^4$, R-squared=0.9992, and RMSE=32.3. A plot of the primes 5, 7, 11,. . ., 127 versus the minimum primes is

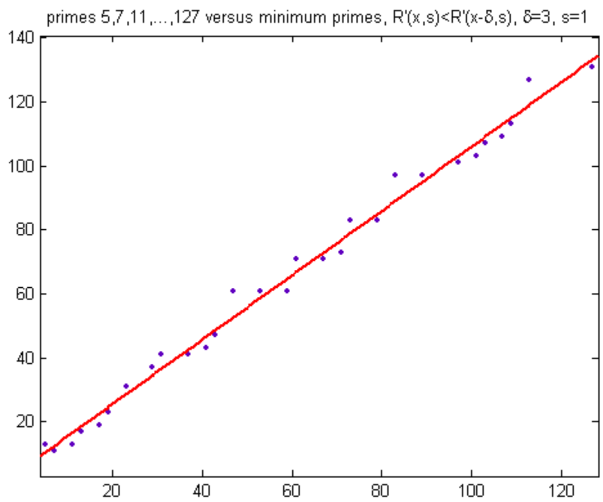

Figure 15: Primes 5, 7, 11,. . ., 127 versus minimum primes

For a linear least-squares fit of the curve $p_1 = 1.006$ with a 95% confidence interval of (0.964, 1.047), $p_2 = 5.532$ with a 95% confidence interval of (2.657, 8.4080), SSE=414.3, R-squared-0.9892, and RMSE=3.917.

For $\delta = 3$, $s = 2$, and $x \leq 50000$, there are 7843 instances where $R_{x,s} < R_{x-\delta,s}$ and only 1960 of these x values are abundant numbers. None of the sparse numbers are primes of the form $6k - 1$. The 5883 sparse numbers include 2 times the 1203 primes 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 103,. . ., 24967 (primes of the form $6k + 1$). There are 1370 primes of the form $6k + 1$ less than or equal to 24967, so 1203/1370 (about 87.81%) of them are included. The sparse numbers include 2^2 times the 675 primes 11, 17, 23, 29, 41, 47, 53, 59, 83, 89, 101,. . ., 12497 (primes of the form $6k - 1$). There are 750 primes of the form $6k - 1$ less than or equal to 12497, so 675/750 (about 90%) of them are included. For prime-power factors of 2, 2^2, 2^3, 2^4, 2^5, 2^6, and 2^7, the numbers of primes are 1203, 675, 352, 194, 98, 47, and 10 respectively, the smallest primes are 7, 11, 19, 41, 67, 131, and 271 respectively, and the largest primes are 24967, 12497, 6247, 3119, 1549, 773, and 379 respectively. The linear least-squares fits of the logarithms of these quantities are almost the same as for $s = 1$.

For $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 3$, $s = 3$ and $x \leq 50000$, there are 439 sparse numbers. These sparse numbers include 3 times the 87 primes 397, 409, 577, 829, 997, 1009, 1249, 1321, 1489, 1657, 1669,. . ., 16417. A plot of the staircase obtained from the 87 primes and Gauss' $G(x)$ curve is

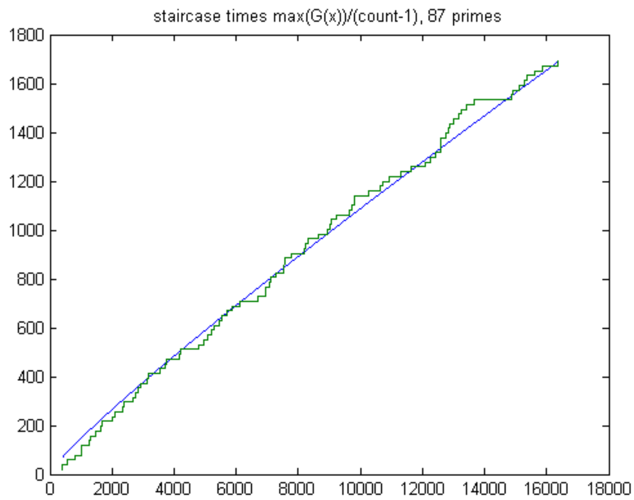

Figure 16: Scaled staircase of primes and $G(x)$

The portion of the staircase consisting of zeros (the first 396 elements) is not included in the graph and the scaling factor is 1691.3/86 (about 19.6663).

