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Some Ordered Metric Spaces for Rational Type Expressions in Fixed Point Result

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Abstract

In this paper, we establish some fixed point theorems satisfying generalized contraction mapping of rational type using a class of pairs of functions satisfying certain assumptions. The main result of this chapter is generalizes and extends the main result of Cabrera et al [1]. Furthermore, our result generalized and extends the corresponding result of [3] and [4] on the context of Ordered Metric Spaces.

Keywords: Mann iteration, fixed point, measurable mappings, Banach space**.**

AMS Subject Classification: 47H10, 47H40.

1.Introduction and Preliminaries

In this paper we prove Some Ordered Metric Spaces for Rational Type Expressions in Fixed Point Result in metric spaces. The theory originated at a relatively later point of time.

An early result in this direction was established by Turinici [9] in ordered metrizable uniform spaces. Application of fixed point result in partially ordered metric spaces was made sub sequentially, for example, by Ran[7] and Reurings to solving matrix equations and by Nito and Rodriguez-Lopez [6] to obtain solutions of certain partial differential equations with periodic boundary conditions. Recently, fixed point theory has developed in partially ordered metric spaces and many mathematicians have obtained several fixed point, common fixed point theorems in the setting of partially ordered metric spaces.

Das and Gupta [2] were the Pioneers in proving fixed point theorems using contractive conditions involving rational expressions. They proved the following fixed point theorem.

Definition1.1: Let $(X,\leq)$ is a partially ordered set and $T:X\to X$ is said to be monotone non-decreasing if for all $x,y\in X$

$$x\leq y\Rightarrow Tx\leq Ty$$

Definition 1.2: [5] A function $\phi:[0,\infty)\to[0,\infty)$ is called an altering distance function, if the following properties are satisfied:

(i) ϕ is monotone increasing and continuous,

(ii) $\phi(t)=0$ if and only if $t=0$.

In this paper, we consider the following class of pairs of functions F.

Definition 1.3: A pair of functions (φ,ϕ) is said to belong to the class F, if they satisfy the following conditions:

(i) $\varphi,\phi:[0,\infty)\to[0,\infty)$;

(ii) for $t,s\in[0,\infty)$, if $\varphi(t)\leq\phi(s)$ then $t\leq s$;

(iii) for (t_n) and (s_n) sequence in $[0,\infty)$ such that $\lim_{n\to\infty}t_n=\lim_{n\to\infty}s_n=a$,

If $\varphi(t_n)\leq\phi(s_n)$ for any $n\in N$, then $a=0$.

Remark 1.1: Note that, if $(\varphi,\phi)\in F$ and $\varphi(t)\leq\phi(t)$, then $t=0$, since we can take $t_n=s_n=t$ for any $n\in N$ and by (iii) we deduce $t=0$.

Now, we present some interesting examples of pairs of functions belonging to the class F.

Example 1.1: [8]. Let $\varphi:[0,\infty)\to[0,\infty)$ be a continuous and increasing function such that $\varphi(t)=0$ if and only if $t=0$ (these functions are known in the literature as altering distance functions).

Let $\phi:[0,\infty)\to[0,\infty)$ be a nondecreasing function such that $\phi(t)=0$ if and only if $t=0$ and suppose that $\phi\leq\varphi$. Then the pair $(\varphi,\varphi-\phi)\in F$. In fact, it is clear that $(\varphi,\varphi-\phi)$ satisfy (i).

To prove (ii), suppose that $t,s\in[0,\infty)$, and $\varphi(t)\leq(\varphi-\phi)(s)$.Then, from

$$\varphi(t)\leq\varphi(s)-\phi(s)\leq\varphi(s).$$

and taking into account the increasing character of φ, we can deduce that $t\leq s$.

In order to prove (iii), we suppose that

$$\varphi(t_n) \le \varphi(s_n) - \phi(s_n) \le \varphi(s_n) \text{ for any } n \in N, \tag{1.1}$$

where $t_n, s_n \in [0,\infty)$, and

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$$

Taking $n \to \infty$ in (7.2.3), we infer that $\lim_{n\to\infty} \phi(s_n) = 0$.

Let us suppose that $a > 0$. Since $\lim_{n\to\infty} s_n = a > 0$, we can find $\varepsilon > 0$ and a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} > \varepsilon$ for any $k \in N$. As ϕ is nondecreasing, we have $\phi(s_{n_k}) > \phi(\varepsilon)$ for any $k \in N$ and, consequently, $\lim_{k\to\infty} \phi(s_{n_k}) \ge \phi(\varepsilon)$. This contradicts the fact that $\lim_{k\to\infty} \phi(s_{n_k}) = 0$. Therefore, $a = 0$.

This proves that $(\varphi, \varphi - \phi) \in F$.

An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$, and $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \le t$ for any $t \in [0,\infty)$.

2. Partial differential Order For Fixed point Theorems in Metric Space.

Theorem 1.1: Let $(X, \le)$ is a partially ordered set and suppose that there exist a metric d on X such that (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying

$$\varphi(d(Tx,Ty)) \le \max\left\{\phi(d(x,y)), \phi\left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right)\right\} \tag{1.1}$$

for all $x, y \in X$ with $x \le y$.

