Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 8, Number 5 (2012), pp. 635-643

http://www.ripublication.com/gjpam.htm

L-Approx Bayesian Estimation of Hazard Function under Precautionary Loss of Weibull Model

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Abstract

This paper describes the unknown parameter, reliability and hazard function of the Weibull distribution-based on failure censored data. The scale parameter of the Weibull distribution is considered with a Natural Conjugate Gamma Prior under the shape parameter is known. The Weibull parameter, reliability and hazard function estimators are derived based on the Precautionary loss Function (PLF) function. Lindley's approximation(L-approx) is used to obtain Approximate Bayes estimators hazard function estimator. The result from Bayesian method is used to compare with Bayes and Maximum likelihood estimate (MLE) methods. The simulation shows that the results from Bayes is Robust for Approximate Bayesian method than MLE in terms of mean square error (MSE).

Keywords: Precautionary loss function, Maximum likelihood estimation, Bayesian estimation, Reliability, Hazard Rate, Lindley approximation(L-Approx), Weibull distribution.

INTRODUCTION

For Bayesian inference, a frequent choice of loss function is a Squared Error loss function. However, Bayesian estimation under this loss function is not frequently discussed, perhaps, because the estimators under symmetric and asymmetric loss function involve integral expressions, which are not analytically solvable. Therefore, one has to use the numerical techniques or certain approximation methods for the solution. One of the most suitable loss function Precautionary loss functions, which is asymmetrical. Lindley's approximation(L-Approx) is the method suitable for solving such problems. There has been a significant amount of research done in statistical inference of several distributions.

The Weibull distribution was introduced by the Swedish physicist Weibull [1959], it has been used in many different fields like material science, engineering, physics, chemistry, meteorology, medicine, pharmacy, economics and business, quality control, biology, geology and geography. The two parameters Weibull distribution is one of the most widely used lifetime models in reliability and survival analysis because of its various shapes of the probability density function(pdf) and its convenient representation of the Reliability and Hazard Function. The estimation of its parameters has been discussed by a number of authors.[Zakerzadeh and Jafari [2014], Doostparast [2006], Modarress, Kaminskiy and Krivtsov [2006], Sun and Berger[1998] and Kundu and Joarder [2006] and Kundu [2007]]. The properties of the Weibull distribution are best described in terms of the hazard function. This tells us how likely something is to fail given that it has survived so far. Weibull distribution has also been extensively used in life testing and reliability probability problems. Estimation and properties of the Weibull distribution is studied by many author's[Kao (1959)].

The Probability density function, Reliability and Hazard rate functions of Weibull distribution are given respectively as

$$f(x) = p\theta x^{(p-1)} \exp(-\theta\, x^{P}) \quad ; \quad x, \theta, p > 0 \qquad (1)$$

$$R(t) = exp(-\theta t^{p}) \quad ; \quad t > 0 \qquad (2)$$

$$H(t) = p\theta t^{(p-1)} \quad ; \quad t > 0 \qquad (3)$$

Where $'\theta'$ is the scale and 'p' is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta}, \theta) = k(\hat{\theta} - \theta)^2$ where $\hat{\theta}$ is the estimate of $\theta,$ the loss function is called quadratic weighed loss function if k=1, we have

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad (4)$$

This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson(1985). Canfield (1970), Basu and Ebrabimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators , especially when , low failure rates are being estimated. These estimators are very useful and simple asymmetric precautionary loss function is

$$\mathrm{L}(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}} \qquad (5)$$

In Bayesian Principle the unknown parameter θ which is treated as random variable assumes a probability distribution known as a priori of θ denoted by g(θ). To start the estimation of parameters we have the prior information about the unknown parameter θ. Different types of prior are used like Noninformative Prior, Natural conjugate prior. To simplify the calculations, statisticians use natural conjugate priors. Usually there is a natural parameter family of distributions such that the posterior distributions also belong to the same family. These priors make the computations much simpler.

The paper deals with the methods to obtain the approximate Bayes estimators of Hazard Function of the Weibull distribution by using Lindley approximation technique for failure censored samples. A bivariate prior density for the parameters, Precautionary Loss function (PLF) is used to obtain the approximate Bayes Estimators.

