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A Note on Multiplicative (Generalized) - (α, β) - Reverse Derivations on Left Ideals in Prime Rings

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Abstract

A mapping G: R→ R (not necessarily additive) is called multiplicative right α-centralizer if T(xy) = α(x)T(y) for all x, y ∈ R. A mapping G: R → R (not necessarily additive) is called multiplicative (generalized) - (α, β) - reverse derivation if there exists a map (neither necessarily additive or derivation) g : R→ R such that G(xy) = G(y)α(x) + β(y)g(x) for all x,y ∈ R, where α and β are automorphisms on R. The main purpose of this paper is to study some algebraic identities with multiplicative (generalized)-(α, β)-reverse derivations and multiplicative right α-centralizer on the left ideal of a prime ring R. The main objective of the present paper is to investigate the following algebraic identities: (i) G(xy) ± T (x)T(y) = 0 (ii) G(xy) ± T (xy) = 0 (iii) G(xy) ± T(xy) ∈ Z(R) (iv) G(xy) ± G(x)T(y) ∈ Z(R) and (v) G(xy) ± T(x)G(y) = 0 for all x, y in an appropriate subset of R.

Keywords: Prime ring, right ideal, multiplicative right α-centralizer, derivation, reverse derivation, generalized derivation, multiplicative (generalized) derivation, multiplicative (generalized) reverse derivation, Multiplicative (generalized) - (α, β) - derivation, Multiplicative (generalized) - (α, β) - reverse derivation.

INTRODUCTION:

Throughout R will represent an associative ring. A ring R is n-torsion free, if nx = 0 implies x = 0 for all x ∈ R, where n > 1 is an integer. A ring R is called prime if aRb = (0) implies either a = 0 or b = 0 and is called semiprime if aRa = (0) implies that a = 0. An additive mapping d : R → R is called a derivation if d(xy) = d(x)y + xd(y) holds for all pairs x, y ∈ R. We make frequent use of following commutator identities:

[xy, z] = [x, z]y + x[y, z] [x, yz] = [x, y]z + y[x, z].

An additive mapping G: R → R is said to be generalized derivation if there exists an associated derivation d: R → R such that G(xy) = G(x)y + x d(y) holds for all pairs x, y ∈ R. Of course,a generalized derivation is a generalization of derivation. Inspired by Martindale [6], in 1991, Daif [2] has given a generalization of derivation as multiplicative derivation which is defined as: a mapping d: R → R (not necessarily additive) is said to be multiplicative derivation if d(xy) = d(x)y + xd(y) for all x, y ∈ R.

Similar type of notation is defined in [5]. Later, in [3] Daif and Tammam have extended this notation to multiplicative generalized derivation as follows: a mapping G : R → R is a multiplicative generalized derivation if there exists a derivation g such that that G(xy) = G(x)y + xg(y) for all x, y ∈ R. If we consider g is any map neither necessarily derivation nor additive, then G is called multiplicative (generalized) derivation. The concept of multiplicative (generalized) derivation covers the concept of multiplicative generalized derivation and multiplicative centralizer (if g = 0). A mapping g from R to R is centralizing if [g(x), x] ∈ Z(R) and particularly commuting if [g(x), x] = 0 for all x ∈ R. A mapping T : R → R (not necessarily additive) is called multiplicative left α- centralizer if T(xy) =T(x)α(y) for all x, y ∈ R. The multiplicative left α-centralizer covers the concept of multiplicative left centralizer, left α-centralizer and centralizers. Posner [8] has initiated to investigate the commutative of rings with derivations. In details he proved that a prime ring R is commutative if there is a nonzero derivation d which is centralizing on R.

Ashraf and Rehman [1] showed that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying either of the properties d(xy) + xy ∈ Z(R) or d(xy) − xy ∈ Z(R) for all x, y ∈ R. A number of generalizations of such kind of commutativity ideas found in [4, 10]. Recently Rehman et al. in [9] introduced a concept of multiplicative (generalized) - (α, β) – derivations in rings as follows: A mapping G : R → R (not necessarily additive) is called multiplicative (generalized) - (α, β) – derivation if there exists a map (neither necessarily additive or derivation) g : R→ R such that G(xy) = G(x)α(y) + β(x)g(y) for all x, y ∈ R. A mapping G : R → R (not necessarily additive) is called multiplicative (generalized) - (α, β) - reverse derivation if there exists a map (neither necessarily additive or derivation) g : R→ R such that G(xy) = G(y)α(x) + β(y)g(x) for all x, y ∈ R, where α and β are automorphisms on R.

Lemma: (Muthana and Alkhamisi [7, Lemma2.1]). Let R be a semiprime ring, I be a nonzero left ideal of R. suppose that $\alpha,$: R → R are two mappings such that β(I) ⊆ I.