3.3. $\delta = 4$

For $\delta = 4$, $M(1) = 1$, $M(2) = 0$, $M(3) = -1$, and $M(4) = -1$ are used to compute $R_{x,s}$. All x values of the form $6p$, $p \neq 2$, $p \neq 3$ where p denotes a prime are "marginally" abundant since the sum of divisors minus $2x$ equals 12.

If $\delta = 4$, $s = 2$ or $s = 4$, and $R'_{x,s} < R'_{x-\delta,s}$, then x is usually an abundant number. For the 1250 $R'_{x,s}$ values, $\delta = 4$, $s = 2$, $x \leq 5000$, there are 365 instances where $R'_{x,s} < R'_{x-\delta,s}$ and in all but 38 instances, x is an abundant number. By Conjecture 2, none are divisible by 3. Of the 327 x values that are abundant, 114 are "marginally" abundant. An empirical result is

Conjecture 6 *The sparse numbers for* $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 4$, $s = 2$ *are of form* $2 \cdot (6k - 1)$.

For $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 4$, $s = 2$, and $x \leq 50000$, 2 times the primes 89, 281, 449, 761, 809, 93, 1013, 1481, 1601,. . ., 24473 are sparse numbers. A plot of the staircase obtained from the 89 primes and Gauss's $G(x)$ curve is

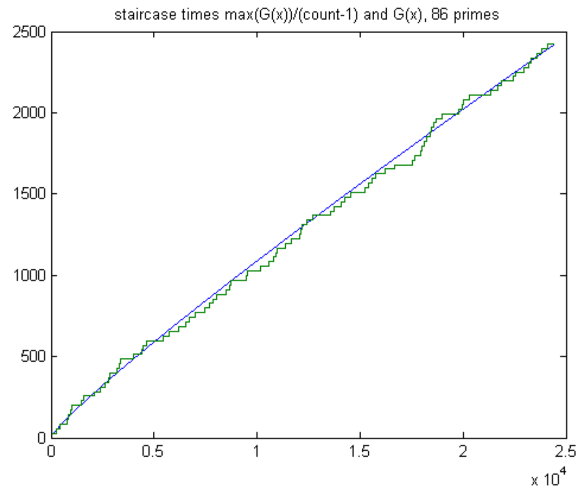

Figure 17: Staircase of primes

The portion of the staircase consisting of zeros (the first 88 elements) is not included in the graph and the scaling factor is 2421.7/85 (about 28.4906).

For the 1250 $R'_{x,s}$ values, $\delta = 4$, $s = 4$, $x \leq 5000$, there are 327 instances where $R'_{x,s} < R'_{x-\delta,s}$ and in all but 76 instances, x is an abundant number.

Conjecture 7 *The sparse numbers for* $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 4$, $s = 4$ *are of the form* $2^k \cdot (6k + 1)$ *(usually* $k = 2$*).*

By Conjecture 2, none are divisible by 3. Also, none of the values are divisible by 5 or 7. For $x \leq 75000$, all the products of the form $2^2 \cdot p^2$ (disregarding $p = 2, 3, 5,$

and 7) except $2^2 \cdot 29^2$ are sparse numbers. For $x = 2^2 \cdot 29^2$, $R'_{x,s} = 160.5169$ and $R'_{x-\delta,s} = 160.5016$.

