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Mathematical and Actuarial Analysis on a Deterministic *SEIR* Model

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Abstract

In this paper, we consider a classical *SEIR* model, a deterministic epidemiological compartment model which can be described as a system of differential equations under a certain set of assumptions. The first part of the paper presents mathematical properties of the solutions. Finding the orbit of the epidemic curves, we can determine the limiting behavior of the solutions. Especially, an algebraic equation containing the initial value of the differential equations and the basic reproductive number is derived to determine the limiting values of the solutions. The second part of the paper explores actuarial quantities associated to the compartment model to formulate the level net premiums of infinite term infectious disease insurance products for the case of continuous premium collection and continuous benefit payments or lump-sum payments.

Keywords: *SEIR* model, compartment model, epidemiology, mathematical analysis, actuarial analysis, infectious disease insurance, reproductive number, differential equations
AMS Subject classification: 62P05, 92D30, 34A34

1. INTRODUCTION

World Health Organization (WHO) declared COVID-19 as a global pandemic on 11 March, 2020. A pandemic is defined as an infectious disease spreads over a wide geographic area passing easily from person to person at the same time. It awakes us the importance of the study of the transmission dynamics of communicable diseases to prevent potential adverse impact caused by that disease.

From an actuarial point of view, a disease can be thought as a risk which can be properly treated under the usual actuarial framework. In fact, actuaries can combine mathematical and statistical models on epidemiology with economical considerations to develop actuarial policy designs. On the other hand, analyzing mathematical models is crucial in the study of infectious disease dynamics to discover the mechanisms and develop controlling strategies by predicting future outbreaks.

In this paper, we will focus on epidemiological compartment models which divide the population of interest into non-overlapping groups, so called compartments, according to each individual's infectious status, then perform compartmental analysis which can be systematically applied to linear or nonlinear systems tracking the movement of individual from one compartment to another. For a brief introduction and basic analysis of compartment models, e.g. see [1] and references therein.

The authors in [2] investigated actuarial applications for *SIR* model, an epidemiological model with three compartments S, I, R and their work motivates our work here after adding one more compartment E leading to *SEIR* model (see Figure 1).

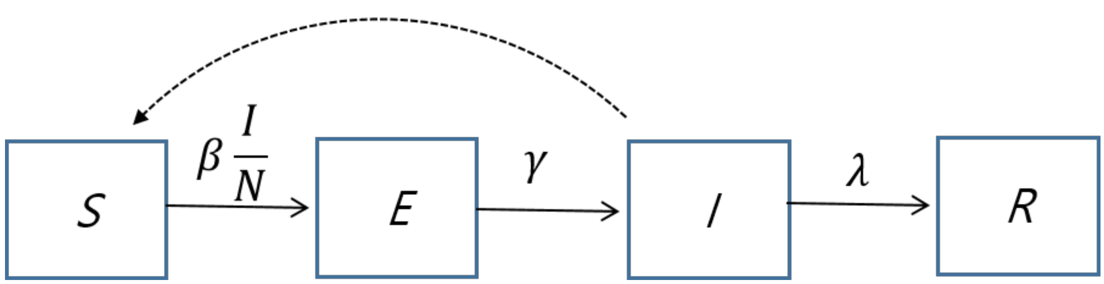


Figure 1: *SEIR* Model

The description of each compartments and parameters of the model is as follows:

- S (Susceptible): part of the population consisting of healthy individuals without immunity subjected to infection,

- E (Exposed): fraction of the population who are infected but not able to transmit the disease (latent phase),
- I (Infective): infective population after the latent phase,
- R (Removed): recovered or removed from the category I due to death/recovery,
- β: infection rate, i.e. the number of individuals an infective person infects each day,
- γ: inverse of the average latent time governing the lag between between having undergone an infectious contact and showing symptoms,
- λ: recovery/removal rate.

