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An Extension of a Variational Inequality in the Simader-Sohr Theorem to a Variable Exponent Sobolev Space and Applications: The Neumann case

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Abstract

In this paper, we shall extend a fundamental variational inequality which is developed by Simader-Sohr to a variable exponent Sobolev space. The inequality is very useful for the existence theory to the Poisson equation with the Neumann boundary conditions in $L^{p(\cdot)}$-framework, where $L^{p(\cdot)}$ denotes a variable exponent Lebesgue space. Furthermore, we can also derive the Helmholtz decomposition theorem in a variable exponent Lebesgue space.

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1. INTRODUCTION

In Simader and Sohr [18], the authors derived a variational inequality of a quadratic form. More precisely, let Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a C^1-boundary $\partial\Omega$ and $1 < p < \infty$. They proved that there exists a positive constant $C = C(p, \Omega) > 0$ such that

$$\|\nabla u\|_{L^p(\Omega)} \leq C \sup_{\substack{v \in W^{1,p'}(\Omega) \\ \nabla v \neq 0}} \frac{|\langle \nabla u, \nabla v \rangle|}{\|\nabla v\|_{L^{p'}(\Omega)}} \text{ for all } u \in W^{1,p}(\Omega), \tag{1.1}$$

where $\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx$ and $\boldsymbol{\nabla}$ denotes the gradient operator. They also considered the case where Ω is an exterior domain and got a variational inequality like as in (1.1).

This inequality has many applications. For example, let $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, then it follows from (1.1) that the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{v} & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = \boldsymbol{v} \cdot \boldsymbol{n} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ denotes the outer unit normal vector to $\partial\Omega$, has a unique (up to an additive constant) solution in a generalized sense. The equation (1.2) plays an essential role for the Helmholtz-Weyl decomposition for vector fields in $\boldsymbol{L}^p(\Omega)$ (cf. [18] and Kozono and Yanagisawa [13]). As an application of (1.1), we can show the Helmholtz decomposition for $\boldsymbol{L}^p(\Omega)$. It is basic for the treatment of Navier-Stokes equation, for example, see Fujiwara and Morimoto [9], Miyakawa [16].

In this paper, we attempt to derive an improvement of the above variational inequality (1.1) to a variable exponent Sobolev space (Theorem 3.3). We restrict ourselves to the case where variational inequality in a bounded domain. Though we follow the arguments of [18], we have to proceed the analysis very carefully. The result brings about the existence theory of weak solutions to the Neumann problem for the Laplacian in the variable exponent Sobolev space, that is, for given functions $f \in (W^{1,p(\cdot)}(\Omega))'$ and $g \in (\mathrm{Tr}(W^{1,p'(\cdot)}\Omega)))'$, where $W^{1,p(\cdot)}(\Omega)$ is a variable exponent Sobolev space, $(W^{1,p(\cdot)}(\Omega))'$ is the dual space of $W^{1,p(\cdot)}(\Omega)$ and $\mathrm{Tr}(W^{1,p(\cdot)}(\Omega))$ denotes the trace space, satisfying a compatibility condition, the following problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial\Omega \end{cases}$$

has a unique (up to an additive constant) weak solution. According to our best knowledge, the result for the Neumann problem in a variable exponent Sobolev space is the most general and new. Furthermore, we show that the Helmholtz decomposition in a variable exponent $\boldsymbol{L}^{p(\cdot)}$-space which also seems to be new.

The study of differential equations with $p(x)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [23]), in electrorheological fluids (Diening [4], Halsey [11], Mihăilescu and Rădulescu [14], Růžička [17]).

For the Dirichlet case of the variational inequality, we will give a result in the future work (cf. Simader [19] for the case $p(\cdot) = p = \mathrm{const.}$).

The paper is organized as follows. In section 2, we give some preliminaries on variable exponent Lebesgue-Sobolev spaces. In section 3, we give main theorems (Theorem 3.3) which is an extension of variational inequality of type (1.1) to a variable exponent Sobolev space. Section 4 is devoted to a proof of main theorem. In section 5, we consider the existence theory of weak solutions for the Poisson equation with the Neumann boundary conditions. Finally, section 6 is devoted to a proof for the Helmholtz decomposition in a variable exponent $\boldsymbol{L}^{p(\cdot)}$-space.

2. PRELIMINARIES

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_d)$ and $\boldsymbol{b} = (b_1, \dots, b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^d a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Occasionally, we also use the same character for matrix values functions. Moreover, for the dual space B' of B (resp. $\boldsymbol{B}'$ of $\boldsymbol{B}$), we denote the duality bracket between B' and B (resp. $\boldsymbol{B}'$ and $\boldsymbol{B}$) by $\langle \cdot, \cdot \rangle_{B',B}$ (resp. $\langle \cdot, \cdot \rangle_{\boldsymbol{B}',\boldsymbol{B}}$).

In this section, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. See Diening et al. [5], Fan and Zhao [8], Fan and Zhang [6], Kováčik and Rácosník [12] and references therein for more detail. Let G be a (Lebesgue) measurable subset of $\mathbb{R}^d$ with the measure $|G| > 0$. Then we define a set of variable exponents by $\mathcal{P}(G) = \{p; G \to [1, \infty); p \text{ is measurable in } G\}$ and for $p \in \mathcal{P}(G)$, define

$$p^- = \operatorname*{ess\,inf}_{x \in G} p(x) \text{ and } p^+ = \operatorname*{ess\,sup}_{x \in G} p(x)$$

and

$$\mathcal{P}_+(G) = \{p \in \mathcal{P}(G); 1 < p^- \leq p^+ < \infty\}.$$

For any measurable function u on G and $p \in \mathcal{P}(G)$, a modular $\rho_{p(\cdot),G}$ is defined by

$$\rho_{p(\cdot),G}(u) = \int_G |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(G) = \left\{u; u \text{ is a measurable function on } G \text{ satisfying } \rho_{p(\cdot),G}(u) < \infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0; \rho_{p(\cdot),G}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then $L^{p(\cdot)}(G)$ is a Banach space.

The following proposition is well known (see Fan et al. [7], Wei and Chen [20], [8], Zhao et al. [22], Yücedağ [21]).

Proposition 2.1. *Let* $p \in \mathcal{P}_+(G)$ *and let* $u, u_n \in L^{p(\cdot)}(G)$ $(n = 1, 2, \ldots)$. *Then we have*

(i) $\|u\|_{L^{p(\cdot)}(G)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot),G}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(G)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(G)} \le \rho_{p(\cdot),G}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(G)}$.

(iii) $\|u\|_{L^{p(\cdot)}(G)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(G)} \le \rho_{p(\cdot),G}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(G)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(G)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot),G}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(G)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot),G}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let* $p \in \mathcal{P}_+(G)$. *For any* $u \in L^{p(\cdot)}(G)$ *and* $v \in L^{p'(\cdot)}(G)$, *we have*

$$\int_G |uv|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(G)} \|v\|_{L^{p'(\cdot)}(G)} \le 2\|u\|_{L^{p(\cdot)}(G)} \|v\|_{L^{p'(\cdot)}(G)},$$

where $p'(\cdot)$ *is the conjugate exponent of* $p(\cdot)$, *that is,* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

When G is a domain (open and connected subset) of $\mathbb{R}^d$ and $p \in \mathcal{P}_+(G)$, we can define a Sobolev space, for an integer $m \ge 0$,

$$W^{m,p(\cdot)}(G) = \{u \in L^{p(\cdot)}(G); \partial^\alpha u \in L^{p(\cdot)}(G) \text{ for } |\alpha| \le m\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(G)} = \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L^{p(\cdot)}(G)}.$$

Of course, $W^{0,p(\cdot)}(G) = L^{p(\cdot)}(G)$. The local Sobolev space is defined by

$$W^{m,p(\cdot)}_{\text{loc}}(G) = \{u;\ \text{for all open subset } U \Subset G, u \in W^{1,p(\cdot)}(U)\},$$

where $U \Subset G$ means that the closure $\overline{U}$ of U is compact and $\overline{U} \subset G$.

For $p \in \mathcal{P}_+(G)$, define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases}$$

Proposition 2.3. *Let $p \in \mathcal{P}_+(G)$ and $m \geq 0$ be an integer. Then we can show the following properties.*

(i) The space $W^{m,p(\cdot)}(G)$ is a separable and reflexive Banach space.

(ii) Let $G = \Omega$ be a bounded domain of $\mathbb{R}^d$. If $q(\cdot) \in \mathcal{P}_+(\Omega)$ satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.

(iii) Let $G = \Omega$ be a bounded domain of $\mathbb{R}^d$. If $p, q \in \mathcal{P}_+(\Omega) \cap C(\overline{\Omega})$ satisfies that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

Let G be a domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary ∂G and $p \in \mathcal{P}_+(G)$. Since $W^{1,p(\cdot)}(G) \subset W^{1,1}_{\mathrm{loc}}(G)$, the trace $u\big|_{\partial G}$ to ∂G of any function u in $W^{1,p(\cdot)}(G)$ is well defined as a function in $L^1_{\mathrm{loc}}(\partial\Omega)$. We define

$$\mathrm{Tr}(W^{1,p(\cdot)}(G)) = \{f; f \text{ is the trace to } \partial G \text{ of a function } F \in W^{1,p(\cdot)}(G)\}$$

equipped with the norm

$$\|f\|_{\mathrm{Tr}(W^{1,p(\cdot)}(G))} = \inf\{\|F\|_{W^{1,p(\cdot)}(G)}; F \in W^{1,p(\cdot)}(G) \text{ satisfying } F\big|_{\partial G} = f \text{ on } \partial G\}$$

for $f \in \mathrm{Tr}(W^{1,p(\cdot)}(G))$. Then $\mathrm{Tr}(W^{1,p(\cdot)}(G))$ is a Banach space. More precisely, see [5, Chapter 12].

For general measurable subset G of $\mathbb{R}^d$, we say that $p \in \mathcal{P}^{\log}(G)$ if $p \in \mathcal{P}_+(G)$ and p has the globally $\log$-Hölder continuity in G and globally $\log$-Hölder decay condition, that is, $p : G \to \mathbb{R}$ satisfies that there exist a constant $C_{\log}(p) > 0$ and $p_\infty \in \mathbb{R}$ such that the following inequlities hold:

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \text{ for all } x, y \in G,$$

and

$$|p(x) - p_\infty| \leq \frac{C_{\log}(p)}{\log(e + |x|)} \text{ for all } x \in G,$$

respectively.

We also write $\mathcal{P}^{\log}_+(G) = \mathcal{P}^{\log}(G) \cap \mathcal{P}_+(G)$.

Proposition 2.4. *If G is a domain of $\mathbb{R}^d$ and $p \in \mathcal{P}^{\log}_+(G)$, then it has an extension $q \in \mathcal{P}^{\log}_+(\mathbb{R}^d)$ with $C_{\log}(q) = C_{\log}(p)$, $q^- = p^-$ and $q^+ = p^+$. If G is unbounded, then additionally $q_\infty = p_\infty$.*

For the proof, see [5, Proposition 4.1.7].

Proposition 2.5. *If* $p \in \mathcal{P}_+^{\log}(\overline{G})$*, then* $\mathcal{D}(G) := C_0^\infty(G)$ *is dense in* $W_0^{1,p(\cdot)}(G) := \{u \in W^{1,p(\cdot)}(G); u\big|_{\partial G} = 0\}$.

For the proof, see [5, Corollary 11.2.4].

Frequently we use the following Poincaré inequality later.

Theorem 2.6. (i) *If* Ω *is a bounded domain of* $\mathbb{R}^d$ *and* $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$*, then there exists a constant* c *depending only on* d *and* $C_{\log}(p)$ *such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c \, \mathrm{diam}(\Omega) \|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \textit{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $\mathrm{diam}(\Omega)$ *denotes the diameter of* Ω.

(ii) *If* Ω *is a bounded domain of* $\mathbb{R}^d$ *with a Lipschitz-continuous boundary* $\partial\Omega$ *and* $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$*, then there exists a constant* c *depending only on* d *and* $C_{\log}(p)$ *such that*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c \, \mathrm{diam}(\Omega) \|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \textit{ for all } u \in W^{1,p(\cdot)}(\Omega),$$

where $\langle u \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega u dx$.

For the proof, see [5, Theorem 8.2.4].

3. THE SPACE $\boldsymbol{E}^{P(\cdot)}(G)$ AND A MAIN THEOREM

This section consists of two subsections. In subsection 3.1, we define a closed subspace $\boldsymbol{E}^{p(\cdot)}(G)$ of $\boldsymbol{L}^{p(\cdot)}(G)$ when G is a domain (not necessarily bounded) of $\mathbb{R}^d$ and $p \in \mathcal{P}_+^{\log}(G)$, and then we consult some properties of $\boldsymbol{E}^{p(\cdot)}(G)$. In subsection 3.2, we state a main theorem in the case where $G = \Omega$ is a bounded domain with a C^1-boundary.