For $R'_{x,s} < R'_{x-\delta,s}$, $\delta = 4$, $s = 4$, and $x \leq 75000$, $11 \cdot 2^2$ times the 83 primes 23, 29, 47, 53, 59, 71, 83, 89, 107, 113,. . ., 1667 are sparse numbers. A plot of the staircase generated from these primes is

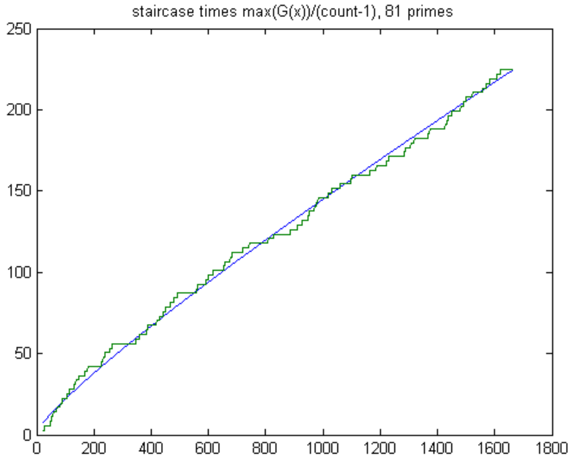

Figure 18: Staircase of primes

The portion of the staircase consisting of zeros (the first 22 elements) is not included in the graph and the scaling factor is 224.5834/80 (about 2.8073). The numbers of primes corresponding to prime-power factors of 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, and 107 are 81, 54, 39, 28, 18, 15, 15, 11, 12, 5, 3, 6, 1 respectively. A plot of the logarithm of these counts is

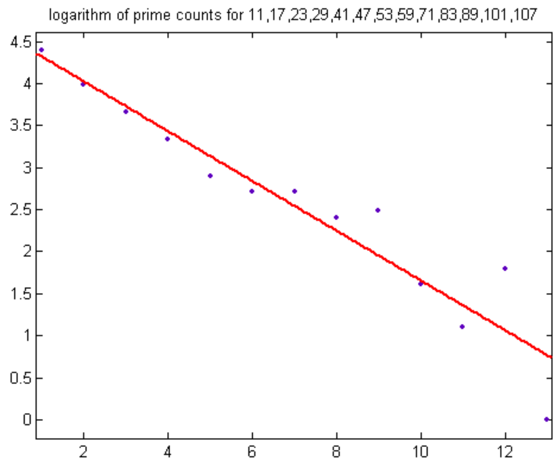

Figure 19: Logarithm of prime counts

For a linear least-squares fit of the curve, the slope is -0.2962 and the intercept is 4.617. The numbers of primes corresponding to prime-power factors of 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, and 127 are 65, 37, 25, 18, 13, 13, 9, 9, 7, 3, 3, 2, and 1 respectively. A plot of the logarithm of these counts is

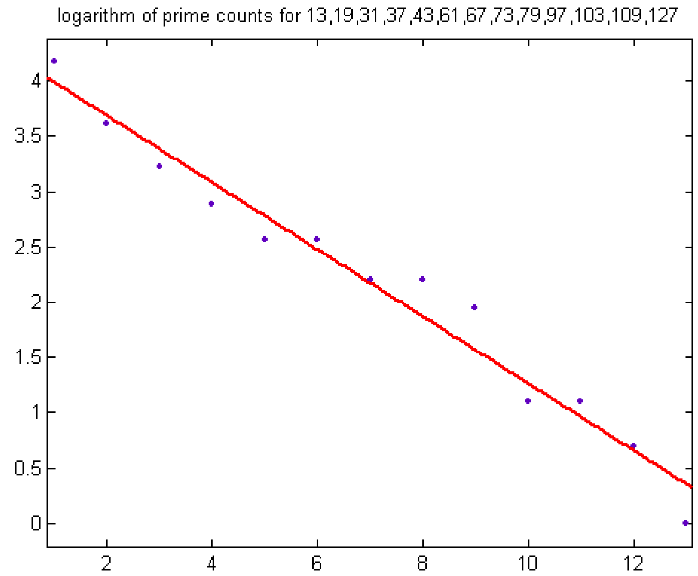

Figure 20: Logarithm of prime counts

For a linear least-squares fit of the curve, the slope is -0.3027, and the intercept is 4.293 (about the same as the above curve.)