Assuming that the population is uniformly mixed, the law of mass action holds and there is no in/out movement from the population, the *SEIR* model can be formulated as a system of differential equations:

$$\begin{aligned} S'(t) &= -\beta I(t)S(t)/N \\ E'(t) &= \beta I(t)S(t)/N - \gamma E(t) \\ I'(t) &= \gamma E(t) - \lambda I(t) \\ R'(t) &= \lambda I(t) \end{aligned}$$

with the initial condition $S(0) = S_0$, $E(0) = E_0$, $I(0) = I_0$ and $R_0 = 0$ so that $S_0 + E_0 + I_0 = N$. Note that $N'(t) = (S(t) + E(t) + I(t) + R(t))' = 0$ and $N(t)$ remains constant, say N.

Since mortality analysis is based on ratios rather than absolute counts, we divide the above systems of equations by N to get:

$$s'(t) = -\beta i(t)s(t) \tag{1}$$
$$e'(t) = \beta i(t)s(t) - \gamma e(t) \tag{2}$$
$$i'(t) = \gamma e(t) - \lambda i(t) \tag{3}$$
$$r'(t) = \lambda i(t) \tag{4}$$

where $s(t) = S(t)/N$, $e(t) = E(t)/N$, $i(t) = I(t)/N$, and $r(t) = R(t)/N$. Note that $s(t) + e(t) + i(t) + r(t) = 1$.

The organization of the remaining sections is as follows. In Section 2, we investigate mathematical properties of the solutions to the differential equations. After recalling the basic reproductive number, various behaviors of epidemic curves are presented. Especially, we find the orbit of epidemic curves to find the limiting values of the solutions with the help of an algebraic equation containing the initial condition and the basic reproductive number. In Section 3, we explore actuarial quantities for an infective disease insurance product to formulate the level net premiums, medical expenses in the form of continuous annuities or lump-sum of benefit payments.

2. MATHEMATICAL ANALYSIS

In this section, we investigate mathematical properties of a deterministic *SEIR* compartment model described in the introduction.

Theorem 2.1. *For the solution* $(s(t), e(t), i(t), r(t))$ *of the system of differential equations (1)–(4) with initial condition* $r(0) = r_0 = 0$ *and* $s(0) + e(0) + i(0) = s_0 + e_0 + i_0 = 1$*, we have the followings:*

(a) The basic reproductive number R_0 *is* β/λ*.*

(b) $s(t)$ *is decreasing to* s_∞*. In fact,*

$$s(t) = s_0 e^{-\int_0^t \beta i(u)du}, \quad \int_0^\infty i(u)du = -\frac{1}{\beta} \ln \frac{s_\infty}{s_0}. \tag{5}$$

(c) $r(t)$ *and* $\tilde{i}(t) \equiv e(t) + i(t)$ *can be represented using* $s(t)$*:*

$$r(t) = \frac{\lambda}{\beta} \ln \frac{s_0}{s(t)}, \quad \tilde{i}(t) \equiv e(t) + i(t) = 1 - s(t) - \frac{\lambda}{\beta} \ln \frac{s_0}{s(t)}$$

Moreover,

$$e(t) = e^{-\gamma t}\left(e_0 - e^{\gamma t} s(t) + s_0 + \int_0^t \gamma e^{\gamma u} s(u)du\right), \tag{6}$$

$$i(t) = 1 - \frac{\lambda}{\beta} \ln \frac{s_0}{s(t)} - e^{-\gamma t}\left(e_0 + s_0 + \int_0^t \gamma e^{\gamma u} s(u)du\right). \tag{7}$$

(d) Both $e(t)$ and $i(t)$ converge to 0 as $t \to \infty$ while s_∞ is bounded below and above by $s_0 e^{-s_0\beta/\lambda}$ and $\lambda/\beta = 1/R_0$, respectively.