3.1. The space $\boldsymbol{E}^{p(\cdot)}(G)$.

Let G be a domain (not necessarily bounded) of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary ∂G and $p \in \mathcal{P}_+^{\log}(\overline{G})$. Define a space

$$\boldsymbol{E}^{p(\cdot)}(G) = \{\nabla v; v \in L_{\mathrm{loc}}^{p(\cdot)}(\overline{G}), \nabla v \in \boldsymbol{L}^{p(\cdot)}(G)\}$$

equipped with the norm $\|\nabla v\|_{\boldsymbol{E}^{p(\cdot)}(G)} = \|\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(G)}$. Here

$$L_{\mathrm{loc}}^{p(\cdot)}(\overline{G}) = \{v; v\big|_{G \cap B} \in L^{p(\cdot)}(G \cap B) \text{ for each ball } B \text{ with } G \cap B \neq \emptyset\}.$$

Proposition 3.1. *Let G be a domain of $\mathbb{R}^d$ $(d \geq 2)$ with a boundary ∂G and let $p \in \mathcal{P}_+^{\log}(\overline{G})$. If $u_j \in W^{1,p(\cdot)}_{\mathrm{loc}}(G)$ $(j = 1, 2, \dots)$ such that $\{\nabla u_j\}_{j=1}^\infty$ is a Cauchy sequence in $\boldsymbol{L}^{p(\cdot)}_{\mathrm{loc}}(G)$, then there exist a sequence $\{c_j\}_{j=1}^\infty \subset \mathbb{R}$ and $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(G)$ such that $u_j - c_j \to u$ in $W^{1,p(\cdot)}_{\mathrm{loc}}(G)$ as $j \to \infty$.*

Proof. Step 1. For $x \in G$, define

$$d_x = \begin{cases} \frac{1}{2}\mathrm{dist}(x, \partial G) & \text{if } \partial G \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

and put $B(x) = \{y \in \mathbb{R}^d; |y - x| < d_x\}$. Then by the Poincaré inequality (Theorem 2.6 (ii)), there exists a constant $C_x = C(d, C_{\log}(p), |B(x)|) > 0$ such that

$$\left\| f - \frac{1}{|B(x)|}\int_{B(x)} f(y)dy \right\|_{L^{p(\cdot)}(B(x)} \leq C_x \|\nabla f\|_{\boldsymbol{L}^{p(\cdot)}(B(x))} \tag{3.1}$$

for all $f \in W^{1,p(\cdot)}(B(x))$. Assume that $u_j \in W^{1,p(\cdot)}_{\mathrm{loc}}(G)$ $(j = 1, 2, \dots)$ such that $\{\nabla u_j\}_{j=1}^\infty$ is a Cauchy sequence in $\boldsymbol{L}^{p(\cdot)}_{\mathrm{loc}}(G)$. If we put $c_j(x) = \frac{1}{|B(x)|}\int_{B(x)} u_j(y)dy$, then it follows from (3.1) that $\{u_j - c_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $L^{p(\cdot)}(B(x))$.

Step 2. Let $x, y \in G$ and $H = B(x) \cap B(y) \neq \emptyset$. Assume that $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \subset \mathbb{R}$ such that $\{u_j - a_j\}_{j=1}^\infty$ converges in $L^{p(\cdot)}(B(x))$ and $\{u_j - b_j\}_{j=1}^\infty$ converges in $L^{p(\cdot)}(B(y))$. According to the generalized Hölder inequality (Proposition 2.2), since

$$\begin{aligned} |(a_j - b_j) - (a_k - b_k)| &= \frac{1}{|H|}\int_H \{(a_j - b_j) - (a_k - b_k)\}dx \\ &\leq \frac{2}{|H|}\|(a_j - b_j) - (a_k - b_k)\|_{L^{p(\cdot)}(H)}\|1\|_{L^{p'(\cdot)}(H)} \\ &= C(H)\|\{(u_j - a_j) - (u_k - a_k)\} \\ &\qquad -\{(u_j - b_j) - (u_k - b_k)\}\|_{L^{p(\cdot)}(H)} \\ &\leq C(H)\|(u_j - a_j) - (u_k - a_k)\|_{L^{p(\cdot)}(H)} \\ &\qquad +\|(u_j - b_j) - (u_k - b_k)\|_{L^{p(\cdot)}(H)} \to 0 \end{aligned}$$

as $j, k \to \infty$ with some constant $C(H)$. Thus the sequence $\{a_j - b_j\}_{j=1}^\infty$ converges in $\mathbb{R}$. Moreover, we see that $u_j - a_j = u_j - b_j + (b_j - a_j)$ converges in $L^{p(\cdot)}(B(y))$.

Step 3. Choose an arbitrary but fixed $x_0 \in G$ and define $c_j = c_j(x_0)$. Put

$$M = \{x \in G; \{u_j - c_j\} \text{ is a Cauchy sequence in } L^{p(\cdot)}(B(x))\}.$$

Then $x_0 \in M$. M is an open subset of G. Indeed, let $x \in M$. Then for any $y \in B(x)$, since $B(x) \cap B(y) \neq \emptyset$, $\{u_j - c_j\}$ is a Cauchy sequence in $L^{p(\cdot)}(B(y))$ from Step 2, so $y \in M$, that is, $B(x) \subset M$. So M is open. M is a closed subset of G. Indeed, let $x_n \in M$ and $x_n \to z \in G$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(z)$ for all $n \geq n_0$. Hence $B(x_{n_0}) \cap B(z) \neq \emptyset$. From Step 2, $\{u_j - c_j\}$ is a Cauchy sequence in $L^{p(\cdot)}(B(z))$, so $z \in M$. Thsu M is a closed subset of G. Since G is connected, we conclude $M = G$.

Step 4. There exist open subsets $\{G_i\}_{i=1}^{\infty}$ such that $G_1 \Subset G_2 \Subset \cdots \Subset G$ and $G = \cup_{i=1}^{\infty} G_i$. Since $\{(u_j - c_j)\big|_{G_i}\}$ is a Cauchy sequence in $W^{1,p(\cdot)}(G_i)$, there exists $\widehat{u}_i \in W^{1,p(\cdot)}(G_i)$ such that $(u_j - c_j)\big|_{G_i} \to \widehat{u}_i$ in $W^{1,p(\cdot)}(G_i)$. Without loss of generality we may assume $\widehat{u}_i\big|_{G_k} = \widehat{u}_k$ for $i \geq k$. Define $u \in W^{1,p(\cdot)}_{\text{loc}}(G)$ by $u(x) = \widehat{u}_k(x)$ if $x \in G_k$. Then the definition is well defined. Then $u_j - c_j \to u$ in $W^{1,p(\cdot)}_{\text{loc}}(G)$ as $j \to \infty$. $\square$

Corollary 3.2. *Let G be a domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary and let $p \in \mathcal{P}^{\log}_{+}(\overline{G})$. Then the space $\boldsymbol{E}^{p(\cdot)}(G)$ is complete with respect to the norm $\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p(\cdot)}(G)} = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(G)}$, so $\boldsymbol{E}^{p(\cdot)}(G)$ is a closed subspace of $\boldsymbol{L}^{p(\cdot)}(G)$. Therefore, $\boldsymbol{E}^{p(\cdot)}(G)$ is a reflexive Banach space.*

Proof. Let $\{\boldsymbol{\nabla} u_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\boldsymbol{E}^{p(\cdot)}(G)$. For any $U \Subset G$, choose a ball B such that $U \subset B$. Then by definition, $u_j \in L^{p(\cdot)}(G \cap B) \subset L^{p(\cdot)}(U)$. Hence $u_j \in W^{1,p(\cdot)}_{\text{loc}}(G)$. By Proposition 3.1, there exists a sequence $\{c_j\} \subset \mathbb{R}$ and $u \in W^{1,p(\cdot)}_{\text{loc}}(G)$ such that $u_j - c_j \to u$ in $W^{1,p(\cdot)}_{\text{loc}}(G)$, so $\boldsymbol{\nabla} u_j \to \boldsymbol{\nabla} u$ in $\boldsymbol{L}^{p(\cdot)}_{\text{loc}}(G)$ as $j \to \infty$. On the other hand, since $\{\boldsymbol{\nabla} u_j\}$ is a Cauchy sequence in $\boldsymbol{L}^{p(\cdot)}(G)$, there exists $\boldsymbol{g} \in \boldsymbol{L}^{p(\cdot)}(G)$ such that $\boldsymbol{\nabla} u_j \to \boldsymbol{g}$ in $\boldsymbol{L}^{p(\cdot)}(G)$ as $j \to \infty$. So we have $\boldsymbol{\nabla} u = \boldsymbol{g} \in \boldsymbol{L}^{p(\cdot)}(G)$. Thus we see that $u \in W^{1,p(\cdot)}_{\text{loc}}(G) \subset L^1_{\text{loc}}(G)$ and $\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(G)$. For any ball B with $G \cap B \neq \emptyset$, there exists a bounded set C such that $B \subset C$ and $\partial(G \cap C)$ is Lipschitz-continuous. Therefore, $u \in L^1_{\text{loc}}(G \cap C)$ and $\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(G \cap C)$. By [5, Corollary 8.26], we have, for any $G' \Subset G$,

$$\begin{aligned} \|u - \langle u \rangle_{G'}\|_{L^{p(\cdot)}(G \cap B)} &\leq \|u - \langle u \rangle_{G'}\|_{L^{p(\cdot)}(G \cap C)} \\ &\leq C(d, C_{\log}(p), G \cap C)\|\boldsymbol{\nabla} u\|_{L^{p(\cdot)}(G \cap C)} \\ &\leq C(d, C_{\log}(p), G \cap C)\|\boldsymbol{\nabla} u\|_{L^{p(\cdot)}(G)} < \infty. \end{aligned}$$

So we get $u \in L^{p(\cdot)}(G \cap B)$. Therefore, $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(G)$. $\square$

3.2. A main theorem.

We are in a position to state a main theorem.

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a C^1-boundary $\partial\Omega$ and $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. Then there exists a constant $C_1 = C_1(p, d, \Omega) > 0$ such that*

$$\|\nabla u\|_{\boldsymbol{E}^{p(\cdot)}(\Omega)} \leq C_1 \sup_{\mathbf{0} \neq \nabla v \in \boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \nabla u, \nabla v \rangle_\Omega|}{\|\nabla v\|_{\boldsymbol{E}^{p'(\cdot)}(\Omega)}} \text{ for all } \nabla u \in \boldsymbol{E}^{p(\cdot)}(\Omega), \tag{3.2}$$

where

$$\langle \nabla u, \nabla v \rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v dx.$$

If Ω is bounded, then it clearly follows that $\boldsymbol{E}^{p(\cdot)}(\Omega) = \{\nabla v; v \in W^{1,p(\cdot)}(\Omega)\}$. Hence we have the following corollary.

Corollary 3.4. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a C^1-boundary $\partial\Omega$ and $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. Then there exists a constant $C_1 = C_1(p, d, \Omega) > 0$ such that*

$$\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq C_1 \sup_{\substack{v \in W^{1,p'(\cdot)}(\Omega) \\ \nabla v \neq \mathbf{0}}} \frac{|\langle \nabla u, \nabla v \rangle_\Omega|}{\|\nabla v\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}} \text{ for all } u \in W^{1,p(\cdot)}(\Omega). \tag{3.3}$$

Remark 3.5. *If Ω is a bounded domain, the authors of [18] showed the result in the case where $p(\cdot) = p$ is a constant such that $1 < p < \infty$.*

4. PROOF OF THEOREM 3.3.

In this section, we give a proof of Theorem 3.3. For the purpose, since we use the localization method, we devide this section into four subsections. In subsection 4.1, we consider the properties in the general domain G. In subsection 4.2, we treat the case $G = \mathbb{R}^d$. Subsection 4.3 devote to the case G is a half-space or a bended half-space. In the last subsection 4.4, we complete the proof of Theorem 3.3.