The Estimators

Let $x_1, x_2, \dots \dots \dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1). The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= (x_{(1)}, x_{(2)}, \dots \dots \dots \dots \dots, x_{(n)})$, where $x_{(i)}$ is the life time of the i^th^ item . Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$$L(x|\theta, p) = \frac{n!}{(n-r)!} (\mathrm{p}\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} \exp(-\delta\theta), \qquad (6)$$

where

$$\delta = \sum_{i=1}^{r} x_i^p + (n-r) x_r^p$$

The logarithm of the likelihood function is

$$\log L(x|\theta, p) \propto r \, log\, p + r \log \theta + (p-1) \sum_{i=1}^{r} \log x_i - \delta\theta, \qquad (7)$$

assuming that 'p' is known, the maximum likelihood estimator $\hat{\theta}_{ML}$of θ can be obtain by using equation (6) as

$$\hat{\theta}_{ML} = r/\delta \qquad (8)$$

If both the parameters p and θ are unknown their MLE's $\hat{p}_{ML}$and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta}{\delta\theta} \log L = \frac{r}{\theta} - \delta = 0 \;, \qquad (9)$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r} \log x_i - \theta\delta_1 = 0, \qquad (10)$$

where

$\delta_1 = \sum_{i=1}^{r} x_i^P \log x_i + (n-r) x_r^P \log x_r$, eliminating θ between the two equations of (9-10) and simplifying we get

$$\hat{p}_{ML} = \frac{r}{\delta^*} \qquad (11)$$

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} logx_i\right]$

Equation (10) may be solved for Newton- Raphson or any suitable iterative Method and this value is substituted in equation (8) by replacing with p get $\hat{p}$ as

$$\hat{\theta}_{ML} = \frac{\frac{r}{\hat{p}_{ML}} + \sum_{i=1}^{r} logx_i}{\sum_{i=1}^{r} x_i^{\hat{p}_{ML}} \; logx_i + (\text{n}-\text{r}) x_r^{\hat{p}_{ML}} \; logx_r}, \qquad (12)$$

The MLE's of R(t) and H(t) are given respectively by equation (2) and (3) after replacing θ and p by $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

Bayes Estimator of θ when shape Parameter 'p' is known.

If p is known assume gamma prior $\gamma(\alpha, \beta)$ as cojugate prior for θ as

$$g(\theta|\underline{x}) = \frac{\beta^\alpha}{\Gamma\alpha} (\theta)^{(\alpha+1)} \exp(-\beta\theta)\,; (\alpha, \beta) > 0, \theta > 0 \;, \qquad (13)$$

The posterior distribution of θ using equation (2) and (12) we get

$$h(\theta|\underline{x}) = \frac{(\delta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)} (\theta)^{(r+\alpha-1)} \exp\left(-\theta(\delta + \beta)\right), \qquad (14)$$

Under General Precautionary Loss Function, the Bayes estimator $\hat{\theta}_{BP}$ of θ using (5) and (13) given by

$$\hat{\theta}_{BP} = \left[\frac{(r+\alpha)(r+\alpha+1)}{(\delta+\beta)}\right]^{\frac{1}{2}} \qquad (15)$$

Bayes Estimator of R(t)

The posterior distribution of R using equation (5) and (13), is given as

$$h(R|t) = \frac{[\text{ c }(\delta+\beta)]^{(r+\alpha)}}{\Gamma(r+\alpha)} (-\log R)^{(r+\alpha-1)} R^{(\text{ c }(\delta+\beta) - 1)} dR; \qquad (16)$$

Where $c = t^{-p}$

The Bayes estimator of R(t) under precautionary loss function

$$\hat{R}_{BP} = \left[1 + \frac{2}{(\delta+\beta)}\right]^{(r+\alpha)} ; \qquad (17)$$

The Bayes Estimate of H(t)

The posterior density at H(t) using equation (3) and (13), is given as

$$h(H|t) = \frac{[(\delta+\beta)c^*]^{(r+\alpha)}}{\Gamma(r+\alpha)} \;. H^{(r+\alpha-1)} \exp\left(-c^*\text{H}(\delta + \beta)\right); \qquad (18)$$

Where $c^* = pt^{(p-1)}$

The Bayes estimator of H(t) under precautionary loss function

$$\widehat{H}_{BP} = \left[\frac{(r+\alpha)(r+\alpha+1)}{c^*(\delta+\beta)}\right]^{\frac{1}{2}} \; ; \qquad (19)$$

The Bayes estimators with θ and p unknown

The joint prior density of θ and p is given by

$$G(\theta|p) = g_1(\theta|p).\,g_2(p)$$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi} p^{-\xi} \theta^{(\xi-1)}.\exp\left[-\left(\frac{\theta}{p}+\frac{p}{\lambda}\right)\right] \; ; (\theta, p, \lambda, \xi) > 0, \quad (20)$$

where

$$g_1(\theta|p) = \frac{1}{\Gamma\xi} p^{-\xi}\, \theta^{(\xi-1)}.\exp\left[-\frac{\theta}{p}\right]; \qquad (21)$$

And

$$g_2(p) = \frac{1}{\lambda}\exp\left(-\frac{p}{\lambda}\right) \quad ; \qquad (22)$$

