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Generalized Focal Curves of Frenet Curves in Three-Dimensional Euclidean Space

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Abstract

Obtaining a new curve from a given parameterized curve in n-dimensional Euclidean space (in short curve in $\mathbb{E}^n$ or nD curve) is an important tool used in differential geometry, computer graphics and computer aided design. An example of such a creation is the unique focal curve of any Frenet nD curve. The aim of this paper is to present a three-stage construction for a new associate curve in the Euclidean 3-space to any Frenet $3D$ curve of class C^4. The new curve is called a generalized focal curve of a Frenet curve in $\mathbb{E}^3$. The first stage of the proposed construction defines a unit-speed Frenet $4D$ curve from a given unit-speed Frenet $3D$ curve. The second stage of the construction is finding the focal $4D$ curve of the considered Frenet $4D$ curve. The final third stage determines a generalized focal curve of the given Frenet $3D$ curve as a projection of the obtained focal $4D$ curve into the Euclidean space $\mathbb{E}^3$. The construction is accompanied by an algorithm and three illustrative examples in which the generalized focal curves are expressed in an explicit form.

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Key Words and Phrases: curvature; torsion; focal curve; Frenet curve; generalized focal curve.

1. INTRODUCTION

A Frenet curve in n-dimensional Euclidean space $\mathbb{E}^n$ ($n \geq 3$) is a parametrically defined curve of class C^n whose derivatives up to order n are linearly independent for any parameter value. Such a curve determines a Frenet frame consisting of a unit tangent vector function $\mathbf{T}$ and $n-1$ unit normal vector functions $\mathbf{N}_i$, $i = 1, \ldots, n-1$. For any parameter value, n-tuple of vectors $\mathbf{T}$, $\mathbf{N}_1, \ldots, \mathbf{N}_{n-1}$ form a positively oriented orthonormal basis of $\mathbb{R}^n$. Any Frenet curve specifies $(n-1)$ real-valued functions $\kappa_1 > 0, \ldots, \kappa_{n-2} > 0$, $\kappa_{n-1} \neq 0$ called curvatures (also known as Euclidean or Frenet curvatures). The relations between the Frenet frame and the curvatures are expressed by so-called Frenet-Serret equations. Two approaches for determining the Frenet frame and computing the curvatures of the Frenet curves were presented by Gluck in [8] and by Banchoff and Lovett in [2].

Finding the focal curve of a given Frenet curve is an important task in differential geometry of curves. The focal curve is a locus of centers of osculating hyperspheres of the original Frenet curve. A parametrization of the focal curve can be obtained by the parametrization, the unit normal vectors, and the curvatures of the original curve. These and other significant properties of the focal curves were discussed in detail by Uribe-Vargas [20]. In particular, he proved that the focal curves of the self-congruent curves are also self-congruent curves. There are additional results that investigate the focal curves of various classes of Frenet curves. Encheva and Georgiev [6] showed that the focal curve of any self-similar curve in $\mathbb{E}^n$ is also a self-similar curve in $\mathbb{E}^n$. Öztürk and Arslan [15] verified that the focal curve of a curve in $\mathbb{E}^n$ with constant curvature ratios is another curve with constant curvature ratios. Öztürk et al. [16] discussed focal curves of so-called k-slant helices in $\mathbb{E}^{m+1}$.

In the last decade, different associated curves of a Frenet curve in $\mathbb{E}^3$ have been extensive studied. Choi and Kim [3] investigated some integral curves called a principal (binormal)-direction curve and a principal (binormal)-donor curve, which are associated curves of a Frenet curve in $\mathbb{E}^3$. They obtained the relationships between curvature and torsion of the original curve and the curvature and torsion of these associated curves. They also characterized special curves, known as general helices and slant helices, in terms of principal-direction curves and principal-donor curves. Recently, Deshmukh et al. [5] derived relationships between several classes of Frenet curves in $\mathbb{E}^3$. One of their interesting results is that there exists a unique circular helix, i.e. curve with constant non-zero curvature and torsion, which is associated with a non-helical curve in $\mathbb{E}^3$ of class C^k, $k \geq 4$, i.e. a curve with a non-constant ratio of torsion to curvature.

