Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 17, Number 2 (2021), pp. 371-387

http://www.ripublication.com/gjpam.htm

Application of Homotopy Analysis Method for Solving the ONCHOCERCIASIS (RIVERBLINDNESS)

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Abstract

In this paper, a mathematical model of Human population and vector population with constant recruitment rates is considered. The model is based on assumption that the susceptible individuals become infected through contact with infected vector (blackfies) and the susceptible vectors (blackfies) become infected through contact with infected individuals. We used Homotopy Analysis Method to establish the existence of optimal control law for Onchocerciasis. This validates the Pontryagins Maximum Principle that is used to obtain the control law in [16]. Also, the validity of homotopy analysis method (HAM) for Onchocerciasis by the application of the homotopy analysis in [10], [12], [12], [13] and [14].

Keyword: Onchocerciasis, Mathematical model, Optimal Control Law, Homotopy Deformation, Zeroth order deformation, Pontryagins Maximum Principle, Homotopy Analysis Method.

1. INTRODUCTION

Homotopy is a fundamental concept in topology and differential geometry. It deals with deformation such as in the case of a circle which can be continuously deformed into an ellipse. The advantages of Homotopy Analysis Method (HAM) are seen in heat transfer [1], Hirota-Satsuma coupled Kdv equation [2], non -linear problems [3, 14, 15], perturbation methods [4,7], analytic approximations and analytic solutions [10, 11, 12]. Onchocerciasis, also known as river blindness, is a parasitic disease caused by Onchocerca volvulus, which is transmitted by a parasite known as simulium species (blackflies) that breed in fast flowing stream and it is endemic in tropical countries like Nigeria. Our aim is to apply Homotopy Analysis Method to solve the Onchocerciasis and provide homotopy analytic solutions to the SI model. This paper is organized as follows. In section 2, we have the statement of the problem with existence of Optimal control law and homotopy deformation were stated. In section 3, the Homotopy Analysis Method (HAM) [1], [8], [13], is applied to construct approximation solution of (2.1). In section 4, our numerical findings are reported and demonstrated the accuracy of the proposed method by considering the exact solution in the absence of drugs and treatment controls. Finally, conclusions are stated in the last section.

2. STATEMENT OF THE PROBLEM

Let us consider a model of a constant population size. The model is based on assumption that the susceptible individuals become infected through contact with infected vector (blackflies) and the susceptible vectors (blackflies) become infected through contact with infected individuals. The infected individuals thus generates infections by shedding the pathogen into the susceptible vector (blackflies) which susceptible individuals subsequently come into contact with it. In the model to be consider below, we have the following variables and parameters which are:

H_S : Human Susceptible

H_I : Human Infected

V_S : Vector Susceptible

V_I : Vector Infected

Ψ : Human recruitment rate

φ : Vector recruitment rate

ξ : Movement rate from H_S to H_I

α : Recovery rate from H_I to H_S

ρ : Transmission rate parameters from (H_I to V_S)

δ : Transmission rate parameters from (V_I to H_S)

b : Bitting rate

m : Movement rate from V_S to V_I

μ : Natural death rate

β : Death rate for vector

σ : Artificial death rate (cause as a result of chronic river blindness diseases)

$u_i, 1 \leq i \leq 4$: Control parameters

a : Prevention treatment

b : Proportion of Mectizan

c : Insecticides

d : Insecticides and others

The corresponding model equations to the compartments H_S, H_I, V_S and V_I are:

$$\begin{aligned} \dot{H}_S &= \Psi + \alpha H_I - b\delta H_S V_I - \mu H_S + a u_1 H_S \\ \dot{H}_I &= b\delta H_S V_I - \alpha H_I - \sigma H_I - \mu H_I + b u_2 H_I \\ \dot{V}_S &= \varphi - b\rho H_I V_S - m V_S - \beta V_S + c u_3 V_S \\ \dot{V}_I &= b\rho H_I V_S - \beta V_I + d u_4 V_I \end{aligned} \tag{2.1}$$

subject to the initial conditions;

$$H_S(0) = H_{S_0}, \;\; H_I(0) = H_{I_0}, \;\; V_S(0) = V_{S_0}, \;\; V_I(0) = V_{I_0},$$

2.1. Existence of Optimal Control Law

Since the model monitors changes in human and blackfly populations with control function, thus we establish the existence of the optimal control law. Now, transform the system of equations (2.1) into matrix notation, we have

$$A = \begin{bmatrix} -(\mu) & \alpha & 0 & 0 \\ 0 & -(\alpha+\sigma+\mu) & 0 & 0 \\ 0 & 0 & -(\beta+m) & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix}$$

$$B = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$$x = \begin{bmatrix} H_S \\ H_I \\ V_S \\ V_I \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 H_S \\ u_2 H_I \\ u_3 V_S \\ u_4 V_I \end{bmatrix}$$

$f^T(x(t)) = (-b\delta H_S V_I, b\delta H_S V_I, -b\rho H_I V_S, b\rho H_I V_S)$
where A and B are real constant matrices of appropriate dimensions.

