Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 18, Number 2 (2022), pp. 409- 424

http://www.ripublication.com/gjpam.htm

Some Common Fixed Point Theorems for Two Self-Maps Satisfying Contractive Inequality of Integral Type in Metric Space

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ABSTRACT

The intent of this manuscript is to prove a common fixed point theorem for two weakly compatible self-maps $\wp$ and $\mathbb{Q}$ on a metric space (M, d^*) satisfying the following contractive inequality of integral type:

$$\int_0^{d^*(\mathbb{Q}\mu,\mathbb{Q}v)} \xi(t)dt \leq \beta\big(d^*(\wp\mu,\wp v)\big) \int_0^{\Delta_1(\wp\mu,\wp v)} \xi(t)dt,$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, v in M,

where

$$\Delta_1(\wp\mu,\wp v) = max\{d^*(\wp\mu,\wp v), d^*(\mathbb{Q}\mu,\wp\mu), d^*(\mathbb{Q}v,\wp v),$$

$$\frac{d^*(\mathbb{Q}\mu,\wp\mu).\, d^*(\mathbb{Q}v,\wp v)}{1 + d^*(\mathbb{Q}\mu,\mathbb{Q}v)}, \frac{d^*(\mathbb{Q}\mu,\wp\mu).\, d^*(\mathbb{Q}v,\ \wp v)}{1 + d^*(\wp\mu,\wp v)}\Bigg\}.$$

Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

Keywords: fixed point, coincidence point, weakly compatible maps, E.A. property, (CLR) property.

2010 MSC: 47H10, 54H25

1. INTRODUCTION

All around this paper we presume that $\mathbb{R}^+ = [0,\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ stands for the set of positive integers and

- $\xi_1 = \{ \xi \mid \xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that ξ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^{\delta} \xi(t)dt > 0$ for each $\delta > 0 \}$,
- $\xi_2 = \{ \tau \mid \tau : \mathbb{R}^+ \to [0,1)$ satisfies that $\lim_{s\to t} \sup \tau(s) < 1$ for each $t \in \mathbb{R}^+\}$,
- $\xi_3 = \{ \tau \mid \tau \in \xi_2$ and $\lim_{s\to +\infty} \sup \tau(s) < 1\}$.

The concept of contractive mapping of integral type was introduced by Branciari [2] in 2002 and obtained the following fixed point result for the mapping:

Theorem 1.1. [2] Let (M, d^*) be a complete metric space and $\hat{T}$ be a self map on M satisfying

$$\int_0^{d^*(\hat{T}\mu,\hat{T}v)} \xi(t)dt \leq \beta \int_0^{d^*(\mu,v)} \xi(t)dt \quad \text{for all } \mu, v \text{ in } M,$$

where $\beta \in (0,1)$ is a constant and $\xi \in \xi_1$. Then $\hat{T}$ has a unique fixed point $b \in M$ such that $\lim_{n\to\infty} \hat{T}^n\mu = b$ for each $\mu \in M$.

Definition 1.2. A coincidence point of a pair of self – maps $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $P\mu = Q\mu$.

A common fixed point of a pair of self - mappings- $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $\hat{P}\mu = \hat{Q}\mu = \mu$.

The concept of weakly compatible mappings was introduced by Jungck [5] in 1996 to study common fixed point theorems:

Definition 1.3. Let (M, d^*) be a metric space. A pair of self – maps $\hat{P}, \hat{Q} : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in M$ such that $\hat{P}\hat{Q}\mu = \hat{Q}\hat{P}\mu$, where μ is coincidence point of $\hat{P}$ and $\hat{Q}$.

In 2002, Aamri and EI Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.4. Let (M, d^*) be a metric space. Two self-maps $\hat{P}$ and $\hat{Q}$on M are said to satisfy the E.A. property, if there exists a sequence $\{\mu_n\}$ in M such that,

$$\lim_{n\to\infty} \hat{P}\mu_n = \lim_{n\to\infty} \hat{Q}\mu_n = t, \text{ for some } t \in M.$$

In 2011, Sintunavarat*et al.* [9] introduced the notion of (CLR) property as follows:

Definition 1.5. Let (M, d^*) be a metric space. Two self-maps $\hat{P}$ and $\hat{Q}$ on M are said to satisfy the (CLR) property, if there exists a sequence $\{\mu_n\}$ in M such that,

$$\lim_{n\to\infty} \hat{P}\mu_n = \lim_{n\to\infty} \hat{Q}\mu_n = \hat{P}\text{ (t) for some } t \in M.$$

Lemma 1.6.[8] let $\xi \in \xi_1$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be a non negative sequence with $\lim_{n\to\infty} \mu_n = b$.

Then,

$$\lim_{n\to\infty}\int_0^{\mu_n}\xi(t)dt=\int_0^{b}\xi(t)dt.$$

Some of the related work can be seen in [3, 6-7].