For $\delta = 2$ and $s = 1$, the $R_{x,s}$ values for $x = 1, 3, 5, \ldots$ are $e^{\gamma} \sum_{i=1}^{1} i - \sum_{i=1}^{1} \sigma(i)$, $e^{\gamma} \sum_{i=2}^{3} i - \sum_{i=2}^{3} \sigma(i)$, $e^{\gamma} \sum_{i=3}^{5} i - \sum_{i=3}^{5} \sigma(i)$,.... Note that $M(1) = 1$. The difference between two successive values is $(e^{\gamma}x - \sigma(x)) + (e^{\gamma}(x-1) - \sigma(x-1)) - (e^{\gamma}\frac{x-1}{2} - \sigma(\frac{x-1}{2}))$. A table of the corresponding Mertens function values for $x = 1$, 3, 5, 7, and 9 is

1				
1	1			
1	1	1		
1	1	1	1	
1	1	1	1	1

A table of the Mertens function values for $\delta = 2$, $s = 2$, and $x = 2$, 4, 6, 8, and 10 is

0	1				
0	1	1			
0	1	1	1		
0	1	1	1	1	
0	1	1	1	1	1

The rows are read from right to left. The 0's correspond to $M(2)$. A table of the Mertens function values for $\delta = 3$, $s = 1$, and $x = 1$, 4, 7, 10, 13, 16, and 19 is

1												
0	1	1										
0	1	1	1	1								
0	0	1	1	1	1	1						
0	0	1	1	1	1	1	1	1				
0	0	0	1	1	1	1	1	1	1	1		
0	0	0	1	1	1	1	1	1	1	1	1	1

A table of the Mertens function values for $\delta = 3$, $s = 2$, and $x = 2$, 5, 8, 11, 14, 17, and 20 is

0	1												
0	1	1	1										
0	0	1	1	1	1								
0	0	1	1	1	1	1	1						
0	0	0	1	1	1	1	1	1	1				
0	0	0	1	1	1	1	1	1	1	1	1		
0	0	0	0	1	1	1	1	1	1	1	1	1	1

A table of the Mertens function values for $\delta = 3$, $s = 3$, and $x = 3$, 6, 9, 12, 15, 18, and 21 is

−1	1	1												
−1	0	1	1	1										
−1	0	1	1	1	1	1								
−1	0	0	1	1	1	1	1	1						
−1	0	0	1	1	1	1	1	1	1	1				
−1	0	0	0	1	1	1	1	1	1	1	1	1		
−1	0	0	0	1	1	1	1	1	1	1	1	1	1	1

The $R_{x,s}$ values can be computed from this information alone.

4. PROPERTIES OF $L_{x,s}$

Conjecture 8 $L'_{x,s} < L'_{x-\delta,s}$, $2|x$, *only if* x *is an abundant number or 6 does not divide* x.

A plot of the x values (abundant numbers) where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 6$, $s = 6$, and $x = 6, 12, 18, \ldots 60000$ is

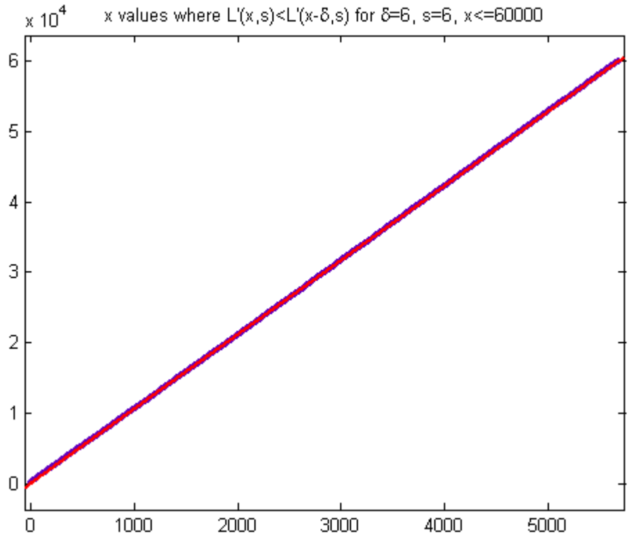

Figure 21: Plot of abundant numbers corresponding to $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 6$, $s = 6$, $x \leq 60000$.

