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# Convergence of Wavelet Expansion at Generalized Continuous Points

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**Abstract**

The purpose of this paper is to discuss convergence of wavelet expansion at generalized continuous points. Our result generalizes the result of Zhao [11]. As an application our result generalizes the results about convergence of wavelet expansion of a function at a continuous point.
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## 1. INTRODUCTION

The rate of convergence of wavelet expansions depends on the smoothness of the expanded function $f \in L^2(\mathbb{R})$. According to Mark A. Kon[4], a phenomenon quite different occurs from function spaces beyond a certain degree of smoothness. In these cases the rate of convergence freezes and fails to improve, no matter what the smoothness of $f$. Such behaviour have been studied in the context of approximation theory. Many results proved that the rate of convergence of wavelets and multiresolution expansions characterizes to the integrable functions, in that supremum error norm, and the necessary and sufficient conditions for the rate of convergence are given in terms of behaviour of scaling functions near the origin and these conditions are called as the special cases to be equivalent to moment conditions and other known conditions determines the rate of convergence (see M. A. Kon and L. A. Raphael [5]).

Wavelets are very important for the localised study of time and frequency. The wavelet bases of $L^2(\mathbb{R})$ consists of many translations and dialations of one or more functions. According to Folland and Gerald [1], a 'wavelet' is a function that exhibits oscillatory behaviour in some interval and then decays rapidly to zero outside this interval. Y. Meyer([6], Ch.2) was among the first to study the convergence of orthogonal wavelet expansions.

In this paper we discuss the rate of convergence of wavelet expansion at a generalized continuous point. Our result generalizes the results of Zhao[11]. For this purpose we introduce a monotonically decreasing and integrable function $P_\beta(x)$ for all natural numbers $\beta$ greater than 1 and generalized continuous point $x \in \mathbb{R}$. We will use following definitions:

### 1.1. Multiresolution Analysis (MRA):

**Definition 1.** *A general structure, called a multiresolution analysis for wavelet bases in $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_m,\ m \in Z\}$ of $L^2(\mathbb{R})$ satisfying the following conditions:*

1. *$V_m \subset V_{m+1}$ for all $m \in Z$;*
2. *$f(.) \in V_m$ if and only if $f(2(.)) \in V_{m+1}$ for all $m \in Z$;*
3. *$\cap_{m \in Z} \overline{V}_m = 0$;*
4. *$\cup_{m \in Z} V_m = L^2(\mathbb{R})$;*
5. *there exists a function $\phi \in V_0$ such that $\phi(. - k),\ k \in Z$ is an orthonormal basis for $V_0$*

### 1.2. Decay of Wavelet and Scaling Functions:

**Definition 2.** *The scaling function $\phi$ and wavelet function $\psi$ have sufficient decay, if they satisfy*

$$\max\{|\phi(x)|, |\psi(x)|\} \leq \frac{C}{(1+|x|^{1+\epsilon})}, \ (\epsilon > 0)$$

*with the mother function $\psi \in L^2(\mathbb{R})$ given by*

$$\hat{\psi}(x) = m_0\left(\frac{x+1}{2}\right)\hat{\phi}\left(\frac{x}{2}\right)e^{i\pi x},$$

*These $\psi \in L^2(\mathbb{R})$ with its normalized integer shifts and scales are given by*

$$\psi_{j,n}(x) = 2^{j/2}\psi(2^j x + n), \ j, n \in \mathbb{Z}$$

*$\phi \in L^2(\mathbb{R})$ is such that $\hat{\phi}$ is bounded with*

$$\hat{\phi}(x) = m_0\left(\frac{x}{2}\right)\hat{\phi}\left(\frac{x}{2}\right)$$

*and $m_0 \in L^2(\mathbb{R})$ is a low pass filter.*

### 1.3. Wavelet Expansion:

If $f \in L^2(\mathbb{R})$. Then

$$f = \sum_{n\in\mathbb{Z}} < f, \phi_{m,n} > \phi_{m,n} + \sum_{j=m}^{\infty}\sum_{n\in\mathbb{Z}} < f, \psi_{j,n} > \psi_{j,n} = f_m + t_m \tag{1.1}$$

is called wavelet expansion, where each $f_m$ in (1.1) is the projection map of $f$ onto $V_m$ defined by,