In this way, new characterizations of slant helices, Salkowski curves, spherical curves and rectifying curves are presented. Associated curves of Frenet curves in $\mathbb{E}^3$ are also discussed in [4], [12], [14] and [17].

The purpose of this paper is to present a new construction of a parameterized curve in $\mathbb{E}^3$ from a given unit-speed Frenet curve of class C^4 in $\mathbb{E}^3$. This construction is divided into three consecutive parts. In the beginning, the given curve in $\mathbb{E}^3$ generates a unit-speed Frenet curve in $\mathbb{E}^4$. After that, the unique focal curve in $\mathbb{E}^4$ of the obtained curve in $\mathbb{E}^4$ can be determined parametrically. Finally, the orthogonal projection of this focal curve into a fixed hyperplane in $\mathbb{E}^4$ is identified with a curve in $\mathbb{E}^3$ called a generalized focal curve of the given Frenet curve in $\mathbb{E}^3$. The transfer from a Frenet curve in $\mathbb{E}^3$ to a Frenet curve in $\mathbb{E}^4$ depends on the type of the original curve. We examine the following three types of Frenet curves in $\mathbb{E}^3$:
1. Non-helical curves, i.e. Frenet curves with non-constant ratio of torsion to curvature;
2. General helices different from circular helices, i.e. Frenet curves with a constant ratio of torsion to curvature, a non-constant curvature and a non-constant torsion;
3. Circular helices,i.e. Frenet curves with a constant non-zero curvature and a constant non-zero torsion.

The proposed construction is explained by a detailed algorithm and suitable examples.

The paper is organized as follows. In the next section, we recall definitions and basic facts concerning the Frenet curves and their focal curves in the Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$. Then, a three-stage construction for determining a parametrized space curve from a given unit-speed Frenet curve in $\mathbb{E}^3$ is presented. The main part of this construction, i.e. the transfer from a Frenet curve in $\mathbb{E}^3$ to a Frenet curve in $\mathbb{E}^4$, is discussed in detail. The construction is accompanied by an algorithm which covers all cases of Frenet curve in $\mathbb{E}^3$. In the sequel, three illustrative examples are given. The paper ends with concluding remarks.

2. FOCAL CURVES OF CURVES IN THE EUCLIDEAN SPACES WITH DIMENSIONS 3 AND 4

In this section we recall some basic definitions, computations and constructions for parameterized curves in the Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$. These curves are also known as $3D$ curves and $4D$ curves.

2.1. Frenet curves in the Euclidean three-space

We consider the Euclidean 3-space $\mathbb{E}^3$ as an affine space with an associated vector space $\mathbb{R}^3$. Any point of $\mathbb{E}^3$ can be represented by its position column vector from $\mathbb{R}^3$. For any two column vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$, the scalar (or dot) product $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$ and the vector cross product $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$ are well known operation.
Let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a curve defined on an interval $I \subseteq \mathbb{R}$ by a vector parametric equation

$$\boldsymbol{\alpha}(t) = \big(x(t),\ y(t),\ z(t)\big)^T, \quad t \in I. \tag{1}$$

Suppose that the coordinate functions $x(t)$, $y(t)$, $z(t)$ have continuous derivatives up to order 3. If the vectors $\boldsymbol{\alpha}'(t) = \big(x'(t), y'(t), z'(t)\big)^T$, $\boldsymbol{\alpha}''(t) = \big(x''(t), y''(t), z''(t)\big)^T$, $\boldsymbol{\alpha}'''(t) = \big(x'''(t), y'''(t), z'''(t)\big)^T$ are linearly independent for any $t \in I$, then $\boldsymbol{\alpha}$ is called a Frenet curve in $\mathbb{E}^3$. For such a curve the norm $\|\boldsymbol{\alpha}'(t)\| = \sqrt{\boldsymbol{\alpha}'(t) \cdot \boldsymbol{\alpha}'(t)}$ is a positive real number, the vector cross product $\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)$ is a vector different form $(0,0,0)^T$ and $\big[\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big] \cdot \boldsymbol{\alpha}'''(t) \neq 0$. Therefore there are determined the curvature of $\boldsymbol{\alpha}$ by

$$\kappa(t) = \frac{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3} > 0 \tag{2}$$

and the torsion of $\boldsymbol{\alpha}$ by

$$\tau(t) = \frac{\big[\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big] \cdot \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|^2} \neq 0 \tag{3}$$