$x \in \mathbb{R}^4$ is the state vector

$u \in \mathbb{R}^4$ is the control vector

f is a non-linear vector function

where $f(0) = 0$, $x_0 \in \mathbb{R}^4$, $x_t \in \mathbb{R}^4$ are the initial and final states, respectively. Thus, the non-linear control system can be written as:

$$\dot{x} = Ax(t) + Bu(t) + f(x(t)) \tag{2.2}$$

$$x(t_o) = x_o, x(T) = x_T$$

where A and B are real constant matrices of appropriate dimensions, $x \in \mathbb{R}^4$ is the state vector and $u \in \mathbb{R}^4$ is the control vector, f is a non-linear vector function where $f(0) = 0$ and $x_o \in \mathbb{R}^4$ is the initial state and $x_T \in \mathbb{R}^4$ is the final state. The optimal control law $u^*(t)$ that minimizes the performance index:

$$J = \frac{1}{2}\int_{t_o}^{T} (x^T(t)Qx(t) + u^T Ru(t))dt \tag{2.3}$$

where $QQ^T = wI_n$,$w \in \mathbb{R}$, and I_n is identity matrix of order n, subject to the system (2.2) where Q is the weight matrix and R is the balancing cost matrix are positive semi-definite and positive definite matrices, respectively.
Given the Pontryagin's maximum principle [16], consider

$$H = x^T(t)Qx(t) + u^T(t)Ru(t) + \dot{x}(t)\lambda(t) \tag{2.4}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}$$

$$-\dot{\lambda} = Qx(t) + A^T\lambda(t) + f_x(x(t))\lambda(t)$$
$$\frac{\partial H}{\partial u} = Ru(t) + B^T\lambda(t)$$

at the optimal time t^*, we obtain the optimal control law

$$u^*(t) = -R^{-1}B^T\lambda^*(t) \tag{2.5}$$

putting (2.5) into (2.2), we obtain

$$\dot{x} = Ax(t) - BR^{-1}B^T\lambda(t) + f(x(t))$$
$$\dot{\lambda} = -Qx(t) - A^T\lambda(t) - f_x(x(t))\lambda(t) \tag{2.6}$$
$$x(t_o) = x_o, x(t_f) = x_T$$

where $f_x = \frac{\partial f}{\partial x}$ and $\lambda \,\epsilon\, \mathbb{R}^4$.

Theorem 1. *Suppose there exists an Hamiltonian function H and adjoint variable λ such that the system of equation in (2.4) holds. Then, there exists an optimal control law $u^*(t)$ such that*

$$u^*(t) = -R^{-1}B\lambda^*(t)$$

where B is a diagonal matrix.

2.2. Homotopy Deformation

In this section, we shall carry out homotopy deformation on the system of equation (2.6). Suppose that there exists functions $k(x)$ and $g(\lambda)$ such that equation (2.6) becomes

$$k(x) = \dot{x}(t) - Ax(t) + BR^{-1}B^T\lambda - f(x(t)) \tag{2.7}$$
$$g(\lambda) = \dot{\lambda}(t) + Qx(t) + A^T\lambda(t) + f_x(x(t))\lambda(t) \tag{2.8}$$

Now, according to Liao [9], considering the zeroth-order deformation

$$\mathfrak{H}(x, \lambda; p) = (1-p)k(x) + pg(\lambda) \tag{2.9}$$

putting (2.7) and (2.8) into equation (2.9), we obtain

$$\mathfrak{H}(x, \lambda; p) = (1-p)(\dot{x}(t) - Ax(t) + BR^{-1}B^T\lambda^T - f(x(t)))+ \\ p[\dot{\lambda}(t) + Qx(t) + A^T\lambda(t) + f_x(x(t))\lambda(t)] \tag{2.10}$$

where $p \in [0,1]$ is called the embedding parameter. Note that $\mathfrak{H}(x,\lambda;p)$ depends on the embedding parameter $p \in [0,1]$. Especially, when $p=0$, we have

$$\mathfrak{H}(x,\lambda;0)=k(x),$$

and when $p=1$, it gives

$$\mathfrak{H}(x,\lambda;1)=g(\lambda),$$

So, as the embedding parameter $p \in [0,1]$ increases from 0 to 1, the real function $\mathfrak{H}(x,\lambda;p)$ varies continuously from $k(x)$ to $g(\lambda)$. In topology, $\mathfrak{H}(x,\lambda;p)$ is called a homotopy, while $k(x)$ and $g(\lambda)$ are called homotopic, denoted by

$$\mathfrak{H}: k(x)\sim g(\lambda).$$

the equation (2.9), can be rewritten in the form

$$\mathfrak{H}(x,\lambda;p)=k(x)+p[g(\lambda)-k(x)], \qquad p \in [0,1]$$

and differentiating partially with respect to parameter p, we have

$$\frac{\partial\mathfrak{H}(x,\lambda;p)}{\partial p}=g(\lambda)-k(x) \tag{2.11}$$