2. MAIN RESULTS

Theorem 2.1. Let (M,d^*) be a metric space and let $\wp$ and $\mathbb{Q}$be two self– maps on Msatisfying the followings:

(2.1) $\mathbb{Q}M \subseteq \wp M$.

(2.2) There exists a continuous mapping $\xi:[0,\infty)\to[0,\infty)$ with $\xi(0)=0$ and $\xi(\alpha)>\alpha$ for all $\alpha>0$ such that:

$$\int_0^{d^*(\mathbb{Q}\mu,\mathbb{Q}v)}\xi(t)dt \le \beta\big(d^*(\wp\mu,\wp v)\big)\int_0^{\Delta_1(\wp\mu,\wp v)}\xi(t)dt,$$

where $(\xi,\beta)\in\ \xi_1\times\xi_3$ and for all μ,v in M,

where

$$\Delta_1(\wp\mu,\wp v)=max\{d^*(\wp\mu,\wp v),d^*(\mathbb{Q}\mu,\wp\mu),d^*(\mathbb{Q}v,\wp v),$$
$$\frac{d^*(\mathbb{Q}\mu,\wp\mu).d^*(\mathbb{Q}v,\wp v)}{1+d^*(\mathbb{Q}\mu,\mathbb{Q}v)},\frac{d^*(\mathbb{Q}\mu,\wp\mu).d^*(\mathbb{Q}v,\ \wp v)}{1+d^*(\wp\mu,\wp v)}\Big\}.$$

If $\wp$ and $\mathbb{Q}$ are weakly compatible and $\wp M$ or $\mathbb{Q}M$ is complete, then$\wp$ and$\mathbb{Q}$ have a unique common fixed point.

Proof: Let $\mu_0$be arbitrary point in M. From (2.1), Since $\mathbb{Q}M\ \subseteq\wp M$. So, we can define a sequence $\{\mu_n\}$such that:

$$\mathbb{Q}\mu_{\rm n}=\wp\mu_{\rm n+1}.$$

Define a sequence $\{v_n\}$ in M by,

$$v_{\rm n}=\ \mathbb{Q}\mu_{\rm n}=\wp\mu_{\rm n+1.} \tag{2.3}$$

If $v_n=v_{n+1}$ for some n in $\mathbb{N}$, then there is nothing to prove.

Now, we assume that $v_n\ne v_{n+1}$ for all n in $\mathbb{N}$.

We prove that

$$\lim_{n\to\infty} d^*(\,v_n,v_{n+1})=0. \tag{2.4}$$

Substituting,$\mu=\mu_n,\ v=\mu_{n+1}$ in (2.2) and using (2.3), we get

$\Delta_1(\wp\mu_n,\wp\mu_{n+1})$= max$\{d^*(\wp\mu_n,\wp\mu_{n+1}),d^*(\mathbb{Q}\mu_n,\wp\mu_n),d^*(\mathbb{Q}\mu_{n+1},\wp\mu_{n+1}),$

$$\frac{d^*(\mathbb{Q}\mu_n,\wp\mu_n).d^*(\mathbb{Q}\mu_{n+1},\wp\mu_{n+1})}{1+d^*(\mathbb{Q}\mu_n,\mathbb{Q}\mu_{n+1})},\frac{d^*(\mathbb{Q}\mu_n,\wp\mu_n).d^*(\mathbb{Q}\mu_{n+1},\ \wp\mu_{n+1})}{1+d^*(\wp\mu_n,\wp\mu_{n+1})}\Bigg\}$$

$$= max\{d^*(v_{n-1},v_n),d^*(v_n,v_{n-1}),d^*(v_{n+1},v_n),\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},v_n)}{1+d^*(v_n,v_{n+1})},$$

$$\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},\ v_n)}{1+d^*(v_{n-1},v_n)}\Bigg\}$$

= max$\{d^*(v_n,v_{n+1}),\ d^*(v_{n-1},v_n)\}$,

since $\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},v_n)}{1+d^*(v_n,v_{n+1})}\leq d^*(v_n,v_{n-1})$ and $\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},v_n)}{1+d^*(v_{n-1},v_n)}\leq d^*(v_{n+1},v_n)$.

If $d^*(v_n,v_{n-1})<d^*(v_{n+1},v_n)$, we have

$$\Delta_1(\wp\mu_n,\wp\mu_{n+1})=d^*(v_{n+1},v_n),$$

and

$$0<\int_0^{d^*(v_n,v_{n+1,})}\xi(t)dt$$

$$=\int_0^{d^*(\mathbb{Q}\mu_n,\mathbb{Q}\mu_{n+1})}\xi(t)dt$$

$$\leq\beta\big(d^*(\wp\mu_n,\wp\mu_{n+1})\big)\int_0^{\Delta_1(\wp\mu_n,\wp\mu_{n+1})}\xi(t)dt,$$

$$=\beta\big(d^*(\wp\mu_n,\wp\mu_{n+1})\big)\int_0^{d^*(v_{n+1},v_n)}\xi(t)dt$$

$$<\int_0^{d^*(v_{n+1},v_n)}\xi(t)dt,$$

which is a contradiction.