Moreover, there are three vectors

$$\begin{aligned} \mathbf{t}(t) &= \tfrac{1}{\|\boldsymbol{\alpha}'(t)\|}\boldsymbol{\alpha}'(t) \\ \mathbf{n}(t) &= \tfrac{1}{\|\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)\|} \cdot \tfrac{1}{\|\boldsymbol{\alpha}'(t)\|} \left\{\big[\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big] \times \boldsymbol{\alpha}'(t)\right\} \\ \mathbf{b}(t) &= \tfrac{1}{\|\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)\|} \big[\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big] \end{aligned} \tag{4}$$

defined at any point $\boldsymbol{\alpha}(t)$ of the curve. These vectors form a so-called Frenet frame which is a positively oriented orthonormal basis, i.e. $\|\mathbf{t}(t)\| = \|\mathbf{n}(t)\| = \|\mathbf{b}(t)\| = 1$, $\mathbf{t}(t) \cdot \mathbf{n}(t) = \mathbf{t}(t) \cdot \mathbf{b}(t) = \mathbf{b}(t) \cdot \mathbf{n}(t) = 0$, and $\big[\mathbf{t}(t) \times \mathbf{n}(t)\big] \cdot \mathbf{b}(t) = 1$. A focal curve (or an evolute) of the Frenet curve $\boldsymbol{\alpha}$ given by (1) is the unique space curve $\boldsymbol{\alpha}^f : I \longrightarrow \mathbb{E}^3$ with a parametrization

$$\boldsymbol{\alpha}^f(t) = \boldsymbol{\alpha}(t) + \frac{1}{\kappa(t)}\mathbf{n}(t) + \left(\frac{1}{\kappa(t)}\right)' \frac{1}{\|\boldsymbol{\alpha}'(t)\|\tau(t)}\mathbf{b}(t). \tag{5}$$

If the Frenet curve $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ is a unit-speed curve, i.e. $\|\boldsymbol{\alpha}'(t)\| = 1$ for any $t \in I$, then usually the parameter "t" is replaced by the parameter "s" and the vector equation

(1) is called an arc-length parametrization of $\boldsymbol{\alpha}$. In this case, the formulas (2), (3), (4) and (5) can be written in a simpler form. In detail, the curvature and torsion are

$$\kappa(s) = \|\boldsymbol{\alpha}''(s)\| > 0 \quad \text{and} \quad \tau(s) = \frac{\left[\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\right] \cdot \boldsymbol{\alpha}'''(s)}{\|\boldsymbol{\alpha}''(s)\|^2} \neq 0, \tag{6}$$

respectively, the Frenet frame is

$$\mathbf{t}(s) = \boldsymbol{\alpha}'(s),\ \mathbf{n}(s) = \frac{1}{\|\boldsymbol{\alpha}''(s)\|}\boldsymbol{\alpha}''(s),\ \mathbf{b}(s) = \frac{1}{\|\boldsymbol{\alpha}''(s)\|}\left[\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\right]. \tag{7}$$

Further, the Frenet-Serret equations for the unit-speed curve $\boldsymbol{\alpha}$

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \quad \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \quad \mathbf{b}'(s) = -\tau(s)\mathbf{n}(t). \tag{8}$$

hold. In addition, the parametrization (5) of the focal curve of the unit-speed curve $\boldsymbol{\alpha}$ becomes

$$\boldsymbol{\alpha}^f(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\mathbf{n}(s) + \left(\frac{1}{\kappa(s)}\right)' \frac{1}{\tau(s)}\mathbf{b}(s). \tag{9}$$