which describes the continuous deformation from $k(x)$ to $g(\lambda)$.
Let subtract equation (2.7) from (2.8) then, we have

$$g(\lambda)-k(x)=\dot{\lambda}(t)+Qx(t)+A^T\lambda(t)+f_x(x(t))\lambda(t)-\dot{x}(t)-Ax(t)+BR^{-1}B^T\lambda-f(x(t))=0$$

given that

$$\frac{\partial h(x,\lambda;p)}{\partial p}=0$$

$$\begin{aligned}g(\lambda)-k(x)=\dot{\lambda}(t)+Qx(t)+A^T\lambda(t)+f_x(x(t))\lambda(t)-&\\ \dot{x}(t)-Ax(t)+BR^{-1}B^T\lambda-f(x(t))&=0\end{aligned}$$

$$g(\lambda)-k(x)=\frac{d}{dt}(\lambda)-Qx(t)+A^T\lambda(t)+f_x(x(t))\lambda(t)-\frac{d}{dt}(x)+Ax(t)+BR^{-1}B^T\lambda-f(x(t))$$

$$\frac{d}{dt}(\lambda-x)+(A-Q)x(t)+(A^T-BR^{-1}B^T\lambda-f_x(x(t)))\lambda(t)+f(x(t))=0$$

let

$$f(x)=(Q+A)x$$

$$f_x(x) = (Q + A)$$

then

$$\frac{d}{dt}(\lambda - x) + (A^T - BR^{-1}B^T + (Q + A))\lambda(t) = 0$$

$$\frac{d\lambda}{dt} + (A^T - BR^{-1}B^T + (Q + A))\lambda(t) = \frac{dx}{dt}$$

by integrating both sides, integrating factors

$$\frac{d}{dt}(\lambda e^{-(A^T+BR^{-1}B^T+(Q+A))t}) = \frac{dx}{dt}(e^{-(A^T+BR^{-1}B^T+(Q+A))t})$$

integrating both sides

$$\int_0^T \left(\frac{d}{dt}(\lambda e^{-(A^T+BR^{-1}B^T+(Q+A))t})\right) dt = \int_0^T \frac{dx}{dt}(e^{-(A^T+BR^{-1}B^T+(Q+A))t})dt \quad (2.12)$$

Let $k = (A^T + BR^{-1}B^T + (Q + A)$. By solving the right hand side equation, we have

$$\int_0^T \frac{dx}{dt} e^{-(k)t} dt = e^{-(k)T} x(T) - x(0) - \int_0^T x(t)(-k)e^{-(k)t} dt$$

$$\int_0^T \frac{dx}{dt} e^{-(k)t} dt = e^{-(k)T} x(T) - x(0) + k\int_0^T x(t)e^{-(k)t} dt$$

$$\int_0^T \frac{dx}{dt} e^{-(k)t} dt = e^{-(k)T} x(T) - \left[x(0) - k\int_0^T x(t)e^{-(k)t} dt\right] \quad (2.13)$$

If

$$x(0) - k\int_0^T x(t)e^{-(k)t} dt = 0$$

then,

$$x(0) = k\int_0^T x(t)e^{-(k)t} dt$$

therefore, equation (2.13) becomes

$$\int_0^T \frac{dx}{dt} e^{-(k)t} dt = e^{-(k)T} x(T)$$

By solving the left hand side equation (2.12), we have

$$\int_0^T \frac{d}{dt}\lambda(t) e^{-(k)t} dt = e^{-(k)T}\lambda(T) - \lambda(0) - \int_0^T \lambda(t)(-k)e^{-(k)t} dt$$

$$\int_0^T \frac{d}{dt}\lambda(t) e^{-(k)t} dt = e^{-(k)T}\lambda(T) - \lambda(0) + k\int_0^T \lambda(t)e^{-(k)t} dt$$

$$\int_0^T \frac{d}{dt}\lambda(t)e^{-(k)t}dt = e^{-(k)T}\lambda(T) - \left[\lambda(0) - k\int_0^T \lambda(t)e^{-(k)t}dt\right] \quad (2.14)$$

If

$$\lambda(0) - k\int_0^T \lambda(t)e^{-(k)t}dt = 0$$

then,

$$\lambda(0) = k\int_0^T \lambda(t)e^{-(k)t}dt$$

therefore, equation (2.14) becomes

$$\int_0^T \frac{d}{dt}\lambda(t)e^{-(k)t}dt = \lambda(T)e^{-(k)T}$$

therefore,

$$\int_0^T \frac{d}{dt}\lambda(t)e^{-(k)t}dt = \int_0^T \frac{dx(t)}{dt}e^{-(k)t}dt$$

$$\lambda(T)e^{-(k)T} = x(T)e^{-(k)T}$$

which implies that

$$\lambda(T) = x(T), \quad for \;\; all \;\; t$$

Theorem 2. *If the adjoint variable λ is such that the system of the equations in (2.4) holds, there exists an optimal control law $u^*(t)$ such that*

$$u^*(t) = -R^{-1}B\lambda^*(t)$$

if $\lambda(t)$ is homotopic to $x(t)$ then the optimal control law $u^*(t)$ is such that

$$u^*(t) = -R^{-1}Bx^*(t)$$

Theorem 3. *If the adjoint variable $\lambda(t;p)$ with embedded parameter is based on the kind of continuous mappings $\lambda(t;p) \longrightarrow \lambda(t)$, then $\lambda(t;p) = x(t)$ as $p \longrightarrow 1$.*