Hence

$$d^*(v_{n+1},v_n)<d^*(v_n,v_{n-1})\qquad(2.5)$$

Hence the sequence $\{d^*(v_n,v_{n+1})\}$ is strictly decreasing and bounded below. Thus, there exists $r\geq 0$, such that

$$\lim_{n\to\infty}d^*(v_n,v_{n+1})=r,\qquad(2.6)$$

Suppose that $r > 0$. Then from (2.6) and lemma 1.6, we get

$$0 < \int_0^r \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(v_{n,}v_{n+1})} \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}\mu_n,\mathbb{Q}\mu_{n+1})} \xi(t)dt$$

$$\leq \limsup_{n\to\infty} \beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big) \int_0^{\Delta_1(\wp\mu_n,\wp\mu_{n+1})} \xi(t)dt$$

$$= \limsup_{n\to\infty}[\beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big)\limsup_{n\to\infty} \int_0^{d^*(v_{n,}v_{n+1})} \xi(t)dt]$$

$$< \limsup_{n\to\infty} \int_0^{d^*(v_{n,}v_{n+1})} \xi(t)dt$$

$$< \int_0^r \xi(t)dt,$$

which is a contradiction. Thus, $r = 0$, which implies that

$$\lim_{n\to\infty} d^*\big(v_{n,}v_{n+1}\big) = 0. \tag{2.7}$$

Next, we prove that $\{v_n\}$is a Cauchy sequence. Suppose that $\{v_n\}$ is not a cauchy sequence. Then there exists $\epsilon > 0$, such that for $k \in \mathbb{N}$, there are $m(k)$, $n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying:

(i) $m(k)$ and $n(k)$ are positive integers.

(ii) $d^*\big(v_{n(k)}, v_{m(k)}\big) > \epsilon$.

(iii) $m(k)$ is the smallest even number such that the condition (ii) holds, that is $d^*\big(v_{n(k)}, v_{m(k)-1}\big) \leq \epsilon$.

Therefore,

$$\epsilon < d^*\big(v_{n(k)}, v_{m(k)}\big),$$
$$\leq d^*\big(v_{n(k)}, v_{m(k)-1}\big) + d^*\big(v_{m(k)-1}, v_{m(k)}\big),$$

$$\leq \epsilon + d^*\big(v_{m(k)-1}, v_{m(k)}\big). \tag{2.8}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^*\big(v_{n(k)}, v_{m(k)}\big) = \epsilon. \tag{2.9}$$

$$\epsilon \leq d^*\big(v_{n(k)-1}, v_{m(k)-1}\big),$$

$$\leq d^*\big(v_{n(k)-1}, v_{m(k)-2}\big) + d^*\big(v_{m(k)-2}, v_{m(k)-1}\big),$$

$$\leq \epsilon + d^*\big(v_{m(k)-2}, v_{m(k)-1}\big).$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^*\big(v_{n(k)-1},\ v_{m(k)-1}\big) = \epsilon. \tag{2.10}$$

Substituting $\mu = \mu_{n(k)},\ v = \mu_{m(k)}$in (2.2), we get

$$\Delta_1\big(\wp\mu_{n(k)}, \wp\mu_{m(k)}\big) = \max\Big\{d^*\big(\wp\mu_{n(k)}, \wp\mu_{m(k)}\big), d^*\big(\mathbb{Q}\mu_{n(k)}, \wp\mu_{n(k)}\big),$$

$$d^*\big(\mathbb{Q}\mu_{m(k)}, \wp\mu_{m(k)}\big), \frac{d^*\big(\mathbb{Q}\mu_{n(k)}, \wp\mu_{n(k)}\big).\, d^*(\mathbb{Q}\mu_{m(k)}, \wp\mu_{m(k)})}{1 + d^*\big(\mathbb{Q}\mu_{n(k)}, \mathbb{Q}\mu_{m(k)}\big)},$$

$$\left.\frac{d^*\big(\mathbb{Q}\mu_{n(k)}, \wp\mu_{n(k)}\big).\, d^*(\mu c_{m(k)},\ \wp\mu_{m(k)})}{1 + d^*\ (\wp\mu_{n(k)}, \wp\mu_{m(k)})}\right\}.$$

$$= \max\Big\{d^*\big(v_{n(k)-1}, v_{m(k)-1}\big), d^*\big(v_{n(k)}, v_{n(k)-1}\big), d^*\big(v_{m(k)}, v_{m(k)-1}\big),$$

$$\left.\frac{d^*\big(v_{n(k)}, v_{n(k)-1}\big).\, d^*(v_{m(k)}, v_{m(k)-1})}{1 + d^*(v_{n(k)}, v_{m(k)})}, \frac{d^*\big(v_{n(k)}, v_{n(k)-1}\big).\, d^*(v_{m(k)},\ v_{m(k)-1})}{1 + d^*\ (v_{n(k)-1}, v_{m(k)-1})}\right\}.$$