There are two important classes of Frenet curves in $\mathbb{E}^3$: the class of circular helices and its extension the class of general helices. Any circular helix has a constant curvature and a constant torsion, and possesses an arc-length parametrization in the form

$$\boldsymbol{\alpha}(s) = \big(p\cos(as), p\sin(as), bs\big)^T, \quad \text{where} \quad a \neq 0, b \neq 0, p \neq 0,\ a^2p^2 + b^2 = 1$$

(see [11, Ch. 2]). An alternative parametrization of an arbitrary unit-speed circular helix is

$$\begin{aligned} &\boldsymbol{\alpha}(s) = \left(\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), \sqrt{\tfrac{1-b^2}{a^2}}\sin(as), bs\right)^T, \\ &\text{where } a \in \mathbb{R} \setminus \{0\},\ b \in \{(-1,0) \cup (0,1)\}. \end{aligned} \tag{10}$$

For such a curve, the curvature and the torsion are

$$\kappa(s) = \sqrt{a^2\left(1 - b^2\right)} \quad \text{and} \quad \tau(s) = ab, \tag{11}$$

respectively, and the right hand side of (9) consists of the first two summands.

Any general helix, or curve of constant slope, is a curve in $\mathbb{E}^3$ whose tangent vectors make a constant angle with a fixed unit vector. Without loss of generality, this fixed vector can be considered to be a vector parallel to the z-axis. Then, the most common parametrization of a unit-speed general helix is

$$\begin{aligned} &\boldsymbol{\alpha}(s) = \big(x(s),\ y(s),\ bs\big)^T, \quad s \in I, \\ &\text{where } I \subseteq \mathbb{R} \text{ is an interval,} \\ &b \in \{(-1,0) \cup (0,1)\} \text{ is a constant, and} \\ &(x'(s))^2 + (y'(s))^2 + b^2 = 1 \text{ for any } s \in I. \end{aligned} \tag{12}$$

A Frenet curve in $\mathbf{E}^3$ is a general helix if and only if it has a constant ratio of the torsion to the curvature (see Lemas 8.18 and 8.19 in [9]). Obviously, all circular helices form a subclass of the class of general helices. Relations between plane curves and general helices are obtained by Izumiya and Takeuchi [10]. Parametric representations of general helices with a given constant angle between the tangent vectors and a fixed unit vector, and a given curvature function are studied by Ali [1]. Some examples of unit-speed general helices are given in [7]. More detailed description of differential geometry of space curves can be found in [2] and [9].

2.2. Frenet curves in the Euclidean four-space

We assume that the Euclidean four-space $\mathbb{E}^4$ is an affine space whose associated vector space $\mathbb{R}^4$ consists of real column four-vectors. In other words, any point $X \in \mathbb{E}^4$ can be identified with its position vector $\boldsymbol{X} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$. The vector space $\mathbb{R}^4$ is equipped with the standard scalar (or dot) product of two vectors. If $\mathbf{U} = (u_1, u_2, u_3, u_4)^T$ and $\mathbf{V} = (v_1, v_2, v_3, v_4)^T$ are four-dimensional vectors, then the scalar product of $\mathbf{U}$ and $\mathbf{V}$ is the real number $\mathbf{U} \cdot \mathbf{V} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ and the norm of the vector $\mathbf{U}$ is $\|\mathbf{U}\| = \sqrt{\mathbf{U} \cdot \mathbf{U}} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$.