Proof 1. *If*

$$\lambda(t;p) \longrightarrow \lambda(t)$$

as $p = 0$

$$\lambda(t;0) = \lambda_0(t)$$

Also, as $p \longrightarrow 1$

$$\lambda(t;1) = \lambda(t)$$

then,

$$\lambda(t;1) = \lambda(t) = x(t)$$

therefore, as $p \longrightarrow 1$

$$\lambda(t;p) = x(t)$$

thus, $\lambda(t)$ is homotopic to $x(t)$

3. HOMOTOPY ANALYSIS METHOD

Since the homotopy deformation carried out in section 2.2 on the system of equations (2.6) is true, we apply Homotopy Analysis Method to our model equations (2.1) by choosing

$$H_S(0) = N_S, \;\; H_I(0) = N_I, \;\; V_S(0) = N_E, \;\; V_I(0) = N_R$$

as initial approximation of $H_S(t), H_I(t), V_S(t)$ and $V_I(t)$. Let $p \in \;\; [0, 1]$ be the embedding parameter. The HAM is based on the kind of continuous mappings,

$$H_S(t,p) \to H_S(t), \;\; H_I(t,p) \to H_I(t), \;\; V_S(t,p) \to V_S(t), \;\; V_I(t,p) \to V_I(t)$$

as the p increases from 0 to 1, $H_S(t,p), \;\; H_I(t,p), \;\; V_S(t,p)$ and $V_I(t,p)$ varies from the initial guess to the solution of the original equation. We also choose some linear operators as

$$\mathcal{L}_1[H_i(t,p)] = \frac{\partial H_i(t,p)}{\partial t}$$
$$\mathcal{L}_2[V_i(t,p)] = \frac{\partial V_i(t,p)}{\partial t}$$

where $i = S, I$
with the

$$\mathcal{L}_1[C_1] = \mathcal{L}_2[C_2] = \mathcal{L}_3[C_3] = \mathcal{L}_4[C_4] = 0$$

where C_1, C_2, C_3 and C_4 are constant. We define the non-linear operators

$$N_1[H_S(t;p)] = \frac{\partial H_S(t,p)}{\partial t} - \Psi - \alpha H_I(t;p) + b\delta H_S(t;p)V_I(t;p) + \mu H_S(t;p) - au_1 H_S(t;p)$$

$$N_2[H_I(t;p)] = \frac{\partial H_I(t,p)}{\partial t} - b\delta H_S(t;p)V_I(t;p) + \alpha H_I(t;p) + \sigma H_I(t;p) + \mu H_I(t;p) - bu_2 H_I(t;p)$$

$$N_3[V_S(t;p)] = \frac{\partial V_S(t,p)}{\partial t} - \varphi - b\rho H_I(t;p)V_S(t;p) + mV_I(t;p) + \beta V_S(t;p) - cu_3 V_S(t;p)$$

$$N_4[V_I(t;p)] = \frac{\partial V_I(t,p)}{\partial t} - b\rho H_I(t;p)V_S(t;p) + \beta V_I(t;p) - du_4 V_I(t;p)$$

Using the embedding parameter p, we have a family of equations

$$(1-p)\mathcal{L}[H_S(t;p) - H_{S_0}(t)] = phJ(t)N_1[H_S(t;p)]$$

$$(1-p)\mathcal{L}[H_I(t;p) - H_{I_0}(t)] = phJ(t)N_2[H_I(t;p)]$$

$$(1-p)\mathcal{L}[V_S(t;p) - V_{S_0}(t)] = phJ(t)N_3[V_S(t;p)]$$

$$(1-p)\mathcal{L}[V_I(t;p) - V_{I_0}(t)] = phJ(t)N_4[V_I(t;p)]$$

subject to the initial conditions

$$H_S(0;p) = H_{S_0},\ \ H_I(0;p) = H_{I_0},\ \ V_S(0;p) = V_{S_0},\ \ V_I(0;p) = V_{I_0}$$