Taking limit as $k \to \infty$ and using (2.7), (2.8), (2.9) and (2.10), we have

$$\lim_{k\to\infty} \Delta_1\big(\wp\mu_{n(k)}, \wp\mu_{m(k)}\big) = \max\{\epsilon, \epsilon, \epsilon, \tfrac{\epsilon.\epsilon}{1+\epsilon}, \tfrac{\epsilon.\epsilon}{1+\epsilon}\}.$$

$= \epsilon.$

And

$$0 < \int_0^{\epsilon} \xi(t)dt$$

$$= \limsup_{\alpha\to\infty} \int_0^{d^*(v_{n(k)}, v_{m(k)})} \xi(t)dt$$

$$= \limsup_{\alpha\to\infty} \int_0^{d^*(\mathbb{Q}c_{n(k)}, \mathbb{Q}c_{m(k)})} \xi(t)dt$$

$$\leq \limsup_{\alpha\to\infty}[\beta d^*\big(\wp\mu_{n(k)},\wp\mu_{m(k)}\big)\int_0^{\Delta_1(\wp\mu_{n(k)},\wp\mu_{m(k)})}\xi(t)dt]$$

$$= \limsup_{\alpha\to\infty}[\beta d^*\big(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1}\big)]\limsup_{\alpha\to\infty}\int_0^{\Delta_1(\wp\mu_{n(k)},\wp\mu_{m(k)})}\xi(t)dt]$$

$$< \int_0^{\epsilon}\xi(t)dt,$$

which is absurd. Hence $\{v_n\}$ is a Cauchy sequence.

Since $\wp M$ is complete, so there exists a point p in $\wp M$ such that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\wp\mu_{n+1} = p = \lim_{n\to\infty}\mathbb{Q}\mu_n \tag{2.11}$$

Since $p \in \wp M$, so we can find q in M such that $\wp q = p$.Now, we claim that $\wp q = \mathbb{Q}q$.

Let, if possible $\wp q \neq \mathbb{Q}q$.

On putting, $\mu = \mu_{n+1}$, $v = q$ in (2.2), we have

$$\Delta_1(\wp\mu_{n+1},\wp q) = \max\{d^*(\wp\mu_{n+1},\wp q), d^*(\mathbb{Q}\mu_{n+1},\wp\mu_{n+1}), d^*(\mathbb{Q}q,\wp q),$$
$$\frac{d^*(\mathbb{Q}\mu_{n+1},\wp\mu_{n+1}).\,d^*(\mathbb{Q}q,\wp q)}{1+d^*(\mathbb{Q}\mu_{n+1},\mathbb{Q}q)},\frac{d^*(\mathbb{Q}\mu_{n+1},\wp\mu_{n+1}).\,d^*(\mathbb{Q}q,\ \wp q)}{1+d^*\ (\wp\mu_{n+1},\wp q)}\bigg\}.$$

Taking limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\Delta_1(\wp\mu_{n+1},\wp q) = max\{d^*(\wp q,\wp\ q), d^*(\wp q,\wp q), d^*(\mathbb{Q}q,\wp q),$$
$$\frac{d^*(\wp q,\wp q).\,d^*(\mathbb{Q}q,\wp q)}{1+d^*(\wp q,\mathbb{Q}q)},\frac{d^*(\wp q,\wp q).\,d^*(\mathbb{Q}q,\wp q)}{1+d^*(\wp q,\wp q)}\bigg\}$$
$$= d^*(\mathbb{Q}q,\wp q).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}q,\wp q)}\xi(t)dt$$

$$= \limsup_{n\to\infty}\int_0^{d^*(\mathbb{Q}q,\mathbb{Q}\mu_{n+1})}\xi(t)dt$$

$$\leq \limsup_{n\to\infty}[\beta d^*(\wp q,\wp\mu_{n+1})\int_0^{\Delta_1(\wp q,\wp\mu_{n+1})}\xi(t)dt]$$

$$= \limsup_{\alpha\to\infty}[\beta d^*(\wp q, \wp\mu_{n+1})]\limsup_{\alpha\to\infty} \int_0^{(\wp q,\wp\mu_{n+1})} \xi(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}q,\wp q)} \xi(t)dt,$$

which is absurd.

Hence, $d^*(\mathbb{Q}q, \wp q) = 0$.Which implies that

$$\wp q = \mathbb{Q}q = p.\text{(2.12)}$$

Therefore, q is a coincidence point of $\wp$ and $\mathbb{Q}$.