For a linear least-squares fit of the curve, $p_1 = 10.58$ with a 95% confidence interval of (10.58, 10.58), $p_2 = -87.82$ with a 95% confidence interval of (-87.82, -84.71), SSE=$2.202 \cdot 10^7$, R-squared=1.0, and RMSE=59.64. The proportion of abundant numbers included for $x \leq 60000$ is 5681/26892 or about 21.13%. The number of marginally abundant numbers included is 375. The first few superabundant numbers are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 10080, 15120, 25200, 27720, 55440, ... Other than 2, 4, and 6, the superabundant numbers are included in the above values. The first few colossally abundant numbers are 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720. The colossally abundant numbers are a subset of the superabundant numbers.

4.1. $\delta = 2$

There are 13760 instances where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 1$, and $x \leq 50000$ and only 3 of the x values are abundant numbers. For the 13757 sparse numbers, 3 times 1926 primes are included. An empirical result is

Conjecture 9 *The usual sequence of primes starting with 5 and multiplied by 3 are sparse numbers for* $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, *and* $s = 1$.

For prime-power factors of 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, and 61, the prime counts are 1926, 941, 630, 353, 295, 159, 141, 95, 64, 56, 39, 32, 16, 7, 4, 2, and 1 respectively. A plot of the reciprocals of 3, 5, 7,. . ., 61 versus the prime counts is

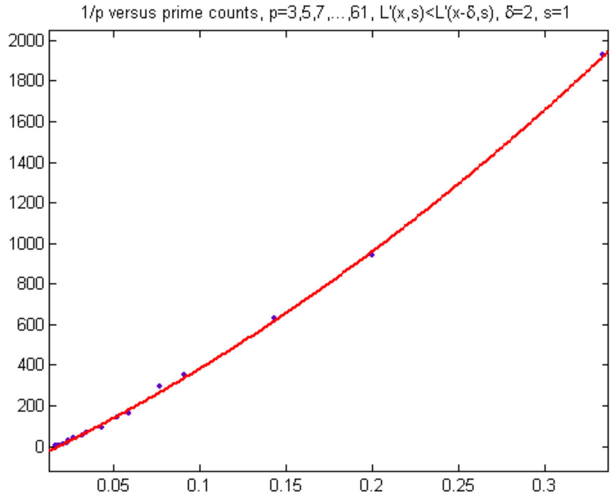

Figure 22: Reciprocal of primes versus prime counts

For a quadratic least-squares fit of the curve, $p_1 = 6025$ with a 95% confidence interval of (5056, 6995), $p_2 = 3983$ with a 95% confidence interval of (3664, 4301), $p_3 = -74.95$ with a 95% confidence of $(-89.21, -60.69)$, SSE=2746, R-squared=0.9993, and RMSE=14.01.

Let p_n denote the nth prime. The asymptotic relation $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$ is logically equivalent to the asymptotic relation $\lim_{x\to\infty} \frac{p_n}{n \log n} = 1$ (see Theorem 4.5 of Apostol's [2] book). A plot of $\frac{p_n}{n \log n}$ for the above primes for a prime-power factor of 5 is

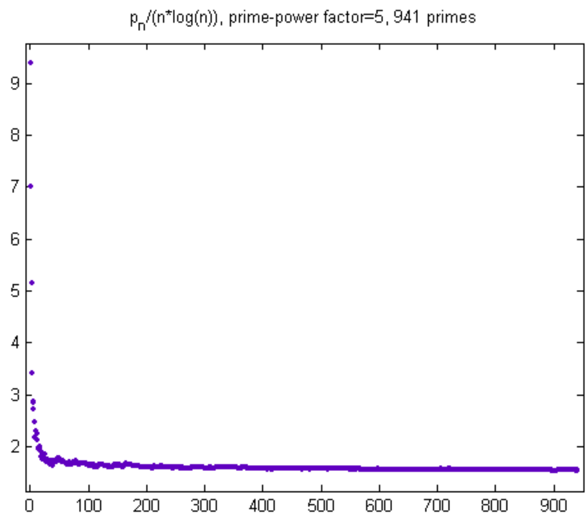

Figure 23: $\frac{p_n}{n \log n}$ values for prime-power factor of 5

Here p_n denotes the nth prime in that group. A plot of the prime-power factors 3, 5, 7,. . ., 43 versus the last $\frac{p_n}{n \log n}$ values is