(e) If $s_0 \le \lambda/\beta$, then $\tilde{i}(t)$ is decreasing to 0. If $s_0 > \lambda/\beta$, then $s_\infty < \lambda/\beta$ and, for t^* where $s(t^*) = \lambda/\beta$,

$$\tilde{i}(t) \equiv e(t) + i(t) = \begin{cases} \textit{increasing} & \textit{if } 0 < t < t^*, \\ 1 - \frac{\lambda}{\beta} + \frac{\lambda}{\beta}\left(\ln\frac{\lambda}{\beta} - \ln s_0\right) & \textit{if } t = t^*, \\ \textit{decreasing} & \textit{if } t > t^*. \end{cases}$$

Moreover,

$$e(t) + i(t) \ge (e(t^*) + i(t^*))e^{\beta(s_\infty - \frac{\lambda}{\beta})(t-t^*)} \quad \textit{for } t \ge t^* \tag{8}$$

(f) s_∞ is the unique solution to the equation with parameters s_0 and $R_0 = \beta/\lambda$:

$$s_\infty - \frac{\lambda}{\beta}\ln s_\infty = 1 - \frac{\lambda}{\beta}\ln s_0, \tag{9}$$

or,

$$s_\infty - \frac{\ln s_\infty}{R_0} = 1 - \frac{\ln s_0}{R_0}.$$

Proof. (a) Following the next generation matrix method (see [1] for details), after linearizing around the disease-free equilibrium $(S^*, E^*, I^*, R^*) = (N, 0, 0, 0)$, a sub-model which contains only the disease compartments, or the vector $\vec{x} = \begin{bmatrix} E \\ I \end{bmatrix}$ can be written as

$$\frac{d\vec{x}}{dt} = (F - V)\vec{x}, \quad F = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \gamma & 0 \\ -\gamma & \lambda \end{bmatrix}.$$

Then the basic reproductive number R_0 is the spectal radius, or the eigenvalue with the largest magnitude of the next generation matrix FV^{-1}. Since the eigenvalues are 0 and β/λ, we obtain $R_0 = \beta/\lambda$.

(b) We solve the differential equation (1) to get (5) which also implies that $s(t)$ is decreasing. Since $s(t)$ is bounded by a natural lower bound of 0, $s(t)$ converges to $s_\infty \ge 0$ as $t \to \infty$.

(c) Following the argument in [3], we can describe the orbits of the epidemic curve. In fact, we take the sum of (2) and (3) to get

$$\tilde{i}'(t) = (e(t) + i(t))' = (\beta s(t) - \lambda) i(t). \tag{10}$$

Divide (10) by (1) to get

$$\frac{d\tilde{i}}{ds} = -1 + \frac{\lambda}{\beta}\frac{1}{s}.$$

Solving this equation leads to

$$\tilde{i}(t) = -s(t) + \frac{\lambda}{\beta}\ln s(t) + C \tag{11}$$

for some constant C which can be determined using the initial condition

$$\tilde{i}(0) = -s_0 + \frac{\lambda}{\beta}\ln s_0 + C$$

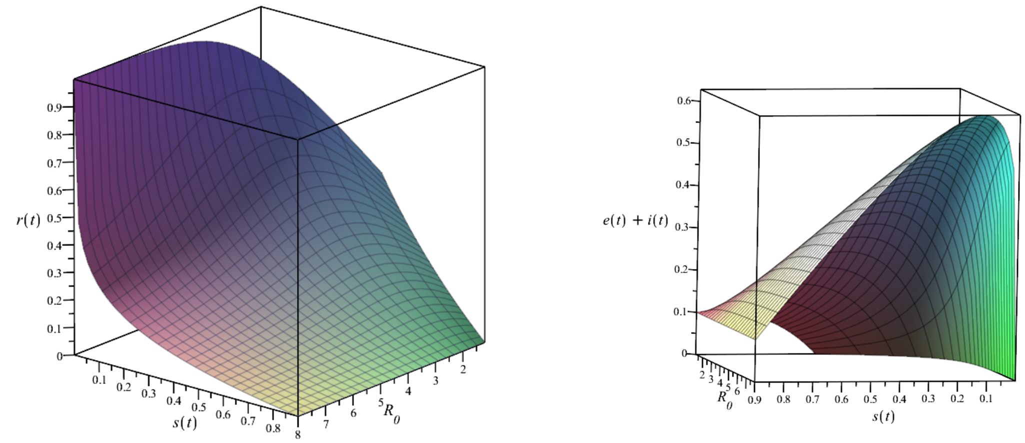
or, $C = \tilde{i}(0) + s_0 - \frac{\lambda}{\beta}\ln s_0$. Substituting this expression to (11) gives

$$\tilde{i}(t) \equiv e(t) + i(t) = 1 - s(t) - \frac{\lambda}{\beta}\ln\frac{s_0}{s(t)}. \tag{12}$$