Now, we show that there exists a common fixed point of$\wp$ and $\mathbb{Q}$. Since $\wp$ and $\mathbb{Q}$ are weakly compatible, by (2.12), we have

$$\mathbb{Q}\wp q = \wp\mathbb{Q}q \text{ and } \mathbb{Q}p = \mathbb{Q}\wp q = \wp\mathbb{Q}q = \wp p.$$

Now, consider

$$\Delta_1(\wp q, \wp p) = max\{d^*(\wp q, \wp p), d^*(\mathbb{Q}q, \wp q), d^*(\mathbb{Q}p, \wp p),$$

$$\frac{d^*(\mathbb{Q}q, \wp q). d^*(\mathbb{Q}p, \wp p)}{1 + d^*(\mathbb{Q}q, \mathbb{Q}p)}, \frac{d^*(\mathbb{Q}q, \wp q). d^*(\mathbb{Q}p, \wp p)}{1 + d^*(\wp q, \wp p)}\Bigg\}$$

$$= max\{d^*(p, \mathbb{Q}p),\ 0 , 0 ,\ 0 , 0 \}$$

$$= d^*(p, \mathbb{Q}p).$$

Now,

$$0 < \int_0^{d^*(p,\mathbb{Q}p)} \xi(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}q,\mathbb{Q}p)} \xi(t)dt$$

$$\leq [\beta d^*(\wp q, \wp p) \int_0^{\Delta_1(\wp q,\wp p)} \xi(t)dt]$$

$$< \int_0^{d^*(p,\mathbb{Q}q)} \xi(t)dt,$$

which is again a contradiction. Hence $\wp p = \mathbb{Q}p = p$.

This implies p is common fixed point of $\wp$ and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then

from (2.2), we have

$$\Delta_1(\wp r, \wp s) = max\{d^*(\wp r, \wp s), d^*(\mathbb{Q}s, \wp s), d^*(\mathbb{Q}r, \wp r), \frac{d^*(\mathbb{Q}s, \wp s). d^*(\mathbb{Q}r, \wp r)}{1 + d^*(\mathbb{Q}p, \mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s, \wp s). d^*(\mathbb{Q}r, \wp r)}{1 + d^*(\wp s, \wp r)}\bigg\}$$

$$= d^*(r, s).$$

And

$$0 < \int_0^{d^*(r,s)} \xi(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \xi(t)dt$$

$$\leq \beta d^*(\wp r, \wp s) \int_0^{\Delta_1(\wp r,\wp s)} \xi(t)dt$$

$$= \beta d^*(r, s) \int_0^{d^*(r,s)} \xi(t)dt$$

$$< \int_0^{d^*(r,s)} \xi(t)dt,$$

which is a contradiction, hence $r = s$.

This proves the uniqueness of the common fixed point. Hence completes the proof of the theorem.

Corollary 2.2. Let Tbe self - map on a metric space (M, d^*) satisfying the followings:

There exists a continuous mapping $\xi: [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ and $\xi(\alpha) > \alpha$ for all

$\alpha > 0$ such that:

$$\int_0^{d^*(T\mu,T\upsilon)} \xi(t)dt \leq \beta\big(d^*(\mu, \upsilon)\big) \int_0^{\Delta_1(\mu,\upsilon)} \xi(t)dt,$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, υ in M.

where

$\Delta_1(\mu, v) = max\left\{d^*(\mu, v), d^*(T\mu, T\mu), d^*(Tv, v), \frac{d^*(T\mu,\mu).d^*(Tv,v)}{1+d^*(T\mu,Tv)}, \frac{d^*(T\mu,\mu).d^*(Tv,v)}{1+d^*(\mu,v)}\right\}.$

If TM is complete, then T has a unique fixed point.

Theorem 2.3. Let $\wp$ *and* $\mathbb{Q}$ be self-maps of a metric space (M, d^*)satisfying (2.2) and the followings:

(2.13)$\wp$ *and* $\mathbb{Q}$ are weakly compatible,

(2.14)$\wp$ and $\mathbb{Q}$ satisfy the E.A. properly.

If either$\wp M$ or $\mathbb{Q}M$ is a complete subspace of M, then$\wp$ and $\mathbb{Q}$have a unique common fixed point in M.

Proof. Since $\wp$ and $\mathbb{Q}$ satisfy the E.A. property, there exists a sequence $\{\mu_n\}$ in M such that

$$\lim_{n\to\infty} \wp\, \mu_n = \lim_{n\to\infty} \mathbb{Q}\, \mu_n = \mu \text{ for some } \mu \text{ in } M. \quad (2.15)$$

Now, suppose that $\wp M$ is complete subspace of M. Then, there exists z in M such that $\mu = \wp z$.