4.1. The case where G is a general (not necessarily bounded) domain.

Definition 4.1. *Let G be a domain of $\mathbb{R}^d$ with a C^1-boundary and $q \in \mathcal{P}^{\log}_+(\overline{G})$.*

(i) *We say that G has the property $Q_1(q)$ if there exists a constant $C_q = C(q, d, G) > 0$ such that*

$$\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)} \leq C_q \sup_{\mathbf{0} \neq \nabla v \in \boldsymbol{E}^{q'(\cdot)}(G)} \frac{|\langle \nabla u, \nabla v \rangle_G|}{\|\nabla v\|_{\boldsymbol{E}^{q'(\cdot)}(G)}} \text{ for all } \nabla u \in \boldsymbol{E}^{q(\cdot)}(G), \tag{4.1}$$

where $\langle \nabla u, \nabla v \rangle_G = \int_G \nabla u \cdot \nabla v dx$.

(ii) *We say that* G *has the property* $Q_2(q)$ *if we define a linear operator* $\sigma_q : \boldsymbol{E}^{q(\cdot)}(\Omega) \to (\boldsymbol{E}^{q'(\cdot)}(G))'$ *by*

$$\sigma_q(\nabla u)(\nabla\phi) = \langle \nabla u, \nabla\phi\rangle_G \text{ for } \nabla u \in \boldsymbol{E}^{q(\cdot)}(G) \text{ and } \nabla\phi \in \boldsymbol{E}^{q'(\cdot)}(G), \tag{4.2}$$

then σ_q *is topologically bijective and there exists a constant* $\widetilde{C_q} = \widetilde{C}(q, d, G) \geq 1$ *such that*

$$\widetilde{C_q}^{-1}\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)} \leq \|\sigma_q(\nabla u)\|_{(\boldsymbol{E}^{q'(\cdot)}(G))'} \leq \widetilde{C_q}\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)} \text{ for all } \nabla u \in \boldsymbol{E}^{q(\cdot)}(G). \tag{4.3}$$

Lemma 4.2. *Let* G *be a domain of* $\mathbb{R}^d$ *with a* C^1*-boundary and* $p \in \mathcal{P}_+^{\log}(\overline{G})$*. Then* G *has the property* $Q_1(q)$ *for* $q = p$ *and* $q = p'$ *if and only if* G *has the property* $Q_2(q)$ *for* $q = p$ *and* $q = p'$*.*

Proof. Suppose that G has the property $Q_1(q)$ for $q = p$ and $q = p'$. Let $\nabla u \in \boldsymbol{E}^{q(\cdot)}(G)$. Then it follows from (4.1) and the generalized Hölder inequality (Proposition 2.2) that

$$\begin{aligned}
C_q^{-1}\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)} &\leq \sup_{\boldsymbol{0}\neq\nabla\phi\in\boldsymbol{E}^{q'(\cdot)}(G)} \frac{|\langle\nabla u, \nabla\phi\rangle_G|}{\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)}} \\
&= \sup_{\boldsymbol{0}\neq\nabla\phi\in\boldsymbol{E}^{q'(\cdot)}(G)} \frac{|\sigma_q(\nabla u)(\nabla\phi)|}{\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)}} \\
&= \|\sigma_q(\nabla u)\|_{(\boldsymbol{E}^{q'(\cdot)}(G))'} \\
&\leq \sup_{\boldsymbol{0}\neq\nabla\phi\in\boldsymbol{E}^{q'(\cdot)}(G)} \frac{2\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)}\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)}}{\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)}} \\
&= 2\|\nabla u\|_{\boldsymbol{E}^{q(\cdot)}(G)}.
\end{aligned}$$

Thus the estimate (4.3) holds and σ_q is linear, continuous and injective.

As in the proof of Proposition 3.1, we can see that $\sigma_q(\boldsymbol{E}^{q(\cdot)}(G))$ is closed subspace of $(\boldsymbol{E}^{q'(\cdot)}(G))'$. If $\sigma_q(\boldsymbol{E}^{q(\cdot)}(G)) \subsetneq (\boldsymbol{E}^{q'(\cdot)}(G))'$, then it follows from the Hahn-Banach theorem that there exists $\boldsymbol{F}'' \in (\boldsymbol{E}^{q'(\cdot)}(G))''$ such that $\boldsymbol{F}'' \neq \boldsymbol{0}$ and $\boldsymbol{F}''\big|_{\sigma_q(\boldsymbol{E}^{q(\cdot)}(G))} = \boldsymbol{0}$. Since $\boldsymbol{E}^{q'(\cdot)}(G)$ is reflexive, there exists uniquely $\nabla\phi \in \boldsymbol{E}^{q'(\cdot)}(G)$ such that $\boldsymbol{F}''(\boldsymbol{F}') = \boldsymbol{F}'(\nabla\phi)$ for all $\boldsymbol{F}' \in (\boldsymbol{E}^{q'(\cdot)}(G))'$ and $\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)} = \|\boldsymbol{F}''\|_{(\boldsymbol{E}^{q'(\cdot)}(G))''} > 0$. For any $\nabla u \in \boldsymbol{E}^{q(\cdot)}(G)$, we have $0 = \sigma_q(\nabla u)(\nabla\phi) = \langle\nabla u, \nabla\phi\rangle_G$. From $Q_1(q')$, we have $\|\nabla\phi\|_{\boldsymbol{E}^{q'(\cdot)}(G)} = 0$. This is a contradiction. Thus σ_q is surjective. By the Banach open mapping theorem, σ_q^{-1} is also continuous.

Conversely, suppose that G has the property $Q_2(q)$ for $q = p$ and $q = p'$. Since $\sigma_{q'}(\boldsymbol{E}^{q'(\cdot)}(G)) = (\boldsymbol{E}^{q(\cdot)}(G))'$ and $Q_2(q')$ holds, for any $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(G)$,

$$\begin{aligned}\|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{q(\cdot)}(G)} &= \sup_{\boldsymbol{0}\neq \boldsymbol{F}'\in(\boldsymbol{E}^{q(\cdot)}(G))'} \frac{|\boldsymbol{F}'(\boldsymbol{\nabla} u)|}{\|\boldsymbol{F}'\|_{(\boldsymbol{E}^{q(\cdot)}(G))'}} \\ &= \sup_{\boldsymbol{0}\neq \boldsymbol{\nabla} v\in\boldsymbol{E}^{q'(\cdot)}(G)} \frac{|\sigma_q(\boldsymbol{\nabla} v)(\boldsymbol{\nabla} u)|}{\|\sigma_q(\boldsymbol{\nabla} v)\|_{(\boldsymbol{E}^{q(\cdot)}(G))'}} \\ &\leq \frac{1}{\widetilde{C_{q'}}} \sup_{\boldsymbol{0}\neq \boldsymbol{\nabla} v\in\boldsymbol{E}^{q'(\cdot)}(G)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_G|}{\|\boldsymbol{\nabla} v\|_{(\boldsymbol{E}^{q(\cdot)}(G))'}}.\end{aligned}$$

So we can get the estimate (4.1). □

Corollary 4.3. *Let G be a domain of $\mathbb{R}^d$ with a C^1-boundary and $p \in \mathcal{P}_+^{\log}(\overline{G})$. If G has the property $Q_1(q)$ for $q = p$ and $q = p'$, then for any $\boldsymbol{F}' \in (\boldsymbol{E}^{q'(\cdot)}(G))'$, there exists a unique $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(G)$ such that $\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla}\phi\rangle = \boldsymbol{F}'(\boldsymbol{\nabla}\phi)$ for all $\boldsymbol{\nabla}\phi \in \boldsymbol{E}^{q'(\cdot)}(G)$. Furthermore, there exists a constant $c = c(q, d, G) \geq 1$ such that*

$$c^{-1}\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{q(\cdot)}(G)} \leq \|\boldsymbol{F}'\|_{(\boldsymbol{E}^{q'(\cdot)}(G))'} \leq c\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{q(\cdot)}(G)}.$$

4.2. The case $G = \mathbb{R}^d$

In this subsection, we consider the case $G = \mathbb{R}^d$. Let $p \in \mathcal{P}_+^{\log}(\mathbb{R}^d)$.

Lemma 4.4. *If we define $M := \{\Delta v; v \in \mathcal{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)\}$, then M is dense in $L^{p(\cdot)}(\mathbb{R}^d)$.*

Proof. Suppose that $\overline{M} \subsetneq L^{p(\cdot)}(\mathbb{R}^d)$, where $\overline{M}$ is the closure of M in $L^{p(\cdot)}(\mathbb{R}^d)$. By the Hahn-Banach theorem, there exists $F' \in (L^{p(\cdot)}(\mathbb{R}^d))'$ with $\|F'\|_{(L^{p(\cdot)}(\mathbb{R}^d))'} > 0$ and $F'|_{\overline{M}} = 0$. Since we can regard $(L^{p(\cdot)}(\mathbb{R}^d))' = L^{p'(\cdot)}(\mathbb{R}^d)$ isometrically, there exists $v \in L^{p'(\cdot)}(\mathbb{R}^d)$ such that $\|v\|_{L^{p'(\cdot)}(\mathbb{R}^d)} = \|F'\|_{(L^{p(\cdot)}(\mathbb{R}^d))'} > 0$ and $F'(w) = \langle v, w\rangle_{\mathbb{R}^d}$ for all $w \in L^{p(\cdot)}(\mathbb{R}^d)$. Since $F'|_{\overline{M}} = 0$, we have $\langle v, \Delta\phi\rangle = \int_{\mathbb{R}^d} v\Delta\phi dx = 0$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$, so $\Delta v = 0$ in $\mathcal{D}'(\mathbb{R}^d)$. By the hypoellipticity of the Laplacian, we can regard that $v \in C^\infty(\mathbb{R}^d)$ (eventually after change of a set of measure zero), so v is harmonic in $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$ fixed, it follows from the second mean value theorem for harmonic functions that

$$v(x) = \frac{1}{|B_R(x)|}\int_{B_R(x)} v(y)dy,$$

where $B_R(x) = \{y \in \mathbb{R}^d : |y - x| < R\}$ and $|B_R(x)|$ denotes the volume of $B_R(x)$. By the generalized Hölder inequality (Proposition 2.2),

$$|v(x)| \leq \frac{2}{|B_R(x)|}\|v\|_{L^{p'(\cdot)}(B_R(x))}\|1\|_{L^{p(\cdot)}(B_R(x))}.$$

Since $\|1\|_{L^{p(\cdot)}(B_R(x))} \le \rho_{p(\cdot),B_R(x)}(1)^{1/p^-} = |B_R(x)|^{1/p^-}$ for large $R > 0$ and $p^- > 1$, we have

$$|v(x)| \le 2|B_R(x)|^{-1+1/p^-}\|v\|_{L^{p'(\cdot)}(\mathbb{R}^d)} \to 0 \text{ as } R \to \infty.$$

Therefore, we have $v(x) = 0$. Since $x \in \mathbb{R}^d$ is arbitrary, $v \equiv 0$ in $\mathbb{R}^d$. This is a contradiction. □

Define $\boldsymbol{\nabla}^2 v = (\partial_i\partial_j v)_{i,j=1,\dots,d}$ for $v \in \mathcal{D}(\mathbb{R}^d)$. Then there exists a constant $C = C(p,d) > 0$ such that

$$C\|\boldsymbol{\nabla}^2 v\|_{\boldsymbol{L}^{p(\cdot)}(\mathbb{R}^d)} \le \|\Delta v\|_{L^{p(\cdot)}(\mathbb{R}^d)} \text{ for all } v \in \mathcal{D}(\mathbb{R}^d). \tag{4.4}$$

For the proof, see [5, Corollary 14.1.7] (cf. when $p(\cdot) = p$ (constant), see Gilbarg and Trudinger [10, Corollary 9.10]).

For $p \in \mathcal{P}_+^{\log}(\mathbb{R}^d)$, we have $\boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d) = \{\boldsymbol{\nabla} u; u \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^d), \boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\mathbb{R}^d)\}$ by definition.

Lemma 4.5. *Let $p, q \in \mathcal{P}_+^{\log}(\mathbb{R}^d)$. If $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(\mathbb{R}^d)$ satisfies*

$$\sup_{0\ne v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p'(\cdot)}(\mathbb{R}^d)}} < \infty, \tag{4.5}$$

then $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$ and there exists a constant $C_1 = C_1(p,d) > 0$ such that

$$\|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)} \le C_1 \sup_{0\ne v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p'(\cdot)}(\mathbb{R}^d)}}. \tag{4.6}$$

In particular, if $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$, then

$$\begin{aligned} \|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)} &\le C_1 \sup_{0\ne v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)}} \\ &\le C_1 \sup_{\boldsymbol{0}\ne \boldsymbol{\nabla} v\in \boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)}}, \end{aligned} \tag{4.7}$$

that is, $\mathbb{R}^d$ has the property $Q_1(p)$ in the Definition 4.1.