Since $s(t) + \tilde{i}(t) = 1 - r(t)$, the equation (11) leads to the following representation:

$$r(t) = \frac{\lambda}{\beta}\ln\frac{s_0}{s(t)}. \tag{13}$$

On the other hand, we can represent the remaining $e(t)$ and $i(t)$ using $s(t)$. In fact, using the integrating factor method, we solve the equation (2) to get (6). Moreover, (12) produces the expression for $i(t)$, or (7). See Figure 2 for the surfaces of $r(t)$ and $e(t) + i(t)$ as functions of R_0 and $s(t)$ when $s_0 = 0.9$. Other values for s_0 will produce similar graphs and we omit here.



(a) $r(t)$ **(b)** $e(t)+i(t)$

Figure 2: $r(t), e(t)+i(t)$ for $s_0 = 0.9$

(d) The representation of $r(t)$ in (13) with the natural upper bound, s_0 gives a lower bound of $s(t)$ as

$$s(t) \geq s_0 e^{-s_0\beta/\lambda} \tag{14}$$

which implies $s_\infty \geq s_0 e^{-s_0\beta/\lambda} > 0$.

Since $s(t)$ converges to $s_\infty > 0$, the representations (6) and (7) guarantee the convergence of $e(t)$ and $i(t)$ to e_∞ and i_∞, respectively. Then the relation (5) leads to $i_\infty = 0$ to make the integration finite. Moreover, if $e_\infty > 0$, then the equation (3) implies that $i(t)$ will be increasing without bound contradicting to the fact $i(t) \leq 1$. Therefore $e_\infty = 0$.

On the other hand, from (10), $\tilde{i}(t) = e(t) + i(t)$ is increasing if $s(t) > \lambda/\beta$ and decreasing if $s(t) < \lambda/\beta$. Since $e_\infty + i_\infty = 0$, we can conclude that $s(t) < \lambda/\beta$ for all sufficiently large $t > 0$ which leads to $s_\infty < \lambda/\beta = 1/R_0$.

(e) Since $s(t)$ is decreasing, if $s_0 \leq \lambda/\beta$, then, from the equation (10), $\tilde{i}'(t) = (e(t)+i(t))'$ is negative for $t > 0$ and $\tilde{i}(t)$ is strictly decreasing to 0.

If $s_0 > \lambda/\beta$ then $\tilde{i}'(t) = (e(t)+i(t))'$ changes the sign from positive to negative at t^* where $s(t^*) = \lambda/\beta$. Therefore $\tilde{i}(t)$ is increasing on the interval $(0, t^*)$, hitting the maximum value of $1 - \frac{\lambda}{\beta} + \frac{\lambda}{\beta}\left(\ln\frac{\lambda}{\beta} - \ln s_0\right)$ at $t = t^*$, then decreasing to 0 as $t \to \infty$.

A lower bound for $\tilde{i}(t)$ comes from (10), i.e. for $t > t^*$,

$$\tilde{i}'(t) = (\beta s(t) - \lambda)i(t) < (\beta s(t) - \lambda)\tilde{i}(t)$$

Now we use the fact $s(t) > s_\infty$ to get the desired inequality (8) using the Gronwall's inequality.

(f) From the expression (12) for $\tilde{i}(t)$, we take the limit $t \to \infty$ to get

$$0 = 1 - s_\infty - \frac{\lambda}{\beta} \ln \frac{s_0}{s_\infty}.$$

This equation can be reorganized to have the form (9). Finally, the uniqueness of s_∞ comes from the condition that $s_\infty < \lambda/\beta$ which makes the function $f(x) = x - (\ln x)/R_0$ monotone decreasing. See Figure 3 for the surface of s_∞ as a function of s_0 and R_0. □