We expand $H_i(t;p), i = S, I$, $V_i(t;p), i = S, I$ by a power series of the embedding parameter p as follows

$$H_S(t;p) = H_{S_0}(t) + \sum_{m=1}^{\infty} H_{S_m}(t)p^m$$

$$H_I(t;p) = H_{I_0}(t) + \sum_{m=1}^{\infty} H_{I_m}(t)p^m$$

$$V_S(t;p) = V_{S_0}(t) + \sum_{m=1}^{\infty} V_{S_m}(t)p^m$$

$$V_I(t;p) = V_{I_0}(t) + \sum_{m=1}^{\infty} V_{I_m}(t)p^m$$

where

$$\begin{aligned} H_{S_m}(t) &= \frac{1}{m!}\frac{\partial^m H_S(t;p)}{\partial p^m}|_{p=0} \\ H_{I_m}(t) &= \frac{1}{m!}\frac{\partial^m H_I(t;p)}{\partial p^m}|_{p=0} \\ V_{S_m}(t) &= \frac{1}{m!}\frac{\partial^m V_S(t;p)}{\partial p^m}|_{p=0} \\ V_{I_m}(t) &= \frac{1}{m!}\frac{\partial^m V_I(t;p)}{\partial p^m}|_{p=0} \end{aligned} \tag{3.1}$$

then, at $p = 1$ the series becomes

$$H_S(t) = H_{S_0}(t) + \sum_{m=1}^{\infty} H_{S_m}(t)$$

$$H_I(t) = H_{I_0}(t) + \sum_{m=1}^{\infty} H_{I_m}(t)$$

$$V_S(t) = V_{S_0}(t) + \sum_{m=1}^{\infty} V_{S_m}(t)$$

$$V_I(t) = V_{I_0}(t) + \sum_{m=1}^{\infty} V_{I_m}(t)$$

from the mth-order deformation of the system equation (3.1), we have

$$\mathcal{L}\left[H_{S_m}(t)-\chi_m H_{S_{m-1}}(t)\right]=hJ(t)\mathbb{R}_m(H_{S_{m-1}}(t)) \tag{3.2}$$

$$\mathcal{L}\left[H_{I_m}(t)-\chi_m H_{I_{m-1}}(t)\right]=hJ(t)\mathbb{R}_m(H_{I_{m-1}}(t)) \tag{3.3}$$

$$\mathcal{L}\left[V_{S_m}(t)-\chi_m V_{S_{m-1}}(t)\right]=hJ(t)\mathbb{R}_m(V_{S_{m-1}}(t)) \tag{3.4}$$

$$\mathcal{L}\left[V_{I_m}(t)-\chi_m V_{I_{m-1}}(t)\right]=hJ(t)\mathbb{R}_m(V_{I_{m-1}}(t)) \tag{3.5}$$

with initial conditions

$$H_{S_m}(0)=H_{I_m}(0)=V_{S_m}(0)=V_{I_m}(0)=0$$

where

$$\mathbb{R}_m(H_{S_{m-1}})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_S(t;p)]}{\partial p^{m-1}}$$

$$\mathbb{R}_m(H_{I_{m-1}})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_I(t;p)]}{\partial p^{m-1}}$$

$$\mathbb{R}_m(V_{S_{m-1}})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[V_S(t;p)]}{\partial p^{m-1}}$$

$$\mathbb{R}_m(V_{I_{m-1}})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_I(t;p)]}{\partial p^{m-1}}$$

and

$$\chi_m=\left\{\begin{array}{ll}0, & m\le 1\\ 1, & m>1\end{array}\right. \tag{3.6}$$

It is good to use $h=-1$. Using $J(t)=1$, the mth-order deformation of the system equations (3.2)-(3.5) for $m\ge 1$ becomes

$$H_{S_m}(t)=\chi_m H_{S_{m-1}}(t)-\int_0^t[\dot{H}_{S_{m-1}}(\tau)-\Psi-\alpha H_{I_{m-1}}(\tau)+b\delta H_{S_{m-1}}(\tau)V_{I_{m-1}}(\tau)$$
$$-.5in+\mu H_{S_{m-1}}(\tau)-au_1H_{S_{m-1}}(\tau)]d\tau$$

$$H_{I_m}(t)=\chi_m H_{I_{m-1}}(t)-\int_0^t[\dot{H}_{I_{m-1}}(\tau)-b\delta H_{S_{m-1}}(\tau)V_{I_{m-1}}(\tau)$$
$$+\alpha H_{I_{m-1}}(\tau)+\sigma H_{I_{m-1}}(\tau)+\mu H_{I_{m-1}}(\tau)-bu_2H_{I_{m-1}}(\tau)]d\tau$$

$$V_{S_m}(t)=\chi_m V_{S_{m-1}}(t)-\int_0^t[\dot{V}_{S_{m-1}}(\tau)-\varphi-b\rho H_{I_{m-1}}(\tau)V_{S_{m-1}}(\tau)$$
$$+mV_{S_{m-1}}(\tau)+\beta V_{S_{m-1}}(\tau)-cu_3V_{S_{m-1}}(\tau)]d\tau$$

$$V_{I_m}(t) = \chi_m V_{I_{m-1}}(t) - \int_0^t [\dot{V}_{I_{m-1}}(\tau) - b\rho H_{I_{m-1}}(\tau) V_{S_{m-1}}(\tau) + \beta V_{I_{m-1}}(\tau) - du_4 V_{I_{m-1}}(\tau)] d\tau$$