Proof. Let $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(\mathbb{R}^d)$. For every $i = 1, \dots, d$, using (4.4),

$$\begin{aligned} \infty &> \sup_{0\ne v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)}} \\ &\ge \sup_{0\ne w\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla}(\partial_i w)\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla}\partial_i w\|_{\boldsymbol{L}^{p'(\cdot)}(\mathbb{R}^d)}} \\ &\ge \sup_{0\ne w\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \partial_i u, \Delta w\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla}^2 w\|_{\boldsymbol{L}^{p'(\cdot)}(\mathbb{R}^d)}} \\ &\ge C \sup_{0\ne w\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \partial_i u, \Delta w\rangle_{\mathbb{R}^d}|}{\|\Delta w\|_{L^{p'(\cdot)}(\mathbb{R}^d)}}, \end{aligned} \tag{4.8}$$

where C is the constant in (4.4). Therefore, the functional $v \mapsto \langle \partial_i u, v\rangle_{\mathbb{R}^d}$ is linear and continuous on the dense subspace M of $L^{p'(\cdot)}(\mathbb{R}^d)$, so the functional is uniquely extended to a continuous linear functional on $L^{p'(\cdot)}(\mathbb{R}^d)$ which is norm preserving. Thus there exists $g \in L^{p(\cdot)}(\mathbb{R}^d)$ such that $\langle \partial_i u, v\rangle_{\mathbb{R}^d} = \langle g, v\rangle_{\mathbb{R}^d}$ for all $v \in M$, that is,

$$\langle \partial_i u, \Delta w\rangle_{\mathbb{R}^d} = \langle g, \Delta w\rangle_{\mathbb{R}^d} \text{ for all } w \in \mathcal{D}(\mathbb{R}^d).$$

If we define $W = \partial_i u - g$, then $\Delta W = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, so W is harmonic in $\mathbb{R}^d$. By the same argument as in the proof of Lemma 4.4, we can regard $W(x) \equiv 0$, so $\partial_i u = g \in L^{p(\cdot)}(\mathbb{R}^d)$. Hence we have $\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\mathbb{R}^d)$. Since $\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\mathbb{R}^d)$ and $u \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^d) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$, for any ball B, it follows from [5, Corollary 8.2.6] that

$$\|u - \langle u\rangle_B\|_{L^{p(\cdot)}(B)} \le C(p, d, B)\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(B)}.$$

This implies that $u \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^d)$, that is, $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$. Since M is dense in $L^{p'(\cdot)}(\mathbb{R}^d)$, using (4.8),

$$\begin{aligned}
\|\partial_i u\|_{L^{p(\cdot)}(\mathbb{R}^d)} &= \sup_{0\neq f\in L^{p'(\cdot)}(\mathbb{R}^d)} \frac{|\langle \partial_i u, f\rangle_{\mathbb{R}^d}|}{\|f\|_{L^{p'(\cdot)}(\mathbb{R}^d)}} \\
&= \sup_{0\neq w\in \mathcal{D}(\mathbb{R}^d)} \frac{|\langle \partial_i u, \Delta w\rangle_{\mathbb{R}^d}|}{\|\Delta w\|_{L^{p'(\cdot)}(\mathbb{R}^d)}} \\
&\le C^{-1} \sup_{0\neq v\in \mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{L^{p'(\cdot)}(\mathbb{R}^d)}}.
\end{aligned}$$

Hence (4.6) holds. □

Corollary 4.6. *Let $p \in \mathcal{P}^{\log}_{+}(\mathbb{R}^d)$. If we define $\boldsymbol{E}^{\infty}(\mathbb{R}^d) = \{\boldsymbol{\nabla} v; v \in \mathcal{D}(\mathbb{R}^d)\}$, then $\boldsymbol{E}^{\infty}(\mathbb{R}^d)$ is dense in $\boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$.*

Proof. Let $\overline{\boldsymbol{E}^{\infty}(\mathbb{R}^d)} \subsetneq \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$. Then by the Hahn-Banach theorem, there exists $\boldsymbol{F}' \in (\boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d))'$ such that $\boldsymbol{F}' \neq \boldsymbol{0}$ and $\boldsymbol{F}'\big|_{\boldsymbol{E}^{\infty}(\mathbb{R}^d)} = \boldsymbol{0}$. For $q = p$ and $q = p'$, since

$$\|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{q(\cdot)}(\mathbb{R}^d)} \le C \sup_{\boldsymbol{0}\neq \boldsymbol{\nabla}\phi\in \boldsymbol{E}^{q'(\cdot)}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla}\phi\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla}\phi\|_{\boldsymbol{E}^{q'(\cdot)}(\mathbb{R}^d)}} \text{ for all } \boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(\mathbb{R}^d)$$

from Lemma 4.5, $\mathbb{R}^d$ satisfies $Q_1(q)$ for $q = p$ and $q = p'$. By Corollary 4.3, there exists uniquely $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)$ such that $\|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)} > 0$ and

$$\boldsymbol{F}'(\boldsymbol{\nabla}\phi) = \langle \boldsymbol{\nabla} u, \boldsymbol{\nabla}\phi\rangle_{\mathbb{R}^d} \text{ for all } \boldsymbol{\nabla}\phi \in \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d).$$

On the other hand, since $\boldsymbol{F}'\big|_{\boldsymbol{E}^{\infty}(\mathbb{R}^d)} = \boldsymbol{0}$, we have

$$0 = \boldsymbol{F}'(\boldsymbol{\nabla}\phi) = \langle \boldsymbol{\nabla} u, \boldsymbol{\nabla}\phi\rangle_{\mathbb{R}^d} \text{ for all } \phi \in \mathcal{D}(\mathbb{R}^d).$$

From (4.7), we have $\boldsymbol{\nabla} u = \boldsymbol{0}$. This is a contradiction. □

4.3. The case where G is a half-space or a bended half-space

In this subsection, we consider the case where G is a half-space or a bended half-space. Let $G = H = \{x = (x', x_d); x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}, x_d < 0\}$ and define $C_0^\infty(\overline{H}) = \{v\big|_H; v \in C_0^\infty(\mathbb{R}^d)\}$.

Lemma 4.7. *Let $p, q \in \mathcal{P}_+^{\log}(\overline{H})$. If $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(H)$ satisfies*

$$\sup_{0 \neq v \in C_0^\infty(\overline{H})} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_H|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(H)}} < \infty,$$

then $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(H)$ and

$$\|\boldsymbol{\nabla} u\|_{\boldsymbol{E}^{p(\cdot)}(H)} \leq C_2 \sup_{0 \neq v \in C_0^\infty(\overline{H})} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_H|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(H)}}, \tag{4.9}$$

where $C_2 = 2C_1$ and C_1 is the constant of the inequality of (4.6).

In particular, if $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(H)$, then (4.9) *holds, that is, H has the property $Q_1(p)$.*

Proof. We reduce this case to the previous case by reflection argument. For any function $v : \mathbb{R}^d \to \mathbb{R}$, define $\overline{v}(x', x_d) = v(x', -x_d)$, and for any function $u : \overline{H} \to \mathbb{R}$, define

$$u^*(x', x_d) = \begin{cases} u(x', x_d) & \text{if } x_d \leq 0, \\ u(x', -x_d) & \text{if } x_d > 0. \end{cases}$$

We note that if $p, q \in \mathcal{P}_+^{\log}(\overline{H})$, then clearly $p^*, q^* \in \mathcal{P}_+^{\log}(\mathbb{R}^d)$. Let $\boldsymbol{\nabla} u \in \boldsymbol{E}^{q(\cdot)}(H)$, that is, $u \in L^{q(\cdot)}_{\mathrm{loc}}(\overline{H})$ and $\boldsymbol{\nabla} u \in \boldsymbol{L}^{q(\cdot)}(H)$. Since

$$\partial_i u^*(x) = \begin{cases} \partial_i u(x) & \text{for } x_d < 0, \\ (-1)^{\delta_{id}}(\partial_i u)(x', -x_d) & \text{for } x_d > 0 \end{cases}$$

for $i = 1, \ldots, d$. Since

$$\begin{aligned} \rho_{q^*(\cdot), \mathbb{R}^d}(|\boldsymbol{\nabla} u^*|) &= \int_{\mathbb{R}^d} |\boldsymbol{\nabla} u^*|^{q^*(x)} dx \\ &= \int_H |\boldsymbol{\nabla} u^*|^{q(x)} dx + \int_{\{x_d > 0\}} |\boldsymbol{\nabla} u(x', -x_d)|^{q(x', -x_d)} dx \\ &= 2\int_H |\boldsymbol{\nabla} u^*|^{q(x)} dx < \infty, \end{aligned}$$

we can see that $\boldsymbol{\nabla} u^* \in \boldsymbol{L}^{q^*(\cdot)}(\mathbb{R}^d)$. For any $v \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned} \langle \boldsymbol{\nabla} u^*, \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d} &= \int_{\{x_d < 0\}} \boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} v(x) dx + \int_{\{x_d > 0\}} \boldsymbol{\nabla} u^*(x) \cdot \boldsymbol{\nabla} v(x) dx \\ &= \int_H \boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} v(x) dx + \int_H \boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} \overline{v}(x) dx. \end{aligned}$$

Hence $\langle \nabla u^*, \nabla v\rangle_{\mathbb{R}^d} = \langle \nabla u, \nabla(v+\overline{v})\rangle_H$. Since $\|\nabla(v+\overline{v})\|_{L^{p'(\cdot)}(H)} \le 2\|\nabla v\|_{L^{(p^*)'(\cdot)}(\mathbb{R}^d)}$ and $v+\overline{v}\in C_0^\infty(\overline{H})$, we have

$$\begin{aligned}\sup_{0\neq v\in\mathcal{D}(\mathbb{R}^d)}\frac{|\langle \nabla u^*, \nabla v\rangle_{\mathbb{R}^d}|}{\|\nabla v\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^d)}} &\le 2\sup_{\substack{v\in\mathcal{D}(\mathbb{R}^d)\\ v+\overline{v}\neq 0}}\frac{|\langle \nabla u, \nabla(v+\overline{v})\rangle_H|}{\|\nabla(v+\overline{v})\|_{L^{p'(\cdot)}(H)}}\\ &\le 2\sup_{0\neq w\in C_0^\infty(\overline{H})}\frac{|\langle \nabla u, \nabla w\rangle_H|}{\|\nabla w\|_{L^{p'(\cdot)}(H)}}<\infty.\end{aligned}$$

By Lemma 4.5, $\nabla u^*\in E^{p^*(\cdot)}(\mathbb{R}^d)$, so $\nabla u\in E^{p(\cdot)}(H)$ and

$$\begin{aligned}\|\nabla u\|_{E^{p(\cdot)}(H)}\le\|\nabla u^*\|_{E^{p^*(\cdot)}(\mathbb{R}^d)} &\le C_1\sup_{0\neq v\in\mathcal{D}(\mathbb{R}^d)}\frac{|\langle \nabla u^*, \nabla v\rangle_{\mathbb{R}^d}|}{\|\nabla v\|_{L^{(p^*)'(\cdot)}(\mathbb{R}^d)}}\\ &\le 2C_1\sup_{0\neq w\in C_0^\infty(\overline{H})}\frac{|\langle \nabla u, \nabla w\rangle_H|}{\|\nabla w\|_{L^{p'(\cdot)}(H)}}.\end{aligned}$$

□

Next we consider the case of a bended half-space. Let $\omega=\omega(x')$ be a C^1-function defined in $\mathbb{R}^{d-1}$ such that $\omega(x')=0$ for $|x'|>R=R(\omega)>0$ and define a bended half space

$$H_\omega=\{x=(x',x_d)\in\mathbb{R}^d : x_d<\omega(x')\}.$$

Lemma 4.8. (i) *Let $G=H_\omega$ and $p\in\mathcal{P}_+^{\log}(\overline{H_\omega})$. Then $E^\infty(H_\omega):=\{\nabla v|_{\overline{H_\omega}}; v\in C_0^\infty(\overline{H_\omega})\}$ is dense in $E^{p(\cdot)}(H_\omega)$.*

(ii) *Let $G=\Omega$, where Ω is a bounded domain with a C^1- boundary and $p\in\mathcal{P}_+^{\log}(\overline{\Omega})$. Then $E^\infty(\Omega):=\{\nabla v|_{\overline{\Omega}}; v\in C_0^\infty(\overline{\Omega})\}$ is dense in $E^{p(\cdot)}(\Omega)$.*