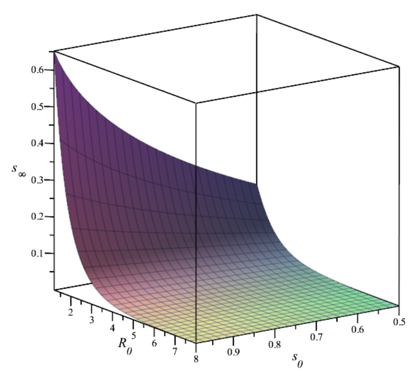


Figure 3: $s_\infty(s_0, R_0)$

We note that, as $s(t)$ can represent all other quantities, any one quantity can represent others as well. For example, if $r(t)$ can be estimated, other quantities will be estimated using the relation:

$$s(t) = s_0 e^{-\frac{\beta}{\lambda} r(t)},$$

$$e(t) = (e_0 + s_0)e^{-\gamma t} - s_0 e^{-\frac{\beta}{\lambda} r(t)} + s_0 \gamma e^{-\gamma t} \int_0^t e^{\gamma u} e^{-\frac{\beta}{\lambda} r(u)} du,$$

$$i(t) = 1 - (e_0 + s_0)e^{-\gamma t} - s_0 \gamma e^{-\gamma t} \int_0^t e^{\gamma u} e^{-\frac{\beta}{\lambda} r(u)} du.$$

In fact, we rewrite the equation (13) to get $s(t)$, while $e(t)$ and $i(t)$ come from the equation (7) and $s(t) + e(t) + i(t) + r(t) = 1$, respectively.

3. ACTUARIAL ANALYSIS

Since $s(t), e(t), i(t)$ and $r(t)$ of the *SEIR* model (1)–(4) are defined as the factions of the population of fixed size N for each compartments S, E, I and R, respectively, we can use those fractions as probabilities of an individual being in the associated compartments to develop an infectious disease insurance product implementing the argument in [2].

We first introduce actuarial symbols following the general principles of international actuarial notation [4] to formulate the necessary quantities for insurance coverages against financial impact caused by infectious diseases.

Definition 3.1. *For an infectious disease insurance product under the SEIR model and the force of interest $\delta > 0$, we define the actuarial present value (APV) of annuities and lump-sum benefit payments.*

(a) APV of continuous premium payments of 1 unit per year for a t-year period from individuals in the class S on average:

$$\bar{a}^{S}_{\overline{t}|} = \int_0^t e^{-\delta x} s(x) dx$$

(b) APV of continuous benefit payments of 1 unit per year for a t-year period to individuals in the class E and I on average, respectively:

$$\bar{a}^{E}_{\overline{t}|} = \int_0^t e^{-\delta x} e(x) dx, \quad \bar{a}^{I}_{\overline{t}|} = \int_0^t e^{-\delta x} i(x) dx$$

(c) APV of unit lump-sum benefit payment to individuals on average at the moment of exposure to the disease:

$$\bar{A}^{E}_{\overline{\infty}|} = \int_0^{\infty} e^{-\delta t} \beta i(t) s(t) dt$$

Note that the force of exposure is defined as $\mu^S_t := -\frac{s'(t)}{s(t)} = \beta i(t)$ so that the probability of being exposed on the infinitesimal time interval $(t, t + dt)$ can be approximated as

$s(t)\mu_t^S = \beta i(t)s(t)$.

In the following theorem, we develop expressions for level premiums based on the equivalence principle: APV of Benefit Outgo = APV of Premium Income. See [2] for the case of SIR model.

Theorem 3.2. *For the SEIR model (1)–(4),*

(a) the level net annual premium for infinite term insurance to be collected continuously from individuals in the class S to pay level benefits continuously while in the class E and I

$$P = \frac{1}{\bar{a}^S_{\overline{\infty}|}}(\bar{a}^E_{\overline{\infty}|} + \bar{a}^I_{\overline{\infty}|}) = \frac{\delta}{\delta+\lambda} \times \frac{i_0 + (\delta+\lambda+\gamma)\bar{a}^E_{\overline{\infty}|}}{s_0 + e_0 - (\delta+\gamma)\bar{a}^E_{\overline{\infty}|}} \tag{15}$$