If either $[xy\alpha(z),\beta(z)] = 0$ or $x[y\alpha(z), \beta(z)] = 0$ for all $x, y, z \in I$, then $I[\alpha(z), \beta(z)] = (0)$ for all $z \in I$.

MAIN RESULTS:

Theorem 1: Let R be a prime ring, I be a nonzero left ideal of R. Suppose that $G : R\rightarrow R$ is multiplicative (generalized) - (α, β) - reverse derivation associated with the map g and $T : R\rightarrow R$ is a multiplicative right α - centralizer. If G and T satisfy $G(xy) \pm T(x)T(y) = 0$ for all $x, y \in I$, then G is multiplicative right α - centralizer on I, where α and β are automorphisms on R.

Proof: We have

$G(xy) \pm T(x)T(y) = 0$ for all $x, y \in I$. (1)

Replacing y by xy, we get

$G(xxy) \pm T(x)T(xy) = 0$ for all $x, y \in I$,

$G(xy)\alpha(x) + \beta(xy)g(x) \pm T(x)\alpha(x)T(y) = 0$ for all $x, y \in I$,

$[G(xy) \pm T(x)T(y)]\alpha(x) + \beta(xy)g(x) = 0$ for all $x, y \in I$.

Using equation (1), we get

$\beta(xy)g(x) = 0$ for all $x, y \in I$,

$\beta(x)\beta(y)g(x) = 0$ for all $x, y \in I$.

Replacing y by ry, we get

$\beta(x)\beta(ry)g(x) = 0$ for all $x, y \in I$, $\beta(x)\beta(r)\beta(y)g(x) = 0$ for all $x, y \in I$.

Replacing r by $\beta^{-1}(r)$, we get

$\beta(x)r\beta(y)g(x) = 0$ for all $x, y \in I$, $r \in R$.

Primeness of R implies that either $\beta(x) = 0$ or $\beta(y)g(x) = 0$, for all $x, y \in I$.

If $\beta(x) = 0$ for all $x \in I$, then $\beta(I) = (0)$. Since I is nonzero left ideal of R, then we get a contradiction. So $\beta(y)g(x) = 0$, for all $x, y \in I$, then $G(xy) = \alpha(x)G(y)$ for all $x, y \in I$, which is the result.

Theorem 2: Let R be a prime ring, I be a nonzero left ideal of R. Suppose that $G : R\rightarrow R$ is multiplicative (generalized) - (α, β) - reverse derivation associated with the map g and $T : R\rightarrow R$ is a multiplicative right α - centralizer. If G and T satisfy $G(xy) \pm T(xy) = 0$ for all $x, y \in I$, then G is multiplicative right α - centralizer on I, where α and β are automorphisms on R.

Proof: We have

$G(xy) \pm T(xy) = 0$ for all $x, y \in I$. (2)

Replacing y by xy, we get

$G(xxy) \pm T(xxy) = 0$ for all $x, y \in I$,

$G(xy)\alpha(x) + \beta(xy)g(x) \pm \alpha(x) T(xy) = 0$ for all $x, y \in I$, (3)

$[G(xy) \pm T(xy)] \alpha(x) + \beta(xy)g(x) = 0$ for all $x, y \in I$.

Using equation (2), we get

$\beta(xy)g(x) = 0$ for all $x, y \in I$,

$\beta(x)\beta(y)g(x) = 0$ for all $x, y \in I$, (4)

Replacing y by ry, we get

$\beta(x)\beta(ry)g(x) = 0$ for all $x, y \in I$,

$\beta(x)\beta(r)\beta(y)g(x) = 0$ for all $x, y \in I$. (5)

Replacing r by $\beta^{-1}(r)$, we get

$\beta(x)r\beta(y)g(x) = 0$ for all $x, y \in I$, $r \in R$.

Primeness of R implies that either $\beta(x) = 0$ or $\beta(y)g(x) = 0$, for all $x, y \in I$.

If $\beta(x) = 0$ for all $x \in I$, then $\beta(I) = (0)$. Since I is nonzero left ideal of R, then we get a contradiction. So $\beta(y)g(x) = 0$, for all $x, y \in I$, then $G(xy) = \alpha(x)G(y)$ for all $x, y \in I$, which is the result.

Theorem 3: Let R be a prime ring, I be a nonzero left ideal of R. Suppose that $G : R\rightarrow R$ is multiplicative (generalized) - (α, β) - reverse derivation associated with the map g and $T : R\rightarrow R$ is a multiplicative right α - centralizer. If G and T satisfy $G(xy) \pm T(xy) \in Z(R)$ for all $x, y\in I$, then $I[g(x), \alpha(x)] = (0)$, where α and β are automorphisms on R with $\beta(I) \subseteq I$.