Proof. For each case, it follows from Proposition 2.4 that we can assume that $p\in\mathcal{P}_+^{\log}(\mathbb{R}^d)$. Let $\nabla v\in E^{p(\cdot)}(G)$. Since ∂G is of class C^1, there exists $\nabla\widetilde{v}\in E^{p(\cdot)}(\mathbb{R}^d)$ such that $\nabla v=\nabla\widetilde{v}$ in H_ω (cf. [5, Theorem 8.5.2] and Miyajima [15, Theorem 6.17]). Then from Corollary 4.6, there exists $\{\phi_j\}\subset C_0^\infty(\mathbb{R}^d)$ such that $\|\nabla\phi_j-\nabla\widetilde{v}\|_{L^{p(\cdot)}(\mathbb{R}^d)}\to 0$ as $j\to\infty$. Thus $\nabla\phi_j|_{\overline{G}}\in E^\infty(G)$ and $\|\nabla\phi_j-\nabla v\|_{L^{p(\cdot)}(G)}\le\|\nabla\phi_j-\nabla\widetilde{v}\|_{L^{p(\cdot)}(\mathbb{R}^d)}\to 0$ as $j\to\infty$. □

Lemma 4.9. *Let $p,q\in\mathcal{P}_+^{\log}(\overline{H_\omega})$. Then there exists a constant $K=K(p,q,d)>0$ such that if $\|\nabla'\omega\|_{L^\infty(\mathbb{R}^{d-1})}=\operatorname{ess\,sup}_{x'\in\mathbb{R}^{d-1}}|\nabla'\omega(x')|\le K$, where $\nabla'=(\partial_{x_1},\dots,\partial_{x_{d-1}})$, then there exists a constant $C(s)=C(s,K,d)>0$ such that*

$$\|\nabla u\|_{L^{s(\cdot)}(H_\omega)}\le C(s)\sup_{0\neq\nabla v\in E^\infty(H_\omega)}\frac{|\langle \nabla u, \nabla v\rangle_{H_\omega}|}{\|\nabla v\|_{L^{s'(\cdot)}(H_\omega)}}$$

for $s(\cdot)=p(\cdot),p'(\cdot),q(\cdot),q'(\cdot)$ and any $\nabla u\in E^{s(\cdot)}(H_\omega)$.

Proof. Define a mapping $y = y(x)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ by $y_i(x) = x_i$ for $i = 1, \dots, d-1$ and $y_d(x) = x_d - \omega(x')$. Then the mapping is of class C^1 and y maps $\overline{H_\omega}$ to $\overline{H}$ bijectively with the Jacobian $J(y(x)) \equiv 1$. Let $x = x(y)$ be the inverse mapping. For $s \in \mathcal{P}_+^{\log}(\overline{H_\omega})$. Denote $(\overline{s})^*$ by an even extension of $\overline{s}$ with respect to y_d, and define $s^*(x) = (\overline{s})^*(y(x))$, Then $s^* \in \mathcal{P}_+^{\log}(\mathbb{R}^d)$. For $\boldsymbol{\nabla} u \in \boldsymbol{E}^{s(\cdot)}(H_\omega)$, define $\overline{u}(y) = u(x(y))$. We use the following notations $\overline{\partial}_i = \partial/\partial y_i$, $\partial_i = \partial/\partial x_i$, $\overline{\boldsymbol{\nabla}} = (\overline{\partial}_1, \dots, \overline{\partial}_d)$, $\boldsymbol{\nabla} = (\partial_1, \dots, \partial_d)$. Then $\partial_i = \overline{\partial}_i - \frac{\partial \omega}{\partial x_i}\overline{\partial}_d$ for $i = 1, \dots, d-1$ and $\partial_d = \overline{\partial}_d$. Hence $\boldsymbol{\nabla} u \in \boldsymbol{E}^{s(\cdot)}(H_\omega)$ if and only if $\overline{\boldsymbol{\nabla}}\overline{u} \in \boldsymbol{E}^{\overline{s}(\cdot)}(H)$ and there exist constants $e_i = e_i(s(\cdot), d) > 0$ $(i = 1, 2)$ independent of ω such that

$$\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{s'(\cdot)}(H_\omega)} \le e_1(1 + \|\boldsymbol{\nabla}'\omega\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})})\|\overline{\boldsymbol{\nabla}}\overline{v}\|_{\boldsymbol{L}^{\overline{s}'(\cdot)}(H)}$$

and

$$\begin{aligned}
&|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{H_\omega}| \\
&= \left| \int_H (\overline{\boldsymbol{\nabla}}'\overline{u} - \boldsymbol{\nabla}'\omega(\overline{\partial}_d\overline{u}), \overline{\partial}_d\overline{u}) \cdot (\overline{\boldsymbol{\nabla}}'\overline{v} - \boldsymbol{\nabla}'\omega(\overline{\partial}_d\overline{v}), \overline{\partial}_d\overline{v}) dy \right| \\
&\ge |\langle \overline{\boldsymbol{\nabla}}\overline{u}, \overline{\boldsymbol{\nabla}}\overline{v}\rangle_H| - e_2\|\boldsymbol{\nabla}'\omega\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})}(1 + \|\boldsymbol{\nabla}'\omega\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})})\|\overline{\boldsymbol{\nabla}}\overline{u}\|_{\boldsymbol{L}^{\overline{s}(\cdot)}(H)}\|\overline{\boldsymbol{\nabla}}\overline{v}\|_{\boldsymbol{L}^{\overline{s}'(\cdot)}(H)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sup_{\boldsymbol{0} \ne \boldsymbol{\nabla} v \in \boldsymbol{E}^\infty(H_\omega)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{H_\omega}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{s'(\cdot)}(H_\omega)}} \\
&\quad \ge \left(e_1(1 + \|\boldsymbol{\nabla}_{x'}\omega\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})})\right)^{-1} \sup_{\boldsymbol{0} \ne \overline{\boldsymbol{\nabla}}\overline{v} \in \boldsymbol{E}^\infty(H)} \frac{|\langle \overline{\boldsymbol{\nabla}}\overline{u}, \overline{\boldsymbol{\nabla}}\overline{v}\rangle_H|}{\|\overline{\boldsymbol{\nabla}}\overline{v}\|_{\boldsymbol{L}^{\overline{s}'(\cdot)}(H)}} \\
&\quad -e_2 e_1^{-1}\|\boldsymbol{\nabla}_{x'}\omega\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})}\|\overline{\boldsymbol{\nabla}}\overline{u}\|_{\boldsymbol{L}^{\overline{s}(\cdot)}(H)}.
\end{aligned}$$

Choose $K \le 1$ so that $(e_1(1+K))^{-1}C_2^{-1} - e_1^{-1}e_2K > 0$ and put

$$C(s(\cdot))^{-1} = \left\{(e_1(1+K))^{-1}C_2^{-1} - e_1^{-1}e_eK\right\}\left(e_1(1+K)\right)^{-1}.$$

Then using Lemma 4.7, we have

$$\sup_{\boldsymbol{0} \ne \boldsymbol{\nabla} v \in \boldsymbol{E}^\infty(H_\omega)} \frac{|\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{H_\omega}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{s'(\cdot)}(H_\omega)}} \ge C(s(\cdot))^{-1}\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{s(\cdot)}(H_\omega)}.$$

This completes the proof. □

4.4. End of the proof of Theorem 3.3

Lemma 4.10. *Let Ω be a bounded domain with a C^1-boundary and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. If $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\Omega)$ satisfies $\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_\Omega = 0$ for all $\boldsymbol{\nabla} v \in \boldsymbol{E}^{p'(\cdot)}(\Omega)$, then $\boldsymbol{\nabla} u = \boldsymbol{0}$.*

Proof. Since $p(x) \geq p^-$, $p'(x) \leq (p^-)'$ and Ω is bounded, $\boldsymbol{E}^{p(\cdot)}(\Omega) \subset \boldsymbol{E}^{p^-}(\Omega)$ and $\boldsymbol{E}^{(p^-)'}(\Omega) \subset \boldsymbol{E}^{p'(\cdot)}(\Omega)$. Hence $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p^-}(\Omega)$ satisfies $\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_\Omega = 0$ for all $\boldsymbol{\nabla} v \in \boldsymbol{E}^{(p^-)'}(\Omega)$. Therefore, it follows from [18, Lemma 3.9] that $\boldsymbol{\nabla} u = \boldsymbol{0}$. □

Proof of Theorem 3.3.

Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^1-boundary $\partial\Omega$. For every $x_0 \in \partial\Omega$, there exist $\rho > 0$ and $\sigma \in C^1(\overline{B_\rho(x_0)})$ such that $\boldsymbol{\nabla}\sigma(x_0) \neq \boldsymbol{0}$, $\Omega \cap B_\rho(x_0) = \{x \in B_\rho(x_0); \sigma(x) < 0\}$ and $\partial\Omega \cap B_\rho(x_0) = \{x \in B_\rho(x_0); \sigma(x) = 0\}$. Then $|\boldsymbol{\nabla}\sigma(x_0)|^{-1}\boldsymbol{\nabla}\sigma(x_0)$ is the exterior unit normal vector at x_0. Hence there exists an orthogonal matrix S such that $S\left(\frac{\boldsymbol{\nabla}\sigma(x_0))}{|\boldsymbol{\nabla}\sigma(x_0)|}\right) = \boldsymbol{e}_d =^t (0, \ldots, 0, 1)$. Define a map $y = y(x) = S(x - x_0)$ from $B_\rho(x_0)$ to $\widehat{B}_\rho(0) := \{y \in \mathbb{R}^d; |y| < \rho\}$ and $\widehat{\sigma}(y) = \sigma(x_0 + S^{-1}y)$, $\Omega_\rho = \Omega \cap B_\rho(x_0)$, $\widehat{\Omega} = S\Omega$ and $\widehat{\Omega}_\rho = \widehat{\Omega} \cap \widehat{B}_\rho(0)$. Then for $s \in \mathcal{P}_+^{\log}(\overline{\Omega})$, $\boldsymbol{\nabla} u \in \boldsymbol{E}^{s(\cdot)}(\Omega_\rho)$ and $\boldsymbol{\nabla} v \in \boldsymbol{E}^{s'(\cdot)}(\Omega_\rho)$, we define $\widehat{s}(y) = s(x_0 + S^{-1}y)$, $\widehat{u}(y) = u(x_0 + S^{-1}y)$ and $\widehat{v}(y) = v(x_0 + S^{-1}y)$. Then we see that $\widehat{s} \in \mathcal{P}_+^{\log}(\overline{\widehat{\Omega}})$,

$$\langle \boldsymbol{\nabla}\widehat{u}, \boldsymbol{\nabla}\widehat{v}\rangle_{\widehat{\Omega}_\rho} = \langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_{\Omega_\rho}$$

and by definition of $L^{s(\cdot)}$-norm, $\|\boldsymbol{\nabla}\widehat{u}\|_{\boldsymbol{L}^{\widehat{s}(\cdot)}(\widehat{\Omega}_\rho)}$ (resp. $\|\boldsymbol{\nabla}\widehat{v}\|_{\boldsymbol{L}^{\widehat{s}(\cdot)}(\widehat{\Omega}_\rho)}$) and $\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{s(\cdot)}(\Omega_\rho)}$ (resp. $\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{s(\cdot)}(\Omega_\rho)}$) are equivalent. Since $\boldsymbol{\nabla}\widehat{\sigma}(0) = |\boldsymbol{\nabla}\sigma(x_0)|\boldsymbol{e}_d \neq \boldsymbol{0}$, using the implicit function theorem, there exist $0 < \rho' < \rho$, $h > 0$ and $\psi \in C^1(\overline{B'_{\rho'}})$, where $\overline{B'_{\rho'}} = \{y' \in \mathbb{R}^{d-1}; |y'| \leq \rho'\}$ such that

$$Z := Z_{\rho',h} = \{y = (y', y_d) \in \mathbb{R}^d; |y'| < \rho', |y_d| < h\} \subset \widehat{B}_\rho(0),$$

$(y', \psi(y')) \in Z$ if $y' \in B'_{\rho'}$, $\widehat{\sigma}(y', \psi(y')) = 0$, $\psi(0) = 0$, $(\boldsymbol{\nabla}'\psi)(0) = \boldsymbol{0}$, where $\boldsymbol{\nabla}' = (\partial_{y_1}, \ldots, \partial_{y_{d-1}})$,

$$\partial\widehat{\Omega} \cap Z = \{y \in Z; y_d = \psi(y')\} \text{ and } \widehat{\Omega} \cap Z = \{y \in Z; y_d < \psi(y')\}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^{d-1})$ such that $\eta(y') = 1$ for $|y'| \leq 1$ and $\eta(y') = 0$ for $|y'| \geq 2$. For $0 < \lambda < \rho'/2$, if we define $\eta_\lambda(y') := \eta(\lambda^{-1}y')$ and

$$\omega_\lambda(y') = \begin{cases} \eta_\lambda(y')\psi(y') & \text{for } |y'| < \rho', \\ 0 & \text{otherwise}, \end{cases}$$

then by $\psi(0) = 0$, $\boldsymbol{\nabla}\psi(0) = \boldsymbol{0}$ and the mean value theeorem, we have

$$\sup\{|\boldsymbol{\nabla}'\omega_\lambda(y')|; y' \in \mathbb{R}^{d-1}\} \to 0 \text{ as } \lambda \to +0.$$