$$\begin{aligned} f_m &= \sum_{n\in\mathbb{Z}} < f, \phi_{m,n} > \phi_{m,n} \\ &= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} 2^m \phi(2^m x - n).\overline{\phi(2^m t - n)} f(t) dt \\ &= \int_{-\infty}^{\infty} q_m(x,t) f(t) dt \end{aligned} \tag{1.2}$$

where $q_m$ is reproducing kernel of $V_m$ given by,

$$q_m(x,t)\& = 2^m q(2^m x, 2^m t) \tag{1.3}$$

and

$$q(x,t) = \sum_{n\in\mathbb{Z}} \phi(x-n)\overline{\phi(t-n)}. \tag{1.4}$$

Similarly each $t_m$ in (1.1) is the projection of $f$ onto $W_m$.

### 1.4. Generalized Continuous Points of Expansion [11]:

**Definition 3.** *Let $f \in L^2(\mathbb{R})$ satisfy*

$$\lim_{h\to 0^+} \sup_{t\in[x-h,x+h]} \left| f(t) - \frac{f(x+0)+f(x-0)}{2} \right| = 0$$

*in some neighbourhood of point $x \in \mathbb{R}$, then $x$ is called a generalized continuous point of $f$.*

## 2. THEOREM AND LEMMAS:

**Lemma 2.1.** [9] Let $\mu$ be a bounded, decreasing and integrable function on $[0, \infty)$. Then for all $x, y \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \mu(|x+k|)\mu(|y+k|) \le C\mu(\frac{|x-y|}{4}) \tag{2.1}$$

where $C$ is a constant depending only on $\mu$.

The proof of the lemma is simple. A little bit more general statements are presented in Kelly *et al.*[3] and M. Skopina [8].

**Lemma 2.2.** Let $\phi$ be a continuous function and

$$|\phi(x)| = O(P_\beta|x|),\ \beta > 1 \tag{2.2}$$

for a monotonically decreasing function $P_\beta(x)$ which is integrable over $\mathbb{R}$ then

$$q(x,t) = O(P_\beta|x-t|) \tag{2.3}$$

**Proof:** Since we know that the kernel $q(x,t)$ is defined by

$$q(x,t) = \sum_{n\in\mathbb{Z}} \phi(x-n)\overline{\phi(t-n)},\ x,t \in \mathbb{R}, \tag{2.4}$$

and $q$ converges locally uniformly, then it is continuous because $\phi$ is given to be continuous. Then using Lemma 2.1

$$\begin{aligned} |q(x,t)| &= \sum_{n\in\mathbb{Z}} |\phi(x-n)\overline{\phi(t-n)}| = \sum_{n\in\mathbb{Z}} |\phi(x-n)||\overline{\phi(t-n)}| \\ &\le \sum_{n\in\mathbb{Z}} P_\beta|x-n|\overline{P_\beta|t-n|} \le KP_\beta|x-t| \end{aligned}$$

which proves the lemma.

**Theorem 2.3.** [9] Let $\phi$ be continuous and satisfy inequality

$$|\phi(x)| \leq \frac{C}{(1+|x|)^{1+\beta}}\ , (x \in \mathbb{R}) \tag{2.5}$$

If $x$ is a generalized continuous point of $f$, then

$$\lim_{m\to\infty} f_m(x) = \frac{f(x+0)+f(x-0)}{2}$$

## 3. MAIN THEOREM AND LEMMA:

**Theorem 3.1.** *Let $\phi$ be continuous and satisfy*

$$|\phi(x)| = O(P_\beta|x|),\ (\beta > 1) \tag{3.1}$$

*where $x$ is a generalized continuous point of $f$. $P_\beta(x)$ is a monotonically decreasing function of $x$ with fixed $\beta$ and is integrable over $\mathbb{R}$. Then*

$$\lim_{m\to\infty} f_m(x) = \frac{f(x+0)+f(x-0)}{2} \tag{3.2}$$

Note:

1. If $f(x+0) = f(x-0) = f(x)$, then $x$ is a point of continuity for $f$.
2. If $P_\beta(|x|) = \frac{C}{(1+|x|)^{1+\beta}}$ , $(x \in \mathbb{R})$. Then we get the result due to S. Zhao and G. Tian [11]. In view of note 1 above our result generalizes the results of G. Walter [10], R. S. Pathak [7] and S. Zhao and H. Cao [12], if $x$ is a point of continuity for $f$.