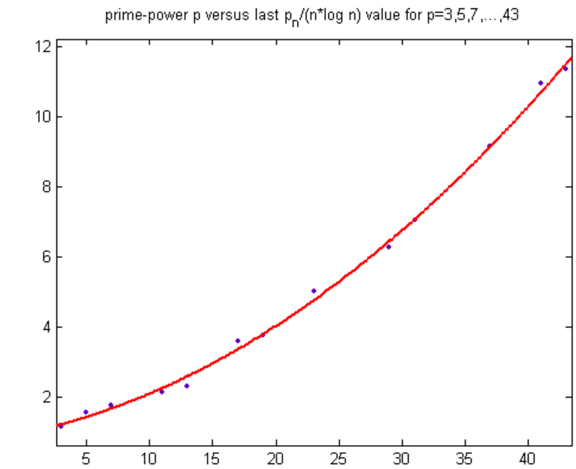

Figure 24: Prime-power factors versus last $\frac{p_n}{n \log n}$ values

For a quadratic least-squares fit of the curve, $p_1 = 0.004008$ with a 95% confidence interval of (0.003175, 0.004841), $p_2 = 0.07318$ with a 95% confidence interval of (0.03384, 0.1125), $p_3 = 0.9541$ with a 95% confidence interval (0.5823, 1.325), SSE=0.372, R-squared=0.9976, and RMSE=0.1929.

Chebyshev's first function $\vartheta(x)$ is defined to be $\sum_{p \leq x} \log p$.
A plot of the x values where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 2$, and x is an abundant number for $x = 2, 4, 6, \ldots, 10000$ is

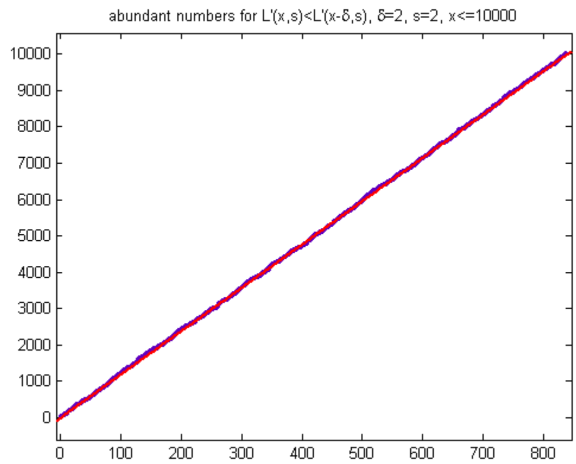

Figure 25: Plot of abundant numbers corresponding to $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 2$, $x \leq 10000$.

For a linear least-squares fit of the curve, $p_1 = 11.92$ with a 95% confidence interval of (11.91, 11.93), $p_2 = -15.34$ with a 95% confidence interval of $(-18.64, -12.04)$, SSE=$4.935 \cdot 10^5$, R-squared=0.9999, and RMSE=24.3. The first few values are 6, 28, 36, 40, 56, 70, 78, 88, 96, 100, 112, 156, 160. . ..

For a linear least-squares fit of the 1925 sparse numbers for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 2$, and $x = 2, 4, 6, \ldots, 10000$, $p_1 = 5.188$ with a 95% confidence interval of (5.187, 5.189), $p_2 = 1.668$ with a 95% confidence interval of (0.6164, 2.72), SSE=$2.659 \cdot 10^5$, R-squared=1.0. and RMSE=11.76.

4.2. $\delta = 3$

Conjecture 10 *The abundant numbers for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, $s = 1$ are a subset of the abundant numbers for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 2$, $s = 2$.*

The first few abundant numbers for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, and $s = 1$ are 28, 40, 70, 88, 112, 160,. . ..