Subsequently, we have

$$\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \mathbb{Q}\mu_n = \mu = \wp z. \quad (2.16)$$

Now, we show that $\wp z = \mathbb{Q}z$.

$$\lim_{n\to\infty} \Delta_1(\wp\mu_n, \wp z) = \lim_{n\to\infty} max\{d^*(\wp\mu_n, \wp z), d^*(\mathbb{Q}\mu_n, \wp\mu_n), d^*(\mathbb{Q}z, \wp z),$$

$$\left.\frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n). d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\mathbb{Q}\mu_n, \mathbb{Q}z)}, \frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n). d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp\mu_n, \wp z)}\right\}$$

$$= max\{d^*(\wp z, \wp z), d^*(\wp z, \wp z), d^*(\mathbb{Q}z, \wp z),$$

$$\left.\frac{d^*(\wp z, \wp z). d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp z, \mathbb{Q}z)}, \frac{d^*(\wp z, \wp z). d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp z, \wp z)}\right\}$$

$$= \max\{0, 0, d^*(\wp z, \mathbb{Q}z), 0, 0\}.$$

$$= d^*(\mathbb{Q}z, \wp z).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}z, \wp z)} \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}z, \mathbb{Q}\mu_n)} \xi(t)dt$$

$$\leq \limsup_{n\to\infty}[\beta d^*(\wp z, \wp\mu_n) \int_0^{\Delta_1(\wp z, \wp\mu_n)} \xi(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}z, \wp z)} \xi(t)dt,$$

which is absurd.

Hence, $d^*(\mathbb{Q}z, \wp z) = 0$. Which implies that

$$\wp z = \mathbb{Q}z$$

Since $\wp$and $\mathbb{Q}$ are weakly compatible. Therefore, $\mathbb{Q}\wp z = \wp\mathbb{Q}z$, implies that,

$$\wp\wp z = \wp\mathbb{Q}z = \mathbb{Q}\wp Z = \mathbb{Q}\mathbb{Q}z.$$

Now, we claim that $\mathbb{Q}z$ is the common fixed point of $\wp$ and$\mathbb{Q}$.

$$\lim_{n\to\infty} \Delta_1(\wp z, \wp\mathbb{Q}z) = max\{d^*(\wp z, \wp\mathbb{Q}z), d^*(\mathbb{Q}z, \wp z), d^*(\mathbb{Q}\mathbb{Q}z, \wp\mathbb{Q}z),$$

$$\left.\frac{d^*(\mathbb{Q}z, \wp z). d^*(\mathbb{Q}\mathbb{Q}z, \wp\mathbb{Q}z)}{1 + d^*(\mathbb{Q}z, \mathbb{Q}\mathbb{Q}z)}, \frac{d^*(\mathbb{Q}z, \wp z). d^*(\mathbb{Q}\mathbb{Q}z, \wp\mathbb{Q}z)}{1 + d^*(\wp z, \wp\mathbb{Q}z)}\right\}.$$

$$= max\{d^*(\mathbb{Q}z, \mathbb{Q}\mathbb{Q}z), 0 , 0, 0 , 0\}.$$

$$= d^*(\mathbb{Q}z, \mathbb{Q}\mathbb{Q}z).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}z, \mathbb{Q}\mathbb{Q}z)} \xi(t)dt$$

$$\leq [\beta d^*(\wp z, \wp\mathbb{Q}z) \int_0^{\Delta_1(\wp z, \wp\mathbb{Q}z)} \xi(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}z, \mathbb{Q}\mathbb{Q}z)} \xi(t)dt,$$

which is absurd.Which implies that

$$\mathbb{Q}z = \mathbb{Q}\mathbb{Q}z = \wp\mathbb{Q}\boldsymbol{z}.$$

Hence $\mathbb{Q}z$ is common fixed point of $\wp$ and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then

from (2.2), we have

$$\Delta_1(\wp r, \wp s) = max\{d^*(\wp r, \wp s), d^*(\mathbb{Q}s, \wp s), d^*(\mathbb{Q}r, \wp r),$$
$$\left.\frac{d^*(\mathbb{Q}s, \wp s).\, d^*(\mathbb{Q}r, \wp r)}{1 + d^*(\mathbb{Q}p, \mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s, \wp s).\, d^*(\mathbb{Q}r, \wp r)}{1 + d^*(\wp s, \wp r)}\right\}$$
$$= d^*(r, s).$$

And

$$0 < \int_0^{d^*(r,s)} \xi(t)dt$$
$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \xi(t)dt$$
$$\leq \beta d^*(\wp r, \wp s) \int_0^{\Delta_1(\wp r,\wp s)} \xi(t)dt$$
$$= \beta d^*(r, s) \int_0^{d^*(r,s)} \xi(t)dt$$
$$< \int_0^{d^*(r,s)} \xi(t)dt,$$

which is a contradiction, hence $r = s$.

This proves the uniqueness of the common fixed point.

Hence completes the proof of the Theorem 2.3.

Theorem2.4. Let (M, d^*)be metric space, let $\wp$ and $\mathbb{Q}$ be self maps on$M$ satisfying (2.2), (2.13) and if$\wp$ and $\mathbb{Q}$ satisfy $(CLR_{\wp})$ property.

Then $\wp$ and $\mathbb{Q}$ have a unique common fixed point in M.