If there exist $x_0 \in X$ such that $x_0 \le Tx_0$ then T has a unique fixed point.

Proof: If $Tx_0 = x_0$, then we have the result. Therefore, we suppose that $x_0 < Tx_0$, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \text{ for every } n \ge 0 \tag{1.2}$$

Since T is non-decreasing, we obtain by induction that

$$x_0 < Tx_0 = Tx_1 = x_2 \le \dots\dots\dots \le Tx_{n-1} = x_n \le Tx_n = x_{n+1} \le \dots\dots\dots\dots \tag{1.3}$$

If there exists $n \ge 1$ such that $x_{n+1} = x_n$, then from (7.3.2), $x_{n+1} = Tx_n = x_n$, that is x_n is a fixed point of T, and the proof is finished.

Now suppose that $x_{n+1} \ne x_n$, that is $d(x_{n+1}, x_n) \ne 0$, for all $n \ge 1$.

Since $x_{n-1} < x_n$ for all $n \geq 1$, from (1.1), we have

$$\varphi(d(x_{n+1}, x_n)) = \varphi(d(Tx_n, Tx_{n-1}))$$

$$\leq \max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, Tx_{n-1})[1+d(x_n, Tx_n)}{1+d(x_n, x_{n-1})}\right)\right\}$$

$$= \max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1+d(x_n, x_{n+1})}{1+d(x_n, x_{n-1})}\right)\right\} \tag{1.4}$$

Now, we distinguish two cases.

Case I. Consider

$$\max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1+d(x_n, x_{n+1})}{1+d(x_n, x_{n-1})}\right)\right\}$$

$$= \phi(d(x_n, x_{n-1})) \tag{1.5}$$

In this case from (1.4), we have

$$\varphi(d(x_{n+1}, x_n)) \leq \phi(d(x_n, x_{n-1})) \tag{1.6}$$

Since $(\varphi, \phi) \in F$, we deduce that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$$

Case II. If $\max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1+d(x_n, x_{n+1})}{1+d(x_n, x_{n-1})}\right)\right\}$

$$= \phi\left(\frac{d(x_{n-1}, x_n)[1+d(x_n, x_{n+1})]}{1+d(x_n, x_{n-1})}\right) \tag{1.7}$$

In this case from (1.4) and since $(\varphi, \phi) \in F$, we get

$$d(x_{n+1}, x_n) \leq \frac{d(x_{n-1}, x_n)[1+d(x_n, x_{n+1})]}{1+d(x_n, x_{n-1})}$$

Since $d(x_{n+1}, x_n) \neq 0$, from the last inequality it follows that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$$

From both cases, we conclude that the sequence $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of non-negative real numbers and is bounded below, there exists $r \geq 0$ such that

$$d(x_{n+1}, x_n) \to r \text{ as } n \to \infty \tag{1.8}$$

Now, we shall show that $r = 0$.

Denote

$$A = \{n \in N : n \ \textit{satisfies} \ (1.5\}$$

$$B = \{n \in N : n \ \textit{satisfies} \ (1.7\}$$

We note that the following.

(1) If $CardA = \infty$, then from (1.4), we can find infinitely natural numbers n satisfying inequality (1.6) and since $\lim_{n\to\infty} d(x_{n+1}, x_n) = d(x_n, x_{n-1}) = r$ and $(\varphi, \phi) \in F$, we have $r = 0$.

(2) If $CardB = \infty$, then from (1.4), we can find infinitely many $n \in N$ such that

$$\varphi(d(x_{n+1}, x_n)) \le \phi\left(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}\right)$$

Since $(\varphi, \phi) \in F$ and using the similar argument to the one used in case II, we obtain

$$d(x_{n+1}, x_n) \le d(x_{n-1}, x_n).\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, x_{n-1})} \tag{1.9}$$

For infinitely many $n \in N$.

Letting $n \to \infty$ in (1.9) and taking into account that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$, we deduce that

$$r \le r.\frac{1 + r}{1 + r}$$

And consequently, we obtain $r = 0$.

Therefore

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0 \tag{1.10}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence.

In contrary case, since $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$, we can find $\varepsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ satisfying

(i) $n(k) > m(k) > k$ for all positive integer k;

(ii) $d(x_{m(k)}, x_{n(k)}) \ge \varepsilon$.

Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k, d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \text{ and } d(x_{m(k)}, x_{n(k)-1}) < \varepsilon .$$

Now

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$$

That is

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq \varepsilon + d(x_{n(k)-1}, x_{n(k)}) .$$

Letting $k \to \infty$ in the above inequality and using (1.10), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon \tag{1.11}$$

Again

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)-1}, x_{n(k)}) .$$

Letting $k \to \infty$ in the above inequality and using (1.10) and (1.11), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon \tag{1.12}$$

Now using contractive condition (1.1), we get

$$\varphi(d(x_{m(k)}, x_{n(k)})) = \varphi(d(Tx_{m(k)-1}, Tx_{n(k)-1}))$$

$$\leq \max\left\{\phi(d(x_{m(k)-1}, x_{n(k)-1})), \phi\left(\frac{d(x_{n(k)-1}, Tx_{n(k)-1})[1 + d(x_{m(k)-1}, Tx_{m(k)-1})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right)\right\}$$

$$= \max\left\{\phi(d(x_{m(k)-1}, x_{n(k)-1})), \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right)\right\} \tag{1.13}$$

Put

$$C = \{k \in N : \varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi(d(x_{m(k)-1}, x_{n(k)-1}))\}$$

$$D = \left\{k \in N : \varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right)\right\} \tag{1.14}$$

By (1.13), we have $cardC = \infty$ or $cardD = \infty$.

Let us suppose that $cardC = \infty$. Then there exists infinitely many $k \in N$ such that

$$\varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi(d(x_{m(k)-1}, x_{n(k)-1}))$$

And since $(\varphi, \phi) \in F$, we have by letting $k \to \infty$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon$$

We infer from (1.11) that $\varepsilon = 0$. Which is a contradiction.

On the other hand, if $cardD = \infty$, then we can find infinitely many $k \in N$ such that

$$\varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right)$$

And since $(\varphi, \phi) \in F$, we obtain from the above inequality

$$d(x_{m(k)}, x_{n(k)}) \leq \frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}$$

Taking $k \to \infty$ and using (1.10) and (1.12), we obtain $\varepsilon \leq 0$.

Which is a contradiction.

Therefore, in both the cases, we obtain a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Next, we will show that u is a fixed point of T.

By the contractive condition (1.1), we obtain

$$\varphi(d(Tu, Tx_n)) \leq \max\left\{\phi(d(u, x_n)), \phi\left(\frac{d(x_n, Tx_n)[1 + d(u, Tu)]}{1 + d(x_n, u)}\right)\right\} \tag{1.15}$$

for any $n \in N$.

Now we distinguish two cases.

(I)There exist infinitely many $n \in N$ such that

$$\varphi(d(Tu, Tx_n)) \leq \phi(d(u, x_n))$$

since $(\varphi, \phi) \in F$, we obtain

$$d(Tu, Tx_n) \leq d(u, x_n)$$

For infinitely many $n \in N$. Since $\lim_{n\to\infty} x_n = u$, letting $n \to \infty$ in the last inequality, we obtain

$$\lim_{n\to\infty} Tx_n = Tu \tag{1.16}$$

Where, to simplify our assumptions, we will denote the subsequence by the same symbol $\{Tx_n\}$. By (1.2),

$$Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} \tag{1.17}$$

$x_n \to u$ in X, this means that

$$Tu = u$$

And therefore u is a fixed point of T.

(II)There exist infinitely many $n \in N$ such that

$$\varphi(d(Tu,Tx_n)) \le \phi\left(\frac{d(x_n,Tx_n)[1+d(u,Tu)]}{1+d(x_n,u)}\right)$$

Again to simplify our considerations, we will denote the subsequence by the same symbol $\{Tx_n\}$.

Since $(\varphi,\phi) \in F$, we deduce that

$$d(Tu,Tx_n) \le \frac{d(x_n,Tx_n)[1+d(u,Tu)]}{1+d(x_n,u)}$$

$$\le \frac{d(x_n,x_{n+1})[1+d(u,Tu)]}{1+d(x_n,u)}$$

For any $n \in N$.

Taking $n \to \infty$, and by using (1.10), we infer (1.16). From the above case, we deduce that u is a fixed point of T. Therefore, in both cases we proved that u is a fixed point of T in X.

To prove uniqueness, let x^* and y^* are two fixed point of T, that is

$x^* = Tx^*$ and $y^* = Ty^*$.

Now by using (1.1), we obtain

$$\varphi(d(x^*,y^*)) = \varphi(d(Tx^*,Ty^*))$$

$$\le \max\left\{\phi(d(x^*,y^*)),\phi\left(\frac{d(y^*,Ty^*)[1+d(x^*,Tx^*)]}{1+d(x^*,y^*)}\right)\right\} \tag{1.18}$$

$$= \max\{\phi(d(x^*, y^*)), \phi(0)\}$$

Again there are two cases.

(I) If $\max\{\phi(d(x^*, y^*)), \phi(0)\} = \phi(d(x^*, y^*))$

Then from (1.18) and Since $(\varphi, \phi) \in F$, with remark 1.1, we obtain ,

$d(x^*, y^*) = 0$ and consequently $x^* = y^*$.

(II) If $\max\{\phi(d(x^*, y^*)), \phi(0)\} = \phi(0)$

Then from (1.18)

$$\phi(d(x^*, y^*)) \le \phi(0)$$

Since $(\varphi, \phi) \in F$, we obtain $d(x^*, y^*) \le 0$.

Therefore $d(x^*, y^*) = 0$ and consequently $x^* = y^*$.

This complete the proof of the theorem.

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