4. NUMERICAL RESULT AND DISCUSSION

Consider the system of equation (2.1), as follows:

$$\dot{H}_S - \Psi - \alpha H_I + b\delta H_S V_I + \mu H_S - au_1 H_S = 0$$
$$\dot{H}_I - b\delta H_S V_I + \alpha H_I + \sigma H_I + \mu H_I - bu_2 H_I = 0$$
$$\dot{V}_S - \varphi + b\rho H_I V_S + m V_S + \beta V_S - cu_3 V_S = 0$$
$$\dot{V}_I - b\rho H_I V_S + \beta V_I - du_4 V_I = 0$$

where $H_S(0) = H_{S_0},\ \ H_I(0) = H_{I_0},\ \ V_S(0) = V_{S_0},\ \ V_I(0) = V_{I_0},$
Also consider:

$$\begin{aligned} L[H_S(t,p)] &= \frac{\partial H_S(t,p)}{\partial t} - \Psi - \alpha H_I + \mu H_S - au_1 H_S \\ L[H_I(t,p)] &= \frac{\partial H_I(t,p)}{\partial t} + \alpha H_I + \sigma H_I + \mu H_I - bu_2 H_I \\ L[V_S(t,p)] &= \frac{\partial V_S(t,p)}{\partial t} - \varphi + m V_S + \beta V_S - cu_3 V_S \\ L[V_I(t,p)] &= \frac{\partial V_I(t,p)}{\partial t} + \beta V_I - du_4 V_I \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} N[H_S(t,p)] &= \frac{\partial H_S(t,p)}{\partial t} - \Psi - \alpha H_I(t,p) + b\delta H_S(t,p) V_I(t,p) + \mu H_S(t,p) - au_1 H_S(t,p) \\ N[H_I(t,p)] &= \frac{\partial H_I(t,p)}{\partial t} - b\delta H_S V_I + \alpha H_I + \sigma H_I + \mu H_I - bu_2 H_I = 0 \\ N[V_S(t,p)] &= \frac{\partial V_S(t,p)}{\partial t} - \varphi + b\rho H_I V_S + m V_S + \beta V_S - cu_3 V_S \\ N[V_I(t,p)] &= \frac{\partial V_I(t,p)}{\partial t} - b\rho H_I V_S + \beta V_I - du_4 V_I \end{aligned} \tag{4.2}$$

then the m-th order deformation of the system of equation (4.1) is:

$$H_{S_m} = \chi_m H_{s_{m-1}} + h \int_0^t R_m(H_{s_{m-1}}) dt, \quad m \geq 1$$

$$H_{I_m} = \chi_m H_{I_{m-1}} + h\int_0^t R_m(H_{I_{m-1}})dt, \quad m \geq 1$$
$$V_{S_m} = \chi_m V_{S_{m-1}} + h\int_0^t R_m(V_{S_{m-1}})dt, \quad m \geq 1 \tag{4.3}$$
$$V_{I_m} = \chi_m V_{I_{m-1}} + h\int_0^t R_m(V_{I_{m-1}})dt, \quad m \geq 1$$

where

$$R_m(H_{S_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_S(t;p)]}{\partial p^{m-1}}$$
$$R_m(H_{I_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_I(t;p)]}{\partial p^{m-1}}$$
$$R_m(V_{S_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[V_S(t;p)]}{\partial p^{m-1}} \tag{4.4}$$
$$R_m(V_{I_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_I(t;p)]}{\partial p^{m-1}}$$

and

$$\chi_m = \{ \begin{matrix} 0, & m \leq 1 \\ 1, & m > 1 \end{matrix} \tag{4.5}$$

Merging equation (4.2), (4.3) and (4.4) we have the following equation

$$H_{S,m+1}(t) = \chi_m H_{S,m}(t) - \int_0^t \Big[\dot{H}_{S,m}(\tau) - \Psi - \alpha H_{I,m}(\tau) + b\delta H_{S,m}(\tau)V_{I,m}(\tau)$$
$$+\mu H_{S,m}(\tau) - au_1 H_{S,m}(\tau)\Big]\, d\tau$$
$$H_{I,m+1}(t) = \chi_m H_{I,m}(t) - \int_0^t \Big[\dot{H}_{I,m}(\tau) - b\delta H_{S,m}(\tau)V_{I,m}(\tau)$$
$$+\alpha H_{I,m}(\tau) + \sigma H_{I,m}(\tau) + \mu H_{I,m}(\tau) - bu_2 H_{I,m}(\tau)\Big]\, d\tau$$
$$V_{S,m+1}(t) = \chi_m V_{S,m}(t) - \int_0^t \Big[\dot{V}_{S,m}(\tau) - \varphi - b\rho H_{I,m}(\tau)V_{S,m}(\tau) \tag{4.6}$$
$$+mV_{S,m}(\tau) + \beta V_{S,m}(\tau) - cu_3 V_{S,m}(\tau)\Big]\, d\tau$$
$$V_{I,m+1}(t) = \chi_m V_{I,m}(t) - \int_0^t \Big[\dot{V}_{I,m}(\tau) - b\rho H_{I,m}(\tau)V_{S,m}(\tau)$$
$$+\beta V_{I,m}(\tau) - du_4 V_{I,m}(\tau)\Big]\, d\tau$$