For $x \leq 10000$, there are 1843 instances where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, and $s = 2$ and 36 of these x values are abundant numbers. The first few of these values are 176, 476, 836, 896, 1184, 1376, 2024,.... None are marginally abundant. For a linear least-squares fit of the values, the slope is about 283, the y-intercept is about -155.8, and R-squared=0.9954. For a linear least-squares fit of the 1807 sparse numbers, the slope is about 5.531, the y-intercept is about -4.051, and R-squared=1.0. A plot of the even sparse numbers for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, and $s = 2$ for $x = 2, 5, 8, . . ., 9999$ is

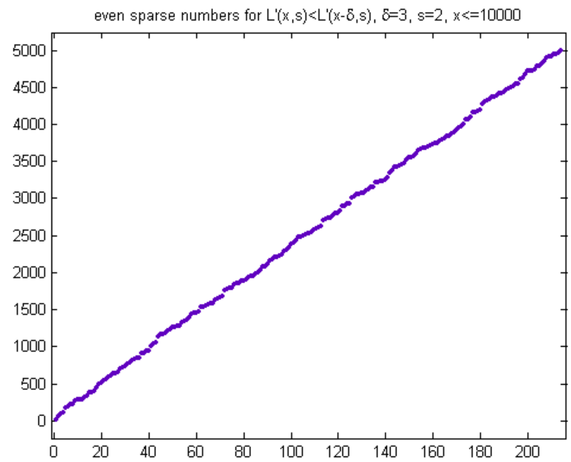

Figure 26: Plot of even sparse numbers corresponding to $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, $s = 2$, $x \leq 10000$.

For a linear least-squares fit of the curve, $p_1 = 5.188$ with a 95% confidence interval of (5.187, 5.189), $p_2 = 1.668$ with a 95% confidence interval of (0.6164, 2.72), SSE=$2.659 \cdot 10^5$, R-squared=1.0, and RMSE=11.76.

For $x \leq 50000$, there are 9280 instances where $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 3$, and $s = 2$ and 187 of these x values are abundant numbers. Of the 9093 sparse numbers, 3575 are the product of two distinct primes (greater than 3). For a linear least-squares fit of these values, $p_1 = 13.98$ with a 95% confidence interval of (13.98, 13.98), $p_2 = -8.278$ with a 95% confidence interval of (-14.63, -1.921), SSE=$3.355 \cdot 10^7$, R-squared=1, and RMSE=96.9. The maximum value is 49967. For a linear least-squares fit of the second Chebyshev function up to $x = 3575$, $p_1 = 0.9983$ with a 95% confidence interval of (0.998, 0.9987), $p_2 = 0.6635$ with a 95% confidence interval of (-0.01235, 1.334), SSE=$3.793 \cdot 10^5$, R-squared=0.9999, and RMSE=10.3. The maximum value is 3573.8. For a linear least-squares fit of the scaled sparse numbers (multiplied by 3573.8/49967), SSE=$1.716 \cdot 10^5$, R-squared=1, and RMSE=6.931. The approximation error in the $\psi(x)$ values is then greater.

4.3. $\delta = 4$

The primes where 2^2 times that prime is a sparse number for $L'_{x,s} < L'_{x-\delta,s}$, $\delta = 4$, $s = 4$, $x \leq 70000$ are 23, 47, 59, 107,139, 149, 167, 179, 223,. . ., 17489. A plot of the staircase generated by the 537 primes and Gauss' $G(x)$ curve is

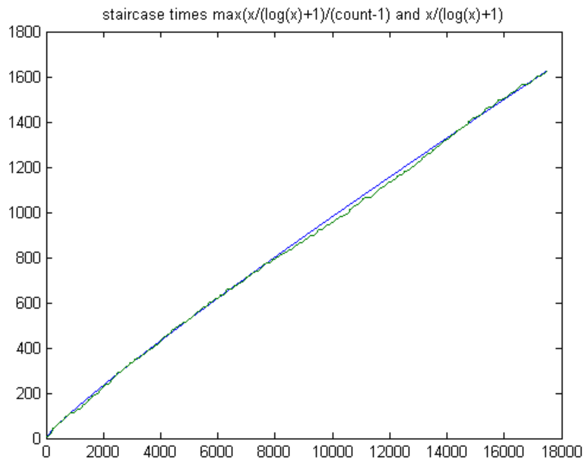

Figure 27: Staircase of primes

The staircase has been scaled by the maximum $G(x)$ value divided by the count minus 1 (about 3.0261).