(b) the level net premium P for infinite term insurance to be collected continuously from individuals in the class S to pay unit lump-sum compensation to individuals at the moment of exposure to the disease

$$P = \frac{\bar{A}^E_{\overline{\infty}|}}{\bar{a}^S_{\overline{\infty}|}} = \frac{(\delta+\gamma)\bar{a}^E_{\overline{\infty}|} - e_0}{\frac{1}{\delta}(s_0+e_0) - \left(1+\frac{\gamma}{\delta}\right)\bar{a}^E_{\overline{\infty}|}}. \tag{16}$$

Proof. (a) From the equation (4), we compute the APV of continuous benefit payments of 1 unit per year for an infinite period to individuals in the class R on average as

$$\bar{a}^R_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} r(t)dt = \int_0^\infty e^{-\delta t}\left(\int_0^t \lambda i(x)dx + r_0\right)dt = \frac{r_0}{\delta} + \frac{\lambda}{\delta}\bar{a}^I_{\overline{\infty}|}$$

where we used the standard integration by parts technique.

Since $s(t)+e(t)+i(t)+r(t) = 1$, computing the APV of continuous benefit payments of 1 unit per year for an infinite period to all individuals in the population, we obtain the relation between continuous annuities of each classes.

$$\bar{a}^S_{\overline{\infty}|} + \bar{a}^E_{\overline{\infty}|} + \left(1+\frac{\lambda}{\delta}\right)\bar{a}^I_{\overline{\infty}|} = \frac{1}{\delta}(1-r_0) = \frac{1}{\delta}(s_0+e_0+i_0). \tag{17}$$

On the other hand, the integration by parts along with the equation (3) leads to

$$\bar{a}^I_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} i(t)dt = \frac{i_0}{\delta} + \frac{\gamma}{\delta}\bar{a}^E_{\overline{\infty}|} - \frac{\lambda}{\delta}\bar{a}^I_{\overline{\infty}|}$$

which gives the relation

$$(\delta + \lambda)\bar{a}^{I}_{\overline{\infty}|} - \gamma \bar{a}^{E}_{\overline{\infty}|} = i_0. \tag{18}$$

Combining (18) and (17), we can find additional relation as

$$\delta \bar{a}^{S}_{\overline{\infty}|} + (\delta + \gamma)\bar{a}^{E}_{\overline{\infty}|} = s_0 + e_0. \tag{19}$$

Finally, substituting (18) and (19) to the premium formula based on the equivalence principle, we obtain the desired result (15).

(b) From the APV formula, Definition 3.1 (c), of unit lump-sum benefit payment to infinite-term policyholder at the moment of exposure to the disease

$$\bar{A}^{E}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} \beta i(t) s(t) dt,$$

we obtain by integration by parts

$$\bar{A}^{E}_{\overline{\infty}|} + \delta \bar{a}^{S}_{\overline{\infty}|} = s_0.$$

Moreover, (19) can be used to replace $\bar{a}^{S}_{\overline{\infty}|}$ with $\bar{a}^{E}_{\overline{\infty}|}$ leading to

$$\bar{A}^{E}_{\overline{\infty}|} + e_0 = (\delta + \gamma)\bar{a}^{E}_{\overline{\infty}|}.$$

Therefore, we have the desired equation (16) for the level net premium P.

□

4. CONCLUSION AND FUTURE WORK

The contribution of this paper is twofold. First, we investigated mathematical properties of the solutions to the system of differential equations describing a deterministic *SEIR* compartment model. Finding the orbit of the epidemic curves, we were able to determine the limiting behavior of the solutions. It turned out that the initial condition and the basic reproductive number determine the limiting values of the solutions controlled by an algebraic equation. Second, we explored actuarial quantities associated to the compartment model to formulate the level net premiums of infinite term infectious disease insurance products for the case of continuous premium collection and continuous benefit payments or lump-sum payments.

Note that the *SEIR* compartment model is formulated under a certain set of assumptions which might be too restrictive to fit actual pandemics. Moreover, the form of the system of differential equations is crucial in deriving mathematical and actuarial properties. Therefore it is needed to extend the argument to more generalized models which can fit different features of the targeted pandemics. We shall pursue such extensions in our future research.

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