Proof: Since $\wp$ and $\mathbb{Q}$ satisfy the $(CLR_{\wp})$ property, there exits a sequence $\{\mu_n\}$ in M such that

$$\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \mathbb{Q}\mu_n = \wp\mu, \qquad (2.17)$$

for some μ in M. First we prove that $\wp\mu = \mathbb{Q}\mu$. Let if possible $\wp\mu \neq \mathbb{Q}\mu$.

On putting $\mu = \mu_n$ and $\nu = \mu$ in (2.2), we have

$$\Delta_1(\wp\mu_n, \wp\mu) = max\{d^*(\wp\mu_n, \wp\mu). d^*(\mathbb{Q}c_n, \wp c_n), d^*(\mathbb{Q}c, \wp c), \frac{d^*(\mathbb{Q}c_n, \wp c_n). d^*(\mathbb{Q}c, \wp c)}{1 + d^*(\mathbb{Q}c_n, \mathbb{Q}c)}, \frac{d^*(\mathbb{Q}c_n, \wp c_n). d^*(\mathbb{Q}c, \wp c)}{1 + d^*(\wp\mu_n, \wp\mu)}\Big\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \Delta_1(\wp\mu_n, \wp\mu) = \lim_{n\to\infty} max\{d^*(\wp\mu, \wp\mu), d^*(\wp\mu, \wp\mu), d^*(\mathbb{Q}\mu, \wp\mu), \frac{d^*(\wp\mu, \wp\mu). d^*(\mathbb{Q}\mu, \wp\mu)}{1 + d^*(\wp\mu, \mathbb{Q}\mu)}, \frac{d^*(\wp\mu, \wp\mu). d^*(\mathbb{Q}\mu, \wp\mu)}{1 + d^*(\wp\mu, \wp\mu)}\Big\}$$

$$= max\{d^*(\wp\mu, \wp\mu), d^*(\wp\mu, \wp\mu), d^*(\mathbb{Q}\mu, \wp\mu), \}.$$

$$= \max\{0, 0, d^*(\wp\mu, \mathbb{Q}\mu), 0, 0\}.$$

$$= d^*(\mathbb{Q}\mu, \wp\mu)$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}\mu, \wp\mu)} \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}\mu, \mathbb{Q}\mu_n)} \xi(t)dt$$

$$\leq \limsup_{n\to\infty} [\beta d^*(\wp\mu, \wp\mu_n) \int_0^{\Delta_1(\wp\mu, \wp\mu_n)} \xi(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}\mu, \wp\mu)} \xi(t)dt,$$

This is possible only when$d^*(\wp\mu, \mathbb{Q}\mu) = 0$. Hence $\wp\mu = \mathbb{Q}\mu$.

Now, let $d = \wp\mu = \mathbb{Q}\mu$.Since $\wp\mathbb{Q}\mu = \mathbb{Q}\wp\mu$,implies that,

$$\wp d = \wp\mathbb{Q}\mu = \mathbb{Q}\wp\mu = \mathbb{Q}d.$$

Now, we claim that $\mathbb{Q}d = d$.

$$\Delta_1(\wp\mu, \wp d) = max\{d^*(\wp\mu, \wp d), d^*(\mathbb{Q}\mu, \wp\mu), d^*(\mathbb{Q}d, \wp d), \frac{d^*(\mathbb{Q}\mu, \wp\mu). d^*(\mathbb{Q}d, \wp d)}{1 + d^*(\mathbb{Q}\mu, \mathbb{Q}d)}, \frac{d^*(\mathbb{Q}\mu, \wp\mu). d^*(\mathbb{Q}d, \wp d)}{1 + d^*(\wp\mu, \wp d)}\Big\}$$

$$= max\{d^*(d, \mathbb{Q}d), 0, 0, 0, 0\}$$

$$= d^*(d, \mathbb{Q}d).$$

And

$$0 < \int_0^{d^*(d,\mathbb{Q}d)} \xi(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}\mu,\mathbb{Q}d)} \xi(t)dt$$

$$\leq [\beta d^*(\wp\mu,\wp d) \int_0^{\Delta_1(\wp\mu,\wp d)} \xi(t)dt]$$

$$< \int_0^{d^*(d,\mathbb{Q}d)} \xi(t)dt,$$

This is possible only when $\mathbb{Q}d = d$. Hence $\wp d = \mathbb{Q}d = d$. So, d is the common fixed point of $\wp$and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then

from (2.2), we have

$$\Delta_1(\wp r,\wp s) = max\{d^*(\wp r,\wp s), d^*(\mathbb{Q}s,\wp s), d^*(\mathbb{Q}r,\wp r),$$
$$\frac{d^*(\mathbb{Q}s,\wp s).\,d^*(\mathbb{Q}r,\wp r)}{1 + d^*(\mathbb{Q}p,\mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s,\wp s).\,d^*(\mathbb{Q}r,\wp r)}{1 + d^*(\wp s,\wp r)}\bigg\}$$

$$= d^*(r,s).$$

And

$$0 < \int_0^{d^*(r,s)} \xi(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \xi(t)dt$$

$$\leq \beta d^*(\wp r,\wp s) \int_0^{\Delta_1(\wp r,\wp s)} \xi(t)dt$$

$$= \beta d^*(r,s) \int_0^{d^*(r,s)} \xi(t)dt$$

$$< \int_0^{d^*(r,s)} \xi(t)dt,$$

which is a contradiction, hence $r = s$.