where $\psi = 0.2568$, $\xi = 0.2192$, $\alpha = 0.1027$, $\beta = 0.0305$, $\rho = 0.2641$, $\delta = 0.1369$, $\mu = 0.0506$, $\sigma = 0.007$, $m = 0.2687$, $H_S = 0.5$, $H_I = 0.5$, $V_S = 0.5$, $V_I = 0.5$. putting all these parameter values into the system (2.1) we have

$$H_{S,m+1}(t) = H_{S,m}(t) - \int_0^t [\dot{H}_{S,m}(\tau) - \Psi + \alpha H_{I,m}(\tau)$$
$$+b\delta H_{S,m}(\tau)V_{I,m}(\tau) + \mu H_{S,m}(\tau) + au_1 H_{S,m}\tau]d\tau$$

$$H_{I,m+1} = H_{I,m}(t) - \int_0^t [\dot{H}_{I,m}(\tau) - b\delta H_{S,m}(\tau)V_{I,m}(\tau) + \alpha H_{I,m}(\tau) + \sigma H_{I,m}(\tau) + \mu H_{I,m}(\tau) - bu_2 H_{I,m}(\tau)]d\tau$$

$$V_{S,m+1} = V_{S,m}(t) - \int_0^t [\dot{V}_{S,m}(\tau) - \varphi + b\rho H_{I,m}(\tau)V_{S,m}(\tau) + mV_{S,m}(\tau) + \beta V_{S,m}(\tau) + cu_3 V_{S,m}(\tau)]d\tau$$

$$V_{I,m+1} = V_{I,m}(t) - \int_0^t [\dot{V}_{I,m}(\tau) - b\rho H_{I,m}(\tau)V_{S,m}(\tau) + \beta V_{I,m}(\tau) - du_4 V_{I,m}(\tau)]ds$$

We get the exact solution for H_S, H_I, V_S and V_I in presence of drugs and control measures.

T(Years)	H_S	H_I	V_S	V_I
0	0.5000000000	0.50000000000	0.500000000	0.5000000000
1	0.5191666542	0.4751821521	0.49791259641	0.4946425418
2	0.5374018231	0.4512126451	0.49526125874	0.4894321341
3	0.5374018234	0.4518663556	0.49526128567	0.48943185418
4	0.5712901231	0.4099109524	0.48849412578	0.47932721422
5	0.5870407234	0.3909685612	0.48448104195	0.4744083924
6	0.6020532312	0.3732668957	0.48011008258	0.4695606152
7	0.6163685693	0.3567213721	0.47542293352	0.4647726323
8	0.630026577	0.34125302224	0.47045801251	0.46003453875
9	0.643062224	0.3267882887	0.46525053239	0.45533778236
10	0.6555099142	0.313258412	0.45983286587	0.45067502569
11	0.7537824294	0.297179497	0.44995275214	0.424049386457

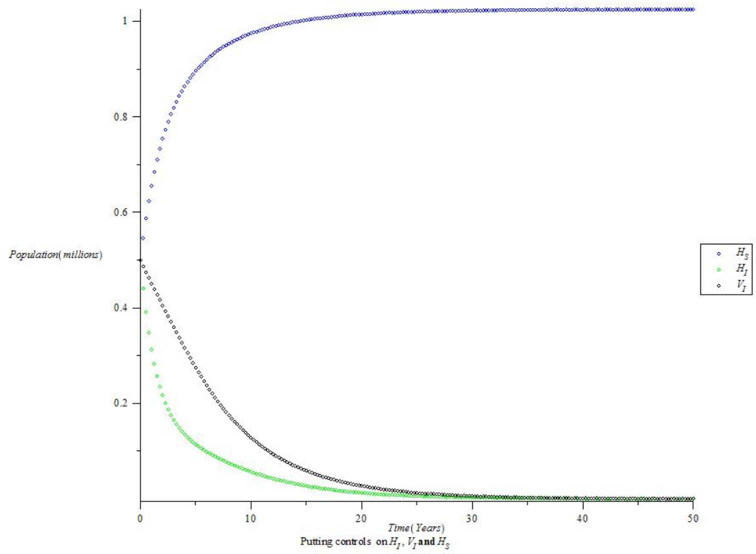

Figure 1: Presence of drugs and control measures for H_S, H_I and V_I

Also we get the exact solution for H_S, H_I, V_S and V_I in the absence of drugs and treatment controls.