Let $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ be a unit-speed curve of class C^4 defined on an interval $I \subseteq \mathbb{R}$ by a vector parametric equation

$$\boldsymbol{\beta}(s) = \big(b_1(s),\ b_2(s),\ b_3(s), b_4(s)\big)^T, \quad s \in I. \tag{13}$$

This means that the coordinate functions $b_i(s)$, $i = 1, 2, 3, 4$ have continuous derivatives up to order 4 and $\boldsymbol{\beta}'(s) = \frac{d}{ds}\boldsymbol{\beta}(s)$ is a unit vector for any $s \in I$. Furthermore, if the vectors $\boldsymbol{\beta}'(s)$, $\boldsymbol{\beta}''(s) = \frac{d}{ds}\boldsymbol{\beta}'(s)$, $\boldsymbol{\beta}'''(s) = \frac{d}{ds}\boldsymbol{\beta}''(s)$, and $\boldsymbol{\beta}^{(4)}(s) = \frac{d}{ds}\boldsymbol{\beta}'''(s)$ are linearly independent for any $s \in I$, then the curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ is called a unit-speed Frenet curve. Equivalently, the curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ given by (13) is a Frenet curve if and only if $\det\Big(\boldsymbol{\beta}'(s), \boldsymbol{\beta}''(s), \boldsymbol{\beta}'''(s), \boldsymbol{\beta}^{(4)}(s)\Big) \neq 0$ for any $s \in I$. Following [8], we may consider four unit vector functions $\mathbf{T}(s) = \boldsymbol{\beta}'(s)$, $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $\mathbf{N}_3(s)$ and three real-valued curvature functions $K_1(s) \neq 0$,$K_2(s) \neq 0$, $K_3(s)$ satisfying following conditions for any value of the parameter s:

(i) the vectors $\mathbf{T}(s)$, $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $\mathbf{N}_3(s)$ form a positively oriented, ordered basis of $\mathbb{R}^4$;

(ii) the so-called Frenet-Serret equations

$$\begin{aligned}\mathbf{T}'(s) &= K_1(s)\mathbf{N}_1(s), \quad \mathbf{N}_1'(s) = -K_1(s)\mathbf{T}(s) + K_2(s)\mathbf{N}_2(s),\\ \mathbf{N}_2'(s) &= -K_2(s)\mathbf{N}_1(s) + K_3(s)\mathbf{N}_3(s), \quad \mathbf{N}_3'(s) = K_3(s)\mathbf{N}_2(s).\end{aligned} \tag{14}$$

hold. According to [2] the Frenet frame along $\boldsymbol{\beta}$ consists of the unit tangent vector $\mathbf{T}(s)$, the first unit normal vector $\mathbf{N}_1(s)$, the second unit normal vector $\mathbf{N}_2(s)$ and the third unit normal vector $\mathbf{N}_3(s)$. As shown in [8], these vectors can be obtained by the Gram-Schmidt orthogonalization process applied to the vectors $\boldsymbol{\beta}'(s)$, $\boldsymbol{\beta}''(s)$, $\boldsymbol{\beta}'''(s)$, and $\boldsymbol{\beta}^{(4)}(s)$. Then, the curvatures $K_1(s)$,$K_2(s)$, and $K_3(s)$ can be computed easily. In [2], Banchoff and Lowett gave explicit formulas for curvatures and proposed another recursive way for determining the Frenet frame. In what follows, we use their approach to examining Frenet curves in $\mathbb{E}^4$.

Suppose that the unit speed Frenet curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ is parameterized by (13). Proposition 3.1.9 in Banchoff and Lovett's book [2] shows that the curvatures of the curve $\boldsymbol{\beta}$ in $\mathbb{E}^4$ can be expressed explicitly by the matrices $B_2(s) = \left(\boldsymbol{\beta}'(s)\boldsymbol{\beta}''(s)\right)$, $B_3(s) = \left(\boldsymbol{\beta}'(s)\boldsymbol{\beta}''(s)\boldsymbol{\beta}'''(s)\right)$, $B_4(s) = \left(\boldsymbol{\beta}'(s)\boldsymbol{\beta}''(s)\boldsymbol{\beta}'''(s)\boldsymbol{\beta}^{(4)}(s)\right)$. More precisely, the first curvature is

$$K_1(s) = \sqrt{\det\left(B_2(s)^T B_2(s)\right)} = \|\boldsymbol{\beta}''(s)\|, \tag{15}$$