The joint posterior density of θ and p is

$$h^*(\theta, p|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\theta^{(\xi+1)}\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right](\mathrm{p}\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} e^{-p\theta}}{\iint \frac{1}{\lambda\Gamma\xi}p^{(r-\xi)}\theta^{(r+\xi+1)} \prod_{i=1}^{r} x_i^{(p-1)}.\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}+\mathrm{p}\theta\right\}\right]d\theta dp} \qquad (23)$$

Approximate Bayes Estimators

The Bayes estimators of a function $\mu = \mu(\theta, p)$ of the unknown parameter θ and p under squared error loss is the posterior mean

$$\hat{\mu}_{ABS} = E(\mu|\underline{x}) = \frac{\iint \mu(\theta,p)G(\theta,p|\underline{x})d\theta dp}{\iint G(\theta,p|\underline{x}).d\theta.dp} \; ; \qquad (24)$$

To evaluate (23) consider the method of Lindley approximation (Lindley (1980))

$$E(\mu(\theta,p)|\underline{x}) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}d\theta}; \qquad (25)$$

Where $(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function

The Lindley approximation for two parameter is given by

$$E(\hat{\mu}(\theta,p)|\underline{x}) = \mu(\theta,p) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}\left[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{\frac{2}{1}}\right], \quad (26)$$

where

$A = \sum_1^2 \sum_1^2 \mu_{ij}\sigma_{ij}$; $l_{\eta\epsilon} = \left(\delta^{(\eta+\epsilon)} l \middle| \delta\theta_1^{\eta}\delta\theta_2^{\epsilon}\right)$; where $(\eta+\epsilon) = 3$ for $i, j = 12$; $\rho_i = (\delta\rho|\delta\theta_i)$;

$$\mu_i = \frac{\delta\mu}{\delta\theta_i} \; ; \quad \mu_{ij} = \frac{\delta^2\mu}{\delta\theta_i\delta\theta_j} \; ; \forall i \neq j \; ;$$

$A_{ij} = \mu_i\sigma_{ij} + \mu_j\sigma_{ji}$; $B_{ij} = (\mu_i\sigma_{ii} + \mu_j\sigma_{ij})\sigma_{ii}$; $C_{ij} = 3\mu_i\sigma_{ii}\sigma_{ij} + \mu_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$;

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\theta_i \delta\theta_j}$.

All the function in above equations are evaluated at MLE of (θ_1, θ_2).In our case $(\theta_1\theta_2) = (\theta, p); So\ \mu(\theta) = \mu(\theta, p)$

To apply Lindley approximation (23), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1,2$,which can be shown to be

$\sigma_{11} = \frac{M}{D}$, $\sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}$,$\sigma_{22} = \frac{r}{D\,\theta^2}$, where $M = \left(\frac{r}{p^2} + \theta\delta_2\right)$; $D = \left[\frac{r}{\theta^2}\left(\frac{r}{p^2} + \theta^2\delta_2\right)\right]$;

$\delta_2 = \sum_{\mathrm{i=1}}^{\mathrm{r}} x_i^p\,(log x_i)^2 + (n-r)x_r^p(log x_r)^2$;

To evaluate ρ_i , take the joint prior $G(\theta|p)$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi} p^{-\xi}\theta^{(\xi-1)}.\exp\left[\left\{-\frac{\theta}{p} + \frac{p}{\lambda}\right\}\right] \ ; (\theta, p, \lambda, \xi) > 0,$$

$$\Rightarrow \rho = \log[G(\theta|p)] = constant - \xi log p - (\xi - 1)log\theta - \frac{\theta}{p} - \frac{p}{\lambda}$$

Therefore

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = \frac{(\xi-1)\theta}{\theta} - \frac{1}{p};\ \text{and}\ \rho_2 = \frac{\theta}{p^2} - \frac{1}{\lambda} - \frac{\xi}{p};$$

Further more

$$l_{21} = 0\,; l_{12} = -\delta_2\,; l_{03} = \frac{2r}{p^3} - \theta\delta_3; and\ \ l_{30} = \frac{2r}{\theta^3};$$

Where $\delta_3 = \sum_{\mathrm{i=1}}^{\mathrm{r}} x_i^p\,(log x_i)^3 + (n-r)x_r^p(log x_r)^3$

By substituting above values in equation (26), yields the Bayes estimator under PLF using Lindley approximation denoted by $\hat{\mu}_{ABS}$

$$\hat{\mu}_{ABPL} = E\big(\mu(\theta,p)\big) = \mu(\theta,p) + \mathrm{Q} + \mu_1 Q_1 + \mu_2 Q_2; \qquad (27)$$