## 4. PROOF OF THE THEOREM:

*Proof.* We have

$$\begin{aligned}\Big|f_m(x) - \frac{f(x+0)+f(x-0)}{2}\Big| &\leq \int_{-\infty}^{\infty} |q_m(x,t)|\Big|f(t) - \frac{f(x+0)+f(x-0)}{2}\Big|dt \\ &= \int_{-\infty}^{\infty} |2^m q(2^m x, 2^m t)|\Big|f(t) - \frac{f(x+0)+f(x-0)}{2}\Big|dt \\ &\leq 2^m C\int_{-\infty}^{\infty} P_\beta|2^m(x-t)|\Big|f(t) - \frac{f(x+0)+f(x-0)}{2}\Big|dt \\ &= I_m\ (say).\end{aligned}$$

As $x$ is a generalized continuous point of $f$, for all $\epsilon > 0$, $\exists\ 0 < \delta < 1$ such that

$$\sup_{t\in(x-\delta,\, x)} \Big|f(t) - \frac{f(x+0)+f(x-0)}{2}\Big| < \epsilon$$

and

$$\sup_{t\in(x,\,x+\delta)}\Big|f(t)-\frac{f(x+0)+f(x-0)}{2}\Big|<\epsilon$$

Considering the above facts, we break $I_m$ into three parts.

$$I_m=\Big(\int_{|t-x|\ge\delta}+\int_{x-\delta}^{x}+\int_{x}^{x+\delta}\Big)2^mCP_\beta|2^m(x-t)|\Big|f(t)-\frac{f(x+0)+f(x-0)}{2}\Big|dt \tag{4.1}$$

Now consider $I_1$,

$$\begin{aligned}|I_1|&\le\int_{|t-x|\ge\delta}2^mCP_\beta|2^m(x-t)||f(t)|dt+2^mC\int_{|t-x|\ge\delta}P_\beta|2^m(x-t)|\Big|\frac{f(x+0)+f(x-0)}{2}\Big|dt\\&\le||f||_1 2^mC\int_{|t-x|\ge\delta}2^mP_\beta|2^m(x-t)|dt+C\Big|\frac{f(x+0)+f(x-0)}{2}\Big|\int_{|t-x|\ge\delta}2^mP_\beta|2^m(x-t)|dt\\&\le\Big(||f||_1+C\Big|\frac{f(x+0)+f(x-0)}{2}\Big|\Big)\int_{|y|\ge 2^m\delta}P_\beta|y|dy\end{aligned} \tag{4.2}$$

For $I_2$ we have

$$\begin{aligned}|I_2|&=\int_{x-\delta}^{x}2^mCP_\beta|2^m(x-t)|\Big|f(t)-\frac{f(x+0)+f(x-0)}{2}\Big|dt\\&\le C\sup_{t\in(x-\delta,\,x)}\Big|f(t)-\frac{f(x+0)+f(x-0)}{2}\Big|\int_{x-\delta}^{x}2^mP_\beta|2^m(x-t)|dt \qquad (4.3)\\&\le C\epsilon\int_0^{2^m\delta}P_\beta|y|dy\end{aligned}$$

similarly

$$|I_3|\le C\epsilon\int_{-2^m\delta}^{0}P_\beta|y|dy \tag{4.4}$$

Combining (4.1) to (4.4) and hypothesis of the theorem, we get the required result. This completes the proof of the theorem. □

## 5. COROLLARIES:

**Corollary 5.1.** *If we take $P_\beta|x|=e^{-\beta|x|}$, then all the conditions of theorem are satisfied. Also*

$$e^{-\beta|x|}\le(|x|)^{-\beta},\ \beta>1.$$

**Corollary 5.2.** *If*

1. $P_\beta|x|=\frac{1}{|x|(\log|x|)^\beta}$, $(\beta>1,\ x>1)$
2. $P_\beta|x|=\frac{1}{|x|(\log|x|)(\log\log|x|)^\beta}$, $(\beta>1, x>1)$

*3.* $P_\beta|x| = \frac{1}{\beta|x|}\log(1 \pm \frac{1}{\beta|x|}),\ (|\beta| > 1, |x| > 1)$

*4.* $P_\beta|x| = \frac{1}{|x|^\beta}\log(1 \pm \frac{1}{|x|^\beta}),\ (|\beta| > 1, |x| > 1)$

*are four monotonically decreasing and integrable functions on* $\mathbb{R}$*. Thus satisfies all the conditions of the theorem.*

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