Thereby if we choose $K > 0$ as in Lemma 4.9 and $\lambda > 0$ is small enough, then $\|\boldsymbol{\nabla}'\omega_\lambda(y')\|_{\boldsymbol{L}^\infty(\mathbb{R}^{d-1})} \leq K$. Choose $r = r(x_0) > 0$ such that $0 < r < \rho'$ and

$B_r(x_0) \subset Z$ and choose $\phi \in C_0^\infty(B_r(x_0))$ such that $0 \le \phi \le 1$, $\phi(x) \equiv 1$ on $B'(x_0) = B_{r/2}(x_0)$. If $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\Omega)$, then $\boldsymbol{\nabla}(\phi u) \in \boldsymbol{E}^{p(\cdot)}(\Omega)$. Here we extend ϕu outside $\Omega \cap B_r(x_0)$ by zero, we may assume that $\widehat{\phi u}$ is defined in H_{ω_λ}. By Lemma 4.10, there exists $C(p(\cdot)) = C(p(\cdot), K, d) > 0$ such that

$$\|\boldsymbol{\nabla}(\phi u)\|_{\boldsymbol{E}^{p(\cdot)}(B_r(x_0)\cap\Omega)} \le C(p(\cdot)) \sup_{\mathbf{0}\neq\boldsymbol{\nabla} v\in \boldsymbol{E}^\infty(H_{\omega_\lambda})} \frac{|\langle \boldsymbol{\nabla}(\widehat{\phi u}), \boldsymbol{\nabla}\widehat{v}\rangle_{H_{\omega_\lambda}}|}{\|\boldsymbol{\nabla}\widehat{v}\|_{\boldsymbol{E}^{p'(\cdot)}(H_{\omega_\lambda})}}.$$

Since $\partial\Omega$ is compact and of class C^1, there exist finitely many $x_i \in \partial\Omega$ and $r_i = r_i(x_i) > 0\,(i = 1, \ldots, M)$ such that $B_i' := B_{r_i/2}(x_i) \subset B_i := B_{r_i}(x_i)$ and $\partial\Omega \subset \cup_{i=1}^M B_i'$. If we choose $\phi_i \in C_0^\infty(B_i)$ such that $0 \le \phi_i \le 1$ and $\phi_i \equiv 1$ on B_i', then

$$\|\boldsymbol{\nabla}(\phi_i u)\|_{\boldsymbol{E}^{p(\cdot)}(B_i\cap\Omega)} \le C_i \sup_{\mathbf{0}\neq\boldsymbol{\nabla}\widehat{v}\in \boldsymbol{E}^\infty(H_{\omega_{\lambda_i}})} \frac{|\langle \boldsymbol{\nabla}(\widehat{\phi_i u}), \boldsymbol{\nabla}\widehat{v}\rangle_{H_{\omega_{\lambda_i}}}|}{\|\boldsymbol{\nabla}\widehat{v}\|_{\boldsymbol{E}^{p'(\cdot)}(H_{\omega_{\lambda_i}})}}. \tag{4.10}$$

Since Ω is bounded, $\Omega_1 = \Omega \setminus (\cup_{i=1}^M B_i')$ is compact. Hence $r := \operatorname{dist}(\Omega_1, \partial\Omega) > 0$, so there exist finitely many $x_i \in \Omega\,(i = M+1, \ldots, N)$ such that $\Omega_1 \subset \cup_{i=M+1}^N B_i'$, where $B_i' = B_{r/2}(x_i)$ and $B_i = B_r(x_i)$. Let $\phi_i \in C_0^\infty(B_i)$ such that $0 \le \phi_i \le 1$, $\phi_i \equiv 1$ on B_i'. By Corollary 4.6, for every $i = M+1, \ldots, N$,

$$\|\boldsymbol{\nabla}(\phi_i u)\|_{\boldsymbol{E}^{p(\cdot)}(B_i)} \le C_i \sup_{0\neq v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla}(\phi_i u), \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)}}. \tag{4.11}$$

Now suppose that the statement of Theorem 3.3 is not true. Then there exists a sequence $\{\boldsymbol{\nabla} u_k\} \subset \boldsymbol{E}^{p(\cdot)}(\Omega)$ such that $\|\boldsymbol{\nabla} u_k\|_{\boldsymbol{E}^{p(\cdot)}(\Omega)} = 1$ and

$$\varepsilon_k := \sup_{\mathbf{0}\neq\boldsymbol{\nabla} v\in\boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \boldsymbol{\nabla} u_k, \boldsymbol{\nabla} v\rangle_\Omega|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\Omega)}} \to 0 \text{ as } k \to \infty. \tag{4.12}$$

Without loss of generality, we may assume that $\int_\Omega u_k dx = 0$. Since $\boldsymbol{E}^{p(\cdot)}(\Omega)$ is a reflexive Banach space, there exist $\boldsymbol{\nabla} u \in \boldsymbol{E}^{p(\cdot)}(\Omega)$ and a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that $\boldsymbol{\nabla} u_k \to \boldsymbol{\nabla} u$ weakly in $\boldsymbol{E}^{p(\cdot)}(\Omega)$. For any $\boldsymbol{\nabla} v \in \boldsymbol{E}^{p'(\cdot)}(\Omega)$, using (4.12), we have

$$\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_\Omega = \lim_{k\to\infty} \langle \boldsymbol{\nabla} u_k, \boldsymbol{\nabla} v\rangle_\Omega = 0.$$

By Lemma 4.10, we have

$$\boldsymbol{\nabla} u = \mathbf{0}. \tag{4.13}$$

From the Poincaré inequality (Theorem 2.6 (ii)),

$$\|u_k\|_{L^{p(\cdot)}(\Omega)} \le C(p(\cdot), d, \Omega)\|\boldsymbol{\nabla} u_k\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

Since $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact from Proposition 2.3. Hence choosing again a subsequence, we may assume that $u_k \to u$ strongly in $L^{p(\cdot)}(\Omega)$. Since $\int_\Omega u_k dx = 0$, we see that $\int_\Omega u dx = 0$. From (4.13), we have $u = 0$, so $u_k \to 0$ strongly in $L^{p(\cdot)}(\Omega)$. Fix any $i = 1, \dots, M, M+1, \dots, N$. If $i = 1, \dots, M$, from (4.10) $C_i^{-1}\|\boldsymbol{\nabla}(\phi_i u_k)\|_{\boldsymbol{L}^{p(\cdot)}(B_i\cap\Omega)} \le d_k^i$, where

$$d_k^i = \sup_{\mathbf{0}\neq\boldsymbol{\nabla}\widehat{v}\in\boldsymbol{E}^\infty(H_{\omega_{\lambda_i}})} \frac{|\langle \boldsymbol{\nabla}(\widehat{\phi_i u}), \boldsymbol{\nabla}\widehat{v}\rangle_{H_{\omega_{\lambda_i}}}|}{\|\boldsymbol{\nabla}\widehat{v}\|_{\boldsymbol{E}^{p'(\cdot)}(H_{\omega_{\lambda_i}})}}.$$

If $i = M+1, \dots, N$, from (4.11), $C_i^{-1}\|\boldsymbol{\nabla}(\phi_i u_k)\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} \le d_k^i$, where

$$d_k^i = \sup_{0\neq v\in\mathcal{D}(\mathbb{R}^d)} \frac{|\langle \boldsymbol{\nabla}(\phi_i u), \boldsymbol{\nabla} v\rangle_{\mathbb{R}^d}|}{\|\boldsymbol{\nabla} v\|_{\boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)}}.$$

For each $k \in \mathbb{N}$, there exist $\boldsymbol{\nabla}\widehat{v}_k \in \boldsymbol{E}^{p(\cdot)}(H_{\omega_{\lambda_i}})$ such that $\|\boldsymbol{\nabla}\widehat{v}_k\|_{\boldsymbol{L}^{p'(\cdot)}(H_{\omega_{\lambda_i}})} = 1$ and

$$0 \le d_k^i - \langle \boldsymbol{\nabla}(\widehat{\phi_i u_k}), \boldsymbol{\nabla}\widehat{v}_k\rangle_{H_{\omega_{\lambda_i}}} \le \frac{1}{k} \text{ for } i = 1, \dots, M,$$

and $\boldsymbol{\nabla} v_k \in \boldsymbol{E}^{p(\cdot)}(\mathbb{R}^d)$ such that $\|\boldsymbol{\nabla} v_k\|_{\boldsymbol{L}^{p'(\cdot)}(\mathbb{R}^d)} = 1$ and

$$0 \le d_k^i - \langle \boldsymbol{\nabla}(\phi_i u_k), \boldsymbol{\nabla} v_k\rangle_{\mathbb{R}^d} \le \frac{1}{k} \text{ for } i = M+1, \dots, N$$

We may assume that $\int_{H_{\omega_{\lambda_i}}} \widehat{v}_k dy = 0$ for $i = 1, \dots, M$ and $\int_{B_i} v_k dx = 0$ for $i = M+1, \dots, N$. By the Poincaré inequality, we conclude

$$\|\widehat{v}_k\|_{W^{1,p'(\cdot)}(H_{\omega_{\lambda_i}})} \le C \text{ for } i = 1, \dots, M \text{ and } \|v_k\|_{W^{1,p'(\cdot)}(B_i)} \le C \text{ for } i = M+1, \dots, N$$

for some constant $C > 0$. Passing to a subsequence, we can assume that there exist $\boldsymbol{\nabla}\widehat{v}_i \in \boldsymbol{E}^{\widehat{p}'(\cdot)}(H_{\omega_{\lambda_i}})$ and $\boldsymbol{\nabla} v_i \in \boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d)$ such that

$$\boldsymbol{\nabla}\widehat{v}_k \to \boldsymbol{\nabla}\widehat{v}_i \text{ weakly in } \boldsymbol{E}^{\widehat{p}'(\cdot)}(H_{\omega_{\lambda_i}}) \text{ and } \widehat{v}_k \to \widehat{v}_i \text{ strongly in } L^{p'(\cdot)}(B_i \cap H_{\omega_{\lambda_i}})$$

for $i = 1, \dots, M$ and

$$\boldsymbol{\nabla} v_k \to \boldsymbol{\nabla} v_i \text{ weakly in } \boldsymbol{E}^{p'(\cdot)}(\mathbb{R}^d) \text{ and } v_k \to v_i \text{ strongly in } L^{p'(\cdot)}(B_i) \text{ for } i = M{+}1, \dots, N.$$

For the brevity of notations, we write $G = H_{\omega_{\lambda_i}}$ for $i = 1, \dots, M$ or $G = \mathbb{R}^d$ for $i = M+1, \dots, N$ and $\widehat{v}_k$, $\widehat{v}_i$ by v_k, v_i. Then we have

$$\begin{aligned} d_k^i &\le \frac{1}{k} + \langle \boldsymbol{\nabla} u_k, \boldsymbol{\nabla}(\phi_i v_k)\rangle_G + \langle u_k \boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla} v_k\rangle_G - \langle \boldsymbol{\nabla} u_k, v_k \boldsymbol{\nabla}\phi_i\rangle_G \\ &\le \frac{1}{k} + \varepsilon_k \|\boldsymbol{\nabla}(\phi_i v_k)\|_{\boldsymbol{L}^{p'(\cdot)}(G)} + |\langle u_k \boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla} v_k\rangle_G| + |\langle \boldsymbol{\nabla} u_k, v_k \boldsymbol{\nabla}\phi_i\rangle_G|. \end{aligned}$$

Since $\mathrm{supp}\boldsymbol{\nabla}\phi_i \subset B_i$, $u_k \to 0$ strongly in $L^{p(\cdot)}(B_i \cap G)$ and $\boldsymbol{\nabla} v_k \to \boldsymbol{\nabla} v_i$ weakly in $\boldsymbol{L}^{p'(\cdot)}(B_i \cap G)$, we have $|\langle u_k \boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla} v_k\rangle_G| \to 0$ as $k \to \infty$. Similarly, we have $|\langle u_k, v_k \boldsymbol{\nabla}\phi_i\rangle_G| \to 0$ as $k \to \infty$. Since $\varepsilon_k \to 0$ as $k \to \infty$ and $\|\boldsymbol{\nabla}(\phi_i v_k)\|_{\boldsymbol{L}^{p'(\cdot)}(G)} \le C$, we have $d_k \to 0$ as $k \to \infty$, so $\|\boldsymbol{\nabla}(\phi_i u_k)\|_{\boldsymbol{L}^{p(\cdot)}(B_i \cap \Omega)} \to 0$ as $k \to \infty$ for $i = 1, \dots, N$. Since $\Omega \subset \cup_{i=1}^N B_i'$, we have

$$\|\boldsymbol{\nabla} u_k\|_{\boldsymbol{E}^{p(\cdot)}(\Omega)} \le \sum_{i=1}^N \|\boldsymbol{\nabla}(\phi_i u_k)\|_{\boldsymbol{L}^{p(\cdot)}(B_i' \cap \Omega)} \le \sum_{i=1}^N \|\boldsymbol{\nabla}(\phi_i u_k)\|_{\boldsymbol{L}^{p(\cdot)}(B_i \cap \Omega)} \to 0$$

as $k \to \infty$. This is a contradiction. This completes the proof of Theorem 3.3.