the second curvature is

$$K_2(s) = \tfrac{1}{(K_1(s))^2}\sqrt{\det\left(B_3(s)^T B_3(s)\right)} \tag{16}$$

and the third curvature is

$$K_3(s) = \frac{1}{(K_1(s))^3\,(K_2(s))^2}\det\left(B_4(s)\right). \tag{17}$$

Note that the condition $\boldsymbol{\beta}$ to be a Frenet curve in $\mathbb{E}^4$ is equivalent to the condition $\boldsymbol{\beta}$ to be a curve of class C^4 with curvatures $K_1(s) > 0$, $K_2(s) > 0$ and $K_3(s) \neq 0$.
The unit speed Frenet curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ defined by (13) has the unit tangent vector

$$\mathbf{T}(s) = \boldsymbol{\beta}'(s)$$

because of $\mathbf{T}(s)\cdot\mathbf{T}(s) = \boldsymbol{\beta}'(s)\cdot\boldsymbol{\beta}'(s) = 1$. Then, $\mathbf{T}(s)\cdot\mathbf{T}'(s) = \boldsymbol{\beta}'(s)\cdot\boldsymbol{\beta}''(s) = 0$, or $\mathbf{T}(s) \perp \mathbf{T}'(s)$ and $\boldsymbol{\beta}'(s) \perp \boldsymbol{\beta}''(s)$. The uniquely determined vector

$$\mathbf{N}_1(s) = \frac{1}{\|\boldsymbol{\beta}''(s)\|}\boldsymbol{\beta}''(s)$$

is the first unit normal vector of $\boldsymbol{\beta}$. From second equation of (14) we derive the second unit normal vector of $\boldsymbol{\beta}$

$$\mathbf{N}_2(s) = \frac{1}{K_2(s)} \left(\mathbf{N}_1'(s) + K_1(s)\mathbf{T}(s)\right).$$

Similarly, from the third equation of (14) the third unit normal vector of $\boldsymbol{\beta}$ can be expressed as

$$\mathbf{N}_3(s) = \frac{1}{K_3(s)} \left(\mathbf{N}_2'(s) + K_2(s)\mathbf{N}_1(s)\right).$$

Thus, the curvatures of a Frenet curve in $\mathbb{E}^4$ can be computed firstly. Then, the vectors of the Frenet frame can be determined as shown above.

Analogously to the three-dimensional case, there is an unique osculating hypersphere at any point of a Frenet curve $\boldsymbol{\beta}$ in $\mathbb{E}^4$. Centers of these osculating hyperspheres form a new curve called a focal curve of $\boldsymbol{\beta}$. Uribe-Vargas [20] investigated focal curves in arbitrary dimension. From his main result it follows that the unit-speed Frenet curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ given by (13) possesses a focal curve with a parametrization

$$\boldsymbol{\beta}^f(s) = \boldsymbol{\beta}(s) + c_1(s)\mathbf{N}_1(s) + c_2(s)\mathbf{N}_2(s) + c_3(s)\mathbf{N}_3(s), \tag{18}$$

where $\mathbf{N}_i(s)$, $i = 1, 2, 3$ are the three unit normal vectors of $\boldsymbol{\beta}$ and the coefficients $c_i(s)$, $i = 1, 2, 3$ called focal curvatures can be expressed by the curvatures $K_1(s)$, $K_2(s)$, $K_3(s)$ of $\boldsymbol{\beta}$. If $K_i(s)$ are non-constant functions, then the focal curvatures are

$$\begin{aligned} c_1(s) &= \frac{1}{K_1(s)}, \qquad c_2(s) = \left(\frac{1}{K_1(s)}\right)' \frac{1}{K_2(s)} \\ c_3(s) &= \left(\frac{c_1(s)}{c_2(s)} c_1'(s) + c_2'(s)\right) \frac{1}{K_3(s)} \\ &= \left\{\frac{K_2(s)}{K_1(s)} + \left[\left(\frac{1}{K_1(s)}\right)' \frac{1}{K_2(s)}\right]'\right\} \frac{1}{K_3(s)}. \end{aligned} \tag{19}$$