Where Q $=\frac{1}{2}[\mu_{11}\sigma_{11} + \mu_{21}\sigma_{21} + \mu_{12}\sigma_{12} + \mu_{22}\sigma_{22}]$;

$$\mathrm{Q}_1 = \frac{1}{\theta^2 D^2}\Big[\frac{M\theta D}{p}(p(\xi-1)-1) + \frac{\theta^2\delta_1 D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\}$$
$$+ \frac{rM^2}{\theta} - \frac{rM\delta_1}{2} - \theta^2\delta_1^2\delta_2 + \frac{r^2}{p^3}\delta_1 - \frac{\theta r\delta_1\delta_3}{2}\Big];$$

$$\mathrm{Q}_2 = \frac{1}{\theta^2 D^2}\Big[\frac{\theta\,\delta_1 D}{p}(p(\xi-1)-\theta) + \frac{\mathrm{r}D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\}$$
$$+ \frac{rM\delta_1}{\theta} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\theta^2 p^3} - \frac{r^2\delta_3}{2\theta}\Big];$$

All the function of right hand side of the equation (27) are to be evaluated for $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

Approximate Bayes Estimate of Hazard Function Under Precautionary Loss function

with equations in (27), the different Approximate Bayes estimators Under PLF using Lindley's approximation given by

Special cases.

substituting $\mu(\theta, p) = H$ in equation (27), we get the Approximate Bayes estimator of Hazard rate H=H(t) as

$$\hat{H}_{ABPL} = H\left[1 + \frac{1}{p\theta\, D}\phi + \frac{Q_1}{\theta} + \frac{(1+plog\ t)}{p}Q_2\right] \text{ at } \left(\hat{\theta}_{ML}, \hat{p}_{ML}\right) \qquad (28)$$

where

$$\phi = \left[\delta_2 + \frac{2r logt}{\theta}(2 + plog\ t)\right]$$

Numerical Calculations and Comparison

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1. The values of and are generated from the equations (21-22) for given ξ=2, and λ=3, which comes out to be θ=0.32 and p=0.31. For these values of and p the Weibull random variates are generated.
2. Taking the different sizes of samples n=25 (10) 65 with failure censoring, MLE's, the Approximate Bayes estimators, and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the tables from (1), for t=2, R(t)=0.68, H(t)=0.024 and hyper parameters of prior distribution α =2 , and β =3.
3. Table(1) presents the Approximate Bayes estimator of hazard rate function H(t) of Weibull density under QLF and MLE's and the respective MSE's for different sample sizes. The estimators have lower efficiency for larger sample sizes. The under $\hat{H}_{APBL}$ under PLF are more efficient than others.

Table (1):-Mean and MSE'S of $R(t)$

$(\lambda = 2, \xi = 3, \theta = 0.32, p = 0.31, t = 2, R(t) = 0.68, H(t) = 0.24)$

n	r	$\hat{H}_{ML}$	$\hat{H}_{BPL}$	$\hat{H}_{ABPL}$
25	15	0.682737	0.928220	0.777489
		[2.77138e-05]	**[1.588131e-05]**	**[1.02131 e-06]**
35	20	0.656174	0.931363	0.791756
		[4.40161e-06]	**[1.319123e-06]**	**[1.08097e-06]**
45	25	0.617673	0.943833	0.794756
		[2.69877e-05]	**[1.878563e-05]**	**[1.01193 e-05]**
55	30	0.810926	0.899141	0.766703
		[1.85604e-06]	**[1.112163e-05]**	**[1.00741 e-06]**
65	35	0.819139	0.619528	0.768143

		[2.3901e-06]	**[8.5114e-06]**	**[1.6458e-06]**

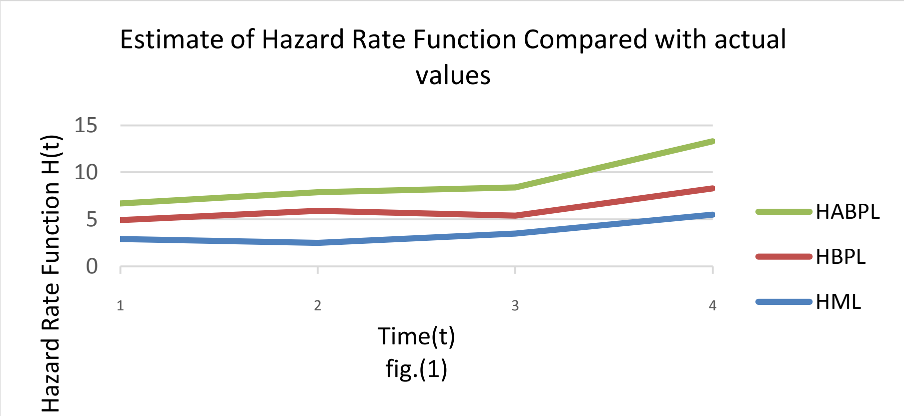

fig.(1)

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