5. POISSON EQUATION WITH THE NEUMANN CONDITIONS

In this section, as an application of Corollary 3.4 we consider the following Poisson equation with the Neumann boundary condition in a bounded domain Ω of $\mathbb{R}^d$ with a C^1-boundary $\partial\Omega$.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

First we consider the normal trace of a vector field.

Lemma 5.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^1-boundary $\partial\Omega$ and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. If $\boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ satisfies $\operatorname{div} \boldsymbol{v} \in (W^{1,p'(\cdot)}(\Omega))'$, then we can define $\boldsymbol{v} \cdot \boldsymbol{n} \in (\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega)))'$ by*

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{\partial\Omega} = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi dx + \langle \operatorname{div} \boldsymbol{v}, \varphi\rangle_{(W^{1,p'(\cdot)}(\Omega))', W^{1,p'(\cdot)}(\Omega)} \text{ for } \varphi \in W^{1,p'(\cdot)}(\Omega). \tag{5.2}$$

Proof. The definition (5.2) is well defined. Indeed, if $\psi \in W^{1,p'(\cdot)}(\Omega)$ satisfies $\psi = \varphi$ on $\partial\Omega$, then $\chi = \psi - \varphi \in W_0^{1,p'(\cdot)}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $W_0^{1,p'(\cdot)}(\Omega)$ from Proposition 2.5, there exists a sequence $\{\chi_j\} \subset \mathcal{D}(\Omega)$ such that $\chi_j \to \chi$ in $W_0^{1,p'(\cdot)}(\Omega)$. We see that

$$\begin{aligned}
&\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi dx + \langle \operatorname{div} \boldsymbol{v}, \chi\rangle_{(W^{1,p'(\cdot)}(\Omega))', W^{1,p'(\cdot)}(\Omega)} \\
&= \lim_{j\to\infty} \left(\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi_j dx + \langle \operatorname{div} \boldsymbol{v}, \chi_j\rangle_{(W^{1,p'(\cdot)}(\Omega))', W^{1,p'(\cdot)}(\Omega)} \right) \\
&= \lim_{j\to\infty} \left(\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi_j dx + \langle \operatorname{div} \boldsymbol{v}, \chi_j\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \right) \\
&= \lim_{j\to\infty} \left(\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi_j dx - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi_j dx \right) = 0.
\end{aligned}$$

Hence

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi dx + \langle \operatorname{div}\boldsymbol{v},\varphi\rangle_{(W^{1,p'(\cdot)}(\Omega))',W^{1,p'(\cdot)}(\Omega)} = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\nabla}\psi dx + \langle \operatorname{div}\boldsymbol{v},\psi\rangle_{(W^{1,p'(\cdot)}(\Omega))',W^{1,p'(\cdot)}(\Omega)}.$$

Let $\varphi \in \operatorname{Tr}(W^{1,p'(\cdot)}(\Omega))$. For any $\psi \in W^{1,p'(\cdot)}(\Omega)$ with $\psi = \varphi$ on $\partial\Omega$, using the generalized Hölder inequality, we have

$$\begin{aligned} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_{\partial\Omega}| &\le \left|\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\nabla}\psi dx\right| + |\langle \operatorname{div}\boldsymbol{v},\psi\rangle_{(W^{1,p'(\cdot)}(\Omega))',W^{1,p'(\cdot)}(\Omega)}| \\ &\le 2\|\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}\|\boldsymbol{\nabla}\psi\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{(W^{1,p'(\cdot)}(\Omega))'}\|\psi\|_{W^{1,p'(\cdot)}(\Omega)} \\ &\le 2(\|\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{(W^{1,p'(\cdot)}(\Omega))'})\|\psi\|_{W^{1,p'(\cdot)}(\Omega)}. \end{aligned}$$

From the definition of $\|\varphi\|_{\operatorname{Tr}(W^{1,p(\cdot)}(\Omega))}$, we see that $\boldsymbol{v}\cdot\boldsymbol{n} \in (\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))'$ and

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{(\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))'} \le 2(\|\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{(W^{1,p'(\cdot)}(\Omega))'}).$$

□

For the brevity of notations, we write

$$\begin{aligned} \langle f,\varphi\rangle_\Omega &= \langle f,\varphi\rangle_{(W^{1,p'(\cdot)}(\Omega))',W^{1,p'(\cdot)}(\Omega)} \text{ for } f \in (W^{1,p'(\cdot)}(\Omega))' \text{ and } \varphi \in W^{1,p'(\cdot)}(\Omega), \\ \langle g,\varphi\rangle_{\partial\Omega} &= \langle g,\varphi\rangle_{(\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))',\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega))} \text{ for } g \in (\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))' \\ &\qquad \text{and } \varphi \in \operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)). \end{aligned}$$

We are in a position to state a main theorem in this section.

Theorem 5.2. *Let Ω be a bounded domain with a C^1-boundary $\partial\Omega$ and $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. Assume that $f \in (W^{1,p'(\cdot)}(\Omega))'$ and $g \in (\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))'$ satisfying the compatibility condition*

$$\langle f,1\rangle_\Omega + \langle g,1\rangle_{\partial\Omega} = 0. \tag{5.3}$$

Then problem (5.1) *has a unique weak solution $[u] \in W^{1,p(\cdot)}(\Omega)/\mathbb{R}$, that is,*

$$\int_\Omega \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}\varphi dx = \langle f,\varphi\rangle_\Omega + \langle g,\varphi\rangle_{\partial\Omega} \textit{ for all } [\varphi] \in W^{1,p'(\cdot)}(\Omega)/\mathbb{R}. \tag{5.4}$$

We note that the right-hand side of (5.4) *is independent of the choice of representative of $[\varphi]$ according to the compatibility condition* (5.3)*.*

Furthermore, there exists a constant $C = C(p(\cdot),d,\Omega) > 0$ such that

$$\|[u]\|_{W^{1,p(\cdot)}(\Omega)} \le C(\|f\|_{(W^{1,p'(\cdot)}(\Omega))'} + \|g\|_{(\operatorname{Tr}(W^{1,p'(\cdot)}(\Omega)))'}). \tag{5.5}$$

For the proof of Theorem 5.2, we apply Corollary 3.4 and the following proposition.

Proposition 5.3. *Let X and M be two reflexive Banach spaces, and X' and M' be their dual spaces, respectively. Assume that $a : X \times M \to \mathbb{R}$ is continuous bilinear form, and let $A \in \mathcal{L}(X, M')$ be the continuous linear operator from X to M' defined by*

$$a(v, w) = \langle Av, w\rangle_{M',M} \text{ for } v \in X, w \in M,$$

and put $V = \ker A$. Then the following statements are equivalent.

(i) *There exists a constant $\beta > 0$ such that*

$$\inf_{0\neq w\in M} \sup_{0\neq v\in X} \frac{a(v,w)}{\|v\|_X\|w\|_M} \geq \beta.$$

(ii) *The operator $A : X/V \to M'$ is a topological isomorphism and $1/\beta$ is the continuity constant of A^{-1}.*

For the proof, see Amrouche and Seloula [1, Theorem 4.2].

Proof of Theorem 5.2.

Define a linear functional l on $W^{1,p'(\cdot)}(\Omega)/\mathbb{R}$ by

$$l([\varphi]) = \langle f, \varphi\rangle_\Omega + \langle g, \varphi\rangle_{\partial\Omega} \text{ for } [\varphi] \in W^{1,p'(\cdot)}(\Omega)/\mathbb{R}. \tag{5.6}$$

We note that from the compatibility condition (5.3), the right-hand side of (5.6) is independent of the choice of representative of $[\varphi]$. For any $\psi \in [\varphi]$,

$$\begin{aligned} |l([\varphi])| = |l([\psi])| &\leq \|f\|_{(W^{1,p'(\cdot)}(\Omega))'}\|\psi\|_{W^{1,p'(\cdot)}(\Omega)} + \|g\|_{(\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega)))'}\|\psi\|_{\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega))} \\ &\leq (\|f\|_{(W^{1,p'(\cdot)}(\Omega))'} + \|g\|_{(\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega)))'})\|\psi\|_{W^{1,p'(\cdot)}(\Omega)}. \end{aligned}$$

Therefore, we can see that $l \in (W^{1,p'(\cdot)}(\Omega)/\mathbb{R})'$ and

$$\|l\|_{(W^{1,p'(\cdot)}(\Omega)/\mathbb{R})'} \leq \|f\|_{(W^{1,p'(\cdot)}(\Omega))'} + \|g\|_{(\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega)))'}. \tag{5.7}$$

Since $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$, we can see that $W^{1,p(\cdot)}(\Omega)/\mathbb{R}$ is a reflexive Banach space and according to the Poincaré inequality (Theorem 2.6 (ii)), $\|[u]\|_{W^{1,p(\cdot)}(\Omega)/\mathbb{R}} \simeq \|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$.

We apply Proposition 5.3 with $X = W^{1,p(\cdot)}(\Omega)/\mathbb{R}$, $M = W^{1,p'(\cdot)}(\Omega)/\mathbb{R}$ and

$$a([u], [v]) = \int_\Omega \nabla u \cdot \nabla v dx \text{ for } [u] \in X, [v] \in M.$$

Using the generalized Hölder inequality, there exists a constant $C > 0$ such that

$$|a([u],[v])| = \left| \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx \right| \leq 2 \|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)} \leq C \|[u]\|_X \|[v]\|_M.$$

Thus a is a continuous bilinear form on $X \times M$. The operator $A : X \to M'$ is defined

$$\langle A([u]), [v]) \rangle_{M',M} = a([u],[v]) = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx \text{ for } [u] \in X, [v] \in M.$$

If $[u] \in \mathrm{Ker} A$, then $0 = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx$ for all $[v] \in M$. By Corollary 3.4, we have $\boldsymbol{\nabla} u = \mathbf{0}$ in Ω, so $u = \text{const.}$, that is, $[u] = 0$. Hence $\mathrm{Ker} A = \{0\}$. Proposition 5.3 (i) follows from Corollary 3.4. Thereby, $A : X \to M'$ is an isomorphism and $\|A^{-1} l\|_X \leq C_1 \|l\|_{M'}$ for all $l \in M'$, where C_1 is the constant of (3.3). Thus for the linear functional l in (5.6), there exists a unique $[u] \in X = W^{1,p(\cdot)}(\Omega)/\mathbb{R}$ such that $A([u]) = l$ and $\|[u]\|_X \leq C_1 \|l\|_{M'}$, that is, (5.4) holds and from (5.7), the estimate (5.5) holds.

Taking $\varphi \in \mathcal{D}(\Omega)$ as a test function of (5.4), we have

$$\langle f, \varphi \rangle_\Omega = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \varphi dx = \langle -\Delta u, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

So we have $-\Delta u = f$ in $(W^{1,p'(\cdot)}(\Omega))'$. Since $\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ and $-\mathrm{div}\, \boldsymbol{\nabla} u = -\Delta u = f \in (W^{1,p'(\cdot)}(\Omega)'$, for any $\varphi \in W^{1,p'(\cdot)}(\Omega)$, we have

$$\begin{aligned} \langle \boldsymbol{n} \cdot \boldsymbol{\nabla} u, \varphi \rangle_{\partial\Omega} &= \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \varphi dx + \langle \mathrm{div}\, \boldsymbol{\nabla} u, \varphi \rangle_{(W^{1,p'(\cdot)}(\Omega))', W^{1,p'(\cdot)}(\Omega)} \\ &= \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \varphi dx - \langle f, \varphi \rangle_\Omega = \langle g, \varphi \rangle_{\partial\Omega}. \end{aligned}$$

Hence $\boldsymbol{n} \cdot \boldsymbol{\nabla} u = \frac{\partial u}{\partial \boldsymbol{n}} = g$ in $(\mathrm{Tr}(W^{1,p'(\cdot)}(\Omega)))'$. This completes the proof of Theorem 5.2.