T(Years)	H_S	H_I	V_S	V_I
0	0.50000000000	0.5000000000	0.50000000000	0.50000000000
1	0.52934031	0.519166151	0.499938891458914	0.538625937553464
2	0.58620527	0.5288636845	0.499858526960259	0.562184915566287
3	0.64074142	0.5322325285	0.497589274825741	0.561559856244223
4	0.66716999	0.558256235	0.488789652566552	0.566099391410710
5	0.71841302	0.562315324	0.485947947943911	0.572801272108869
6	0.76758927	0.552364235	0.480710490281711	0.574461073081517
7	0.82356822	0.574256235	0.474961930481187	0.622015900898057
8	0.75235452	0.632342961	0.468776084589429	0.63466587793962
9	0.72365420	0.6523252544	0.462219358438643	0.6563203866081
10	0.65232354	0.696878213	0.455351266668502	0.7255163390864
11	0.64258241	0.725463254	0.448225242081361	0.9462810433853

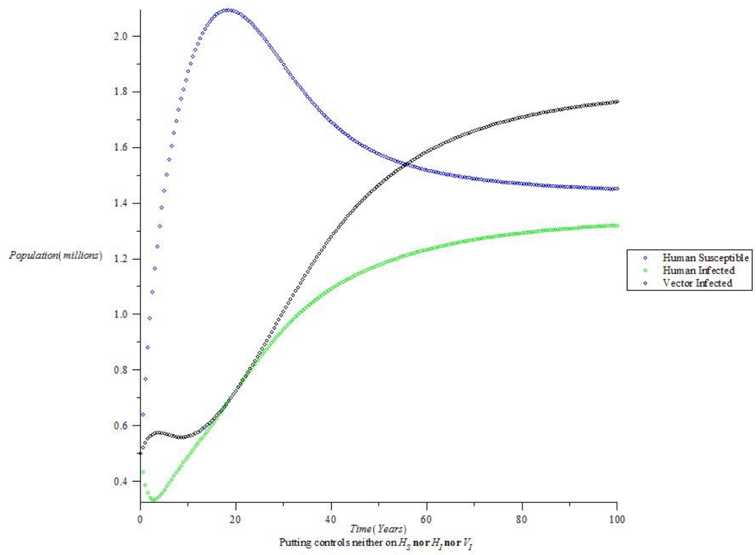

Figure 2: Absent of drugs and treatment control H_S, H_I and V_I

5. CONCLUSION

This study highlighted the compartmental mathematical model of Onchocerciasis (Riverblindness) with optimal control strategies to examing or investigate the transmission dynamics of the disease. Using Homotopy Analysis Method coupled with Pontryagins Maximum Principle condition for existence of the optimality system was derived. The result of the optimal control suggest that combination of preventive measures, treatment and spraying of insecticides at a time will eliminate the disease. Using Homotopy analysis method, conditions were derived for disease free and endemic equilibrium. Numerical analyses of the model were carried out to investigate the effect of some key disease on the transmission dynamics of the disease using Mapple 17 software. The result of our numerical analyses reveal that the vector bitting rate and the contact rate contribute most significantly to the transmission of the disease.

REFERENCES

1. Abbasbandy, S.: The application of the homotopy analysis method to nonlinear equations arising in heat transfer. Phys. Lett. A. 360, 109 â€“ 113 (2006).

2. Abbasbandy, S.: The application of homotopy analysis method to solve a generalized Hirota - Satsuma coupled KdV equation. Phys. Lett. A. 361, 478 â€“ 483(2007).

3. Armstrong, M.A.: Basic Topology (Undergraduate Texts in Mathematics). Springer,New York (1983).

4. Cole, J.D.: Perturbation Methods in Applied Mathematics. Blaisdell Publishing Company, Waltham (1992).

5. Fitzpatrick, P.M.: Advanced Calculus. PWS Publishing Company, New York (1996).

6. Hilton, P.J.: An Introduction to Homotopy Theory. Cambridge University Press, Cambridge (1953).

7. Hinch, E.J.: Perturbation Methods. In series of Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge (1991).

8. Liang, S.X., Jeffrey, D.J.: Comparison of homotopy analysis method and homotopy perturbation method through an evalution equation. Commun. Nonlinear Sci. Numer. Simulat. 14, 4057 â€“ 4064 (2009).

9. Liao, S.J.: The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. PhD dissertation, Shanghai Jiao Tong University (1992).

10. Liao, S.J.: A kind of approximate solution technique which does not depend upon small parameters (II) â€“ An application in fluid mechanics. Int. J. Nonlin. Mech.32, 815 â€“ 822 (1997).

11. Liao, S.J.: An explicit, totally analytic approximation of Blasius viscous flow problems. Int. J. Nonlin. Mech. 34, 759 â€“ 778 (1999a).

12. Liao, S.J.: A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate. J. Fluid Mech. 385, 101 â€“ 128 (1999b).

13. Liao, S.J.: Beyond Perturbation â€“ Introduction to the Homotopy Analysis Method.Chapman & Hall/ CRC Press, Boca Raton (2003b).

14. Liao, S.J.: An optimal homotopy-analysis approach for strongly nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simulat. 15, 2003 â€“ 2016 (2010b).

15. Liao, S.J., Tan, Y.: A general approach to obtain series solutions of nonlinear differential equations. Stud. Appl. Math. 119, 297 â€“ 355 (2007).

16. Wendell Fleming, Raymond Rishel: Deterministic and Stochastic Optimal Control- Springer-Verlag New York (1975)