5. CONCLUSION

Except for 2 and 3, the usual prime sequence 2, 3, 5, 7, 11,. . . is generated by a sparse number sequence. The superabundant numbers are not random in the sense that abundant numbers can be systematically interpolated between them so that they increase almost linearly (see Figure 21). Certain sequences of sparse numbers consisting of the product of two distinct primes greater than 3 are mostly linear and more accurately approximate a straight line than comparable sequences of the second Chebyshev function. Subsequences of primes in sparse numbers give staircases approximately equal to a multiple of Gauss' $G(x)$ function for larger δ values than investigated here. Much of the theoretical machinery of the proof of the prime number theorem does not depend on what the asymptotic limit of $\frac{\pi(x)\log x}{x}$ is. Broadening the definition of what a prime sequence is (as in the above) would lead to variants of the prime number theorem.

6. METHODS

C code for computing $L_{x,s}$ and $R_{x,s}$ and finding abundant and sparse numbers when the values do not increase monotonically is as follows.

```
#include <math.h>
#include <stdio.h>
extern char *malloc();
// compute sum of divisors
unsigned int numdiv(unsigned int a) {
    unsigned int i,sum;
    sum=0;
    for (i=1; i<=a; i++) {
        if (a==(a/i)*i)
            sum=sum+i;
        }
    return sum;
    }
unsigned int max=20000; // maximum x value
unsigned int start=1; // beginning x value
unsigned int delta=2; // increment
unsigned int swap=1; // if set, output L(x,s) results instead of R(x,s)
unsigned int out=2; // set to 1 for abundant numbers
                        // set to 2 for sparse numbers
void main() {
    int sum,t,*m;
    unsigned int i,j,k,index,count;
    double temp,tempsum,tempsum1,oldsum;
    int newsum,tmpsum,newsum1,tmpsum1;
    FILE *Outfp;
    Outfp = fopen("out7fd.dat","w");
    m=(int*) malloc(400004);
    if (m==NULL) {
        printf("not enough memory");
```

```
        return;
        }
    if (max>20000) {
        printf("x value too large");
        return;
        }
    // compute Mertens function
    m[0]=1;
    for (index=2; index<=max; index++) {
        sum=0;
        for (i=2; i<=(index/3); i++)
            sum=sum+m[index/i-1];
        sum=sum+(index+1)/2;
        t=1-sum;
        m[index-1]=t;
        }
    count=0;
    temp=exp(.57721566490153286060); // e^γ
    oldsum=0.0;
    j=1;
    for (index=start; index<=max; index+=delta) {
        newsum=0;
        tmpsum=0;
        newsum1=0;
        tmpsum1=0;
        for (i=1; i<j; i++) {
            newsum1=newsum1+m[index/i-1]*i;
            tmpsum1=tmpsum1+m[index/i-1]*(int)numdiv(i);
            }
        tempsum1=(double)newsum1*temp;
        tempsum1=tempsum1-(double)tmpsum1;
        for (i=j; i<=index; i++) {
            newsum=newsum+m[index/i-1]*i;
            tmpsum=tmpsum+m[index/i-1]*(int)numdiv(i);
            }
        tempsum=(double)newsum*temp;
        tempsum=tempsum-(double)tmpsum;
        j=j+1;
        if (swap==0)
            tempsum=tempsum/(double)index;
        else
            tempsum=tempsum1/(double)index;
        printf(" %d %d %e \n ",j-1,index,tempsum);
```

```
    if (tempsum<oldsum) {
        k=numdiv(index);
        if (k<(index*2)) { // not an abundant number
            if (out==2) {
                fprintf(Outfp," %d, \n",index);
                count=count+1;
                }
            }
        else {
            if (out==1) {
                fprintf(Outfp," %d,\n",index);
                count=count+1;
                }
            }
        }
    oldsum=tempsum;
    }
printf("count=%d",count);
fclose(Outfp);
return;
}
```

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