6. THE HELMHOLTZ DECOMPOSITION

As the next application of Corollary 3.4, we consider the Helmholz decomposition. Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^1-boundary $\partial\Omega$ and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. Then $\boldsymbol{E}^{p(\cdot)}(\Omega) = \{\boldsymbol{\nabla} v; v \in W^{1,p(\cdot)}(\Omega)\}$ and define

$$\begin{aligned} \boldsymbol{\mathcal{D}}_\sigma(\Omega) &= \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega); \mathrm{div}\, \boldsymbol{\varphi} = 0 \text{ in } \Omega\}, \\ \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega) &= \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } \boldsymbol{L}^{p(\cdot)}(\Omega). \end{aligned}$$

Lemma 6.1. *If $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ satisfies*

$$\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} dx = 0 \textit{ for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega),$$

then there exists $\pi \in W^{1,p(\cdot)}(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{\nabla} \pi$ in Ω.

Proof. If $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ satisfies

$$\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} dx = 0 \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega),$$

then clearly $\boldsymbol{u} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$ and by the simplified version of the de Rham theorem (cf. Aramaki [2], [3]), there exists $\pi \in L^{p(\cdot)}(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Since $\boldsymbol{\nabla}\pi = \boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, we see that $\pi \in W^{1,p(\cdot)}(\Omega)$. □

We state the following main theorem in this section.

Theorem 6.2 (The Helmholtz decomposition)**.** *Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^1-boundary $\partial\Omega$ and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. Then for every $\boldsymbol{f} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, there exist uniquely $\boldsymbol{u} \in \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega)$ and $\boldsymbol{\nabla}\pi \in \boldsymbol{E}^{p(\cdot)}(\Omega)$ such that*

$$\boldsymbol{f} = \boldsymbol{u} + \boldsymbol{\nabla}\pi,$$

that is, $\boldsymbol{L}^{p(\cdot)}(\Omega) = \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega) \oplus \boldsymbol{E}^{p(\cdot)}(\Omega)$. Furthermore,

$$\|\boldsymbol{\nabla}\pi\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \le C_1 \|\boldsymbol{f}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \textit{ and } \|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \le (C_1+1)\|\boldsymbol{f}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)},$$

where C_1 is the constant in Corollary 3.4.

Proof. Step 1. If $\boldsymbol{u} \in \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega)$, then $\langle \boldsymbol{u}, \boldsymbol{\nabla}\psi \rangle_{\boldsymbol{L}^{p(\cdot)}(\Omega), \boldsymbol{L}^{p'(\cdot)}(\Omega)} = 0$ for all $\boldsymbol{\nabla}\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega)$.

Indeed, if $\boldsymbol{u} \in \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega)$, then there exists a sequence $\{\boldsymbol{u}_j\}_{j=1}^\infty \subset \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\|\boldsymbol{u}_j - \boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \to 0$ as $j \to \infty$. Then for all $\boldsymbol{\nabla}\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega)$,

$$\langle \boldsymbol{u}_j, \boldsymbol{\nabla}\psi \rangle = -\langle \operatorname{div} \boldsymbol{u}_j, \psi \rangle = 0.$$

Hence

$$\langle \boldsymbol{u}, \boldsymbol{\nabla}\psi \rangle = \lim_{j\to\infty} \langle \boldsymbol{u}_j, \boldsymbol{\nabla}\psi \rangle = 0 \text{ for all } \boldsymbol{\nabla}\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega).$$

Step 2. $\boldsymbol{L}_\sigma^{p(\cdot)}(\Omega) \cap \boldsymbol{E}^{p(\cdot)}(\Omega) = \{\boldsymbol{0}\}$.

Indeed, let $\boldsymbol{u} = \boldsymbol{\nabla} v \in \boldsymbol{L}_\sigma^{p(\cdot)}(\Omega) \cap \boldsymbol{E}^{p(\cdot)}(\Omega)$. By Step 1, we can see that $0 = \langle \boldsymbol{u}, \boldsymbol{\nabla}\psi \rangle = \langle \boldsymbol{\nabla} v, \boldsymbol{\nabla}\psi \rangle$ for all $\boldsymbol{\nabla}\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega)$. By Corollary 3.4, $\boldsymbol{\nabla} v = \boldsymbol{0}$, so $\boldsymbol{u} = \boldsymbol{0}$.

Step 3. $\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega) \oplus \boldsymbol{E}^{p(\cdot)}(\Omega)$ is a closed subspace of $\boldsymbol{L}^{p(\cdot)}(\Omega)$.

First we note that $\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)$ and $\boldsymbol{E}^{p(\cdot)}(\Omega)$ are closed subspaces of $\boldsymbol{L}^{p(\cdot)}(\Omega)$. Suppose that $\boldsymbol{f}_j = \boldsymbol{u}_j + \boldsymbol{\nabla} v_j$, where $\boldsymbol{u}_j \in \boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)$ and $\boldsymbol{\nabla} v_j \in \boldsymbol{E}^{p(\cdot)}(\Omega)$, and $\boldsymbol{f}_j \to \boldsymbol{f}$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$ as $j \to \infty$. By Corollary 3.4 and Step 1,

$$\begin{aligned}
\|\boldsymbol{\nabla}(v_j - v_k)\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} &\leq C_1 \sup_{\boldsymbol{0}\neq\boldsymbol{\nabla}\psi\in\boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \boldsymbol{\nabla}(v_j - v_k), \boldsymbol{\nabla}\psi\rangle|}{\|\boldsymbol{\nabla}\psi\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}} \\
&= C_1 \sup_{\boldsymbol{0}\neq\boldsymbol{\nabla}\psi\in\boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \boldsymbol{f}_j - \boldsymbol{f}_k, \boldsymbol{\nabla}\psi\rangle|}{\|\boldsymbol{\nabla}\psi\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}} \\
&\leq C_1 \|\boldsymbol{f}_j - \boldsymbol{f}_k\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \to 0 \text{ as } j, k \to \infty.
\end{aligned}$$

Hence $\{\boldsymbol{\nabla} v_j\}$ is a Cauchy sequence in $\boldsymbol{E}^{p(\cdot)}(\Omega)$ which is a closed subspace of $\boldsymbol{L}^{p(\cdot)}(\Omega)$. Therefore, there exists $\boldsymbol{\nabla} v \in \boldsymbol{E}^{p(\cdot)}(\Omega)$ such that $\boldsymbol{\nabla} v_j \to \boldsymbol{\nabla} v$ in $\boldsymbol{E}^{p(\cdot)}(\Omega)$. Thus $\boldsymbol{u}_j = \boldsymbol{f}_j - \boldsymbol{\nabla} v_j \to \boldsymbol{u} := \boldsymbol{f} - \boldsymbol{\nabla} v$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$. Since $\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)$ is a closed subspace of $\boldsymbol{L}^{p(\cdot)}(\Omega)$, we see that $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)$ and $\boldsymbol{f} = \boldsymbol{u} + \boldsymbol{\nabla} v \in \boldsymbol{L}^{p(\cdot)}_\sigma(\Omega) \oplus \boldsymbol{E}^{p(\cdot)}(\Omega)$.

Step 4. $\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega) \oplus \boldsymbol{E}^{p(\cdot)}(\Omega) = \boldsymbol{L}^{p(\cdot)}(\Omega)$.

Suppose that $\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega) \oplus \boldsymbol{E}^{p(\cdot)}(\Omega) \subsetneq \boldsymbol{L}^{p(\cdot)}(\Omega)$. By the Hahn-Banach theorem, there exists $\boldsymbol{F}' \in (\boldsymbol{L}^{p(\cdot)}(\Omega))'$ such that $\boldsymbol{F}' \neq \boldsymbol{0}$ and $\boldsymbol{F}'\big|_{\boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)\oplus\boldsymbol{E}^{p(\cdot)}(\Omega)} = \boldsymbol{0}$. Since we can identify $(\boldsymbol{L}^{p(\cdot)}(\Omega))'$ with $\boldsymbol{L}^{p'(\cdot)}(\Omega)$, there exists $\boldsymbol{u} \in \boldsymbol{L}^{p'(\cdot)}(\Omega)$ such that $\boldsymbol{F}'(\boldsymbol{g}) = \langle \boldsymbol{u}, \boldsymbol{g}\rangle_\Omega$ for all $\boldsymbol{g} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ and $\|\boldsymbol{u}\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)} = \|\boldsymbol{F}'\|_{(\boldsymbol{L}^{p(\cdot)}(\Omega))'} > 0$. For any $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega) \subset \boldsymbol{L}^{p(\cdot)}_\sigma(\Omega)$, $\boldsymbol{F}'(\boldsymbol{\phi}) = 0 = \langle \boldsymbol{u}, \boldsymbol{\phi}\rangle$. By Lemma 6.1, there exists $\pi \in W^{1,p'(\cdot)}(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\pi$. Since

$$\langle \boldsymbol{\nabla}\pi, \boldsymbol{\nabla}\psi\rangle = \langle \boldsymbol{u}, \boldsymbol{\nabla}\psi\rangle = \boldsymbol{F}'(\boldsymbol{\nabla}\psi) = 0 \text{ for all } \boldsymbol{\nabla}\psi \in \boldsymbol{E}^{p(\cdot)}(\Omega),$$

it follows from Corollary 3.4 that $\boldsymbol{\nabla}\pi = \boldsymbol{0}$, so $\boldsymbol{u} = \boldsymbol{0}$. This is a contradiction.

Step 5. Estimate.

Let $\boldsymbol{f} = \boldsymbol{u} + \boldsymbol{\nabla}\pi$, $\boldsymbol{f} \in \boldsymbol{L}^{p(\cdot)}(\Omega), \boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}_\sigma(\Omega), \boldsymbol{\nabla}\pi \in \boldsymbol{E}^{p(\cdot)}(\Omega)$. By Corollary 3.4 and Step 1,

$$\begin{aligned}
\|\boldsymbol{\nabla}\pi\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} &\leq C_1 \sup_{\boldsymbol{0}\neq\boldsymbol{\nabla}\psi\in\boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \boldsymbol{\nabla}\pi, \boldsymbol{\nabla}\psi\rangle|}{\|\boldsymbol{\nabla}\psi\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}} \\
&= C_1 \sup_{\boldsymbol{0}\neq\boldsymbol{\nabla}\psi\in\boldsymbol{E}^{p'(\cdot)}(\Omega)} \frac{|\langle \boldsymbol{f}, \boldsymbol{\nabla}\psi\rangle|}{\|\boldsymbol{\nabla}\psi\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&\leq C_1 \|\boldsymbol{f}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}
\end{aligned}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} = \|(\boldsymbol{u} + \boldsymbol{\nabla}\pi) - \boldsymbol{\nabla}\pi\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} = \|\boldsymbol{f} - \boldsymbol{\nabla}\pi\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq (C_1 + 1)\|\boldsymbol{f}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

□

We have the following characterization of $\boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega)$.

Corollary 6.3. *We have* $\boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega) = \boldsymbol{W}^{p(\cdot)}(\Omega)$, *where*

$$\boldsymbol{W}^{p(\cdot)}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}(\Omega); \langle \boldsymbol{v}, \nabla\psi\rangle = 0 \textit{ for all } \nabla\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega)\}.$$

Proof. By Step 1 in the proof of Theorem 6.2, we see that $\boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega) \subset \boldsymbol{W}^{p(\cdot)}(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{W}^{p(\cdot)}(\Omega)$. Then by Theorem 6.2, we can write

$$\boldsymbol{u} = \boldsymbol{v} + \nabla\pi, \text{ where } \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega), \nabla\pi \in \boldsymbol{E}^{p(\cdot)}(\Omega).$$

Using again Step 1 in the proof of Theorem 6.2, we have

$$\langle \nabla\pi, \nabla\psi\rangle = \langle \boldsymbol{u}, \nabla\psi\rangle - \langle \boldsymbol{v}, \nabla\psi\rangle = 0 \text{ for all } \nabla\psi \in \boldsymbol{E}^{p'(\cdot)}(\Omega).$$

By Corollary 3.4, $\nabla\pi = \boldsymbol{0}$, so $\boldsymbol{u} = \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega)$. This means that $\boldsymbol{W}^{p(\cdot)}(\Omega) \subset \boldsymbol{L}^{p(\cdot)}_{\sigma}(\Omega)$. □

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