This proves the uniqueness of the common fixed point.

Hence completes the proof of the Theorem 2.4.

Example 2.5let $M = R^+$ be equipped with the metric space and $d^*(c,d) = |c - d|$ for all $c, d \in M$. Define $\wp, \mathbb{Q} : M \to M$ by

$$\wp c = c$$

$$\mathbb{Q}c = \frac{c}{2} + \frac{1}{2}.$$

Clearly, $\mathbb{Q}M \subset \wp M$.

Let $\{\mu_n\}$be a sequence in M such that $\{\mu_n\} = \frac{n+1}{n}$ for each n. Also, let $\xi: [0, \infty) \to [0, \infty)$ be defined by:

$$\xi(t) = 2t$$

Clearly, $\wp(1) = \mathbb{Q}(1) = 1$ and $\wp\mathbb{Q}(1) = \mathbb{Q}\wp(1) = 1$, this shows that $\wp$ and $\mathbb{Q}$ are weakly compatible and let $c, d \in M$.

Now, we shall prove the inequality (2.2) of the theorem.

$$d^*(\mathbb{Q}c, \mathbb{Q}d) = \frac{1}{2}|c - d|$$

Clearly,

$$\int_0^{d^*(\mathbb{Q}c,\mathbb{Q}d)} \xi(t)dt = \frac{1}{4}(c - d)^2 \tag{2.18}$$

Now,

$$\Delta_1(\wp c, \wp d) \geq d^*(\wp c, \wp d)$$

And

$$d^*(\wp c, \wp d) = |c - d|$$

This implies

$$\int_0^{d^*(\wp c,\wp d)} \xi(t)dt = (c - d)^2 \tag{2.19}$$

From (2.16) and (2.17), we can conclude that

$$\int_0^{d^*(\mathbb{Q}c,\mathbb{Q}d)} \xi(t)dt \leq \beta\big(d^*(\wp c, \wp d)\big) \int_0^{\Delta_1(\wp c,\wp d)} \xi(t)dt,$$

since $\xi(t) < 1$ for all t.

Hence (2.2) holds.

Now, $\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \mathbb{Q}\mu_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right) + \frac{1}{2} = 1$, where $1 \in M$.

This implies $\wp$ and $\mathbb{Q}$ satisfies E.A. property.

Also, $\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \mathbb{Q}\mu_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right) + \frac{1}{2} = 1 = \wp(1)$, where $1 \in M$.

This implies $\wp$ and $\mathbb{Q}$satisfies $(CLR_{\wp})$ property.

Hence all the properties of theorem 2.1, 2.3 and 2.4 are satisfied. Here 1 is the common fixed point of $\wp$ and $\mathbb{Q}$.

REFERENCES

[1] Aamri M., Moutawakil D.EI., some common fixed point theorems under strict constructive conditions, J. Math. Anal. Appl. **27**(1)(2002), 181-188.

[2] Branciari A., A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29(2002), 531-536.

[3] Dahiya A., Rani A., Jyoti K., Common fixed point for Generalized –(ψ, α, β)-weakly contractive mappings in dislocated metric spaces, Global journal of pure and applied mathematics. **13**(7) (2017), 3067-3081.

[4] Feng C., Liu N., Shim S. H., Jung C. Y., on common fixed point theorems of weakly compatible mappings satisfying contractive inequalities of integral type, Nonlinear functional analysis and Applications **26**(2)(2011), 393-409.

[5] Jungck G., common fixed points for non - continuous non - self - maps on non - metric spaces. Far East J. Math.Sci.**4**(2) (1996), 199-212.

[6] Kumar M., Kumar P. and Kumar S., Some common fixed point theorems using (CLRg)-property in cone metric spaces, Advances in Fixed Point Theory, **2**(3)(2012), 340–356.

[7] Kumar M., Kumar P. and Kumar S., Common fixed point for weakly contractive maps, Journal of Analysis and Number theory, **3**(1)(2015), 47-54.

[8] Liu Z., Li X., Kang S.M. and Cho S.Y., fixed point theorems for mappings satisfying contractive conditions of integral type and applications, Fixed point theory Appl., **2011**(2011), paper no.64, 18 pages.

[9] Sintunavarat W. and Kumam P, Common Fixed Point Theorems for a pair of weakly compatible mappings in Fuzzy metric spaces, Hindawi Publishing corporation, journal of applied mathematics, vol. 2011, article ID-637958.