Proof: We have

$G(xy) \pm T(xy) \in Z(R)$ for all $x, y \in I$. (6)

Replacing y by xy, we get

$G(xxy) \pm T(xxy) \in Z(R)$ for all $x, y \in I$,

$G(xy)\alpha(x) + \beta(xy)g(x) \pm \alpha(x) T(xy) \in Z(R)$ for all $x, y \in I$,

Which implies that

$[G(xy) \pm T(xy)] \alpha(x) + \beta(xy)g(x) \in Z(R)$ for all $x, y \in I$.

Now, from equation (6), we get

$[\beta(x)\beta(y)g(x), \alpha(x)] = 0$ for all $x, y \in I$.

$\beta(I) \subseteq I$ implies that

$[xyg(x), \alpha(x)] = 0$ for all $x, y \in I$.

An application of Lemma infer that

$I[g(x), \alpha(x)] = (0)$ for all $x \in I$,

which is the required result.

Theorem 4: Let R be a prime ring, I be a nonzero left ideal of R. Suppose that $G : R \to R$ is multiplicative (generalized) - (α, β) - reverse derivation associated with the map g and $T : R \to R$ is a multiplicative right α - centralizer. If G and T satisfy $G(xy) \pm G(x)T(y) \in Z(R)$ for all $x, y \in I$, then $I[g(x), \alpha(x)] = (0)$, where α and β are automorphisms on R with $\beta(I) \subseteq I$.

Proof: We have

$G(xy) \pm G(x)T(y) \in Z(R)$ for all $x, y \in I$. (7)

Replacing y by xy, we get

$G(xxy) \pm G(x)T(xy) \in Z(R)$ for all $x, y \in I$.

$G(xy)\alpha(x) + \beta(xy)g(x) \pm G(x)\alpha(x)T(y) \in Z(R)$ for all $x, y \in I$,

Which implies that

$[G(xy) \pm G(x)T(y)]\alpha(x) + \beta(xy)g(x) \in Z(R)$ for all $x, y \in I$.

Now, from equation (7), we get

$[\beta(x)\beta(y)g(x), \alpha(x)] = 0$ for all $x, y \in I$.

$\beta(I) \subseteq I$ implies that

$[xyg(x), \alpha(x)] = 0$ for all $x, y \in I$.

An application of Lemma yields that

$I[g(x), \alpha(x)] = (0)$ for all $x \in I$,

which is the result.

Theorem 5: Let R be a prime ring, I be a nonzero left ideal of R. Suppose that $G : R \to R$ is multiplicative (generalized) (α, β) – reverse derivation associated with the map g and $T : R \to R$ is a multiplicative right α – centralizer. If G and T satisfy $G(xy) \pm T(x)G(y) = 0$ for all $x, y \in I$, then either $G(xy) = \pm xG(y)$ or $G(xy) = G(x)\alpha(y)S$, where α and β are automorphisms onR with $\beta(I) \subseteq I$.

Proof: We have

$G(xy) \pm T(x)G(y) = 0$ for all $x, y \in I$. (8)

Replacing y by xy, we get

$G(xxy) \pm T(x)G(xy) = 0$ for all $x, y \in I$.

$G(xy)\alpha(x) + \beta(xy)g(x) \pm T(x)[G(y)\alpha(x) + \beta(y)g(x)] = 0$ for all $x, y \in I$, Which implies that

$G(xy)\alpha(x) + \beta(xy)g(x) \pm T(x)G(y)\alpha(x) \pm T(x)\beta(y)g(x) = 0$ for all $x, y \in I$,

$[G(xy) \pm T(x)G(y)]\alpha(x) + \beta(xy)g(x) \pm T(x)\beta(y)g(x) = 0$ for all $x, y \in I$.

Now, from equation (8), we get

$\beta(xy)g(x) \pm T(x)\beta(y)g(x) = 0$ for all $x, y \in I$.

$\beta(x)\beta(y)g(x) \pm T(x)\beta(y)g(x) = 0$ for all $x, y \in I$.

$[\beta(x) \pm T(x)]\beta(y)g(x) = 0$ for all $x, y \in I$.

$[\beta(x) \pm T(x)]\beta(y)g(x) = 0$ for all $x, y \in I$.

Since $\beta(I) \subseteq I$, we can reword the above expression as

$$[x \pm T(x)]\, y\, g(x) = 0 \text{ for all } x, y \in I.$$

Since R is a prime ring, we conclude that either $T(x) = \mp x$ or $g(x) = 0$ by Brauer's trick. If $T(x) = \mp x$, use (8) to get $G(xy) = \pm xG(y)$ for all $x, y \in I$. Next if $g(x) = 0$, then $G(xy) = G(x)\alpha(y)$ for all $x, y \in I$, which is the result.

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