It is shown in [20] and [13] that the curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ is spherical if and only if

$$(c_1(s))^2 + (c_2(s))^2 + (c_3(s))^2 \tag{20}$$

is a positive constant function. As consequence of (19), if the curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ has constant curvatures $K_i(s)$, $i = 1, 2, 3$, then its focal curvatures are also constant functions determined as follows: $c_1(s) = \frac{1}{K_1(s)}$, $c_2(s) = 0$ and $c_3(s) = \frac{K_2(s)}{K_1(s)K_3(s)}$. Therefore by (20), such a curve is spherical. In [11, Ch. 2]), it is shown that any $4D$ Frenet curve with constant curvatures possesses an arc-length parametrization in the form

$$\begin{gathered} \boldsymbol{\beta}(s) = \left(p\cos(as), p\sin(as), q\cos(bs), q\sin(bs)\right)^T, \\ \text{where} \quad \mathrm{p, q} \neq 0, \quad \mathrm{a, b} \neq 0, \quad \mathrm{a} \neq \mathrm{b}, \quad \mathrm{a^2p^2 + b^2q^2 = 1}. \end{gathered} \tag{21}$$

3. A CONSTRUCTION OF A GENERALIZED FOCAL CURVE OF A FRENET SPACE CURVE

Let $I \subseteq \mathbb{R}$ be an interval containing zero and let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a Frenet curve of class C^4 with an arc-length parametrization

$$\boldsymbol{\alpha}(s) = \big(x(s),\ y(s),\ z(s)\big)^T, \quad s \in I. \tag{22}$$

Under above assumptions, for any $s \in I$, the curvature $\kappa(s)$ of $\boldsymbol{\alpha}$ is a positive real number, the torsion $\tau(s)$ of $\boldsymbol{\alpha}$ is a nonzero real number and the Frenet frame $\big(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\big)$ is well-defined. Using the Frenet-Serret equations (8) we have

$$\begin{array}{rclcl} \boldsymbol{\alpha}'(s) & = & \mathbf{t}(s), & & \boldsymbol{\alpha}''(s) = \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \\ \boldsymbol{\alpha}'''(s) & = & \mathbf{t}''(s) & = & -\kappa(s)^2\mathbf{t}(s) + \kappa'(s)\mathbf{n}(s) + \kappa(s)\tau(s)\mathbf{b}(s), \\ \boldsymbol{\alpha}^{(4)}(s) & = & \mathbf{t}'''(s) & = & -3\kappa(s)\kappa'(s)\mathbf{t}(s) \\ & & & & + \left[-\kappa(s)^3 + \kappa''(s) - \kappa(s)\tau(s)^2\right]\mathbf{n}(s) \\ & & & & + \left[2\kappa'(s)\tau(s) + \kappa(s)\tau'(s)\right]\mathbf{b}(s). \end{array} \tag{23}$$

Then we calculate the dot products

$$\begin{array}{l} \boldsymbol{\alpha}'(s)\cdot\boldsymbol{\alpha}'(s) = 1,\ \boldsymbol{\alpha}'(s)\cdot\boldsymbol{\alpha}''(s) = 0,\ \boldsymbol{\alpha}'(s)\cdot\boldsymbol{\alpha}'''(s) = -\left(\kappa(s)\right)^2, \\ \boldsymbol{\alpha}'(s)\cdot\boldsymbol{\alpha}^{(4)}(s) = -3\kappa(s)\kappa'(s), \\ \boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}''(s) = \left(\kappa(s)\right)^2,\ \boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}'''(s) = \kappa(s)\kappa'(s), \\ \boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}^{(4)}(s) = \kappa(s)\kappa''(s) - \left(\kappa(s)\right)^4 - \left(\kappa(s)\right)^2\left(\tau(s)\right)^2, \\ \boldsymbol{\alpha}'''(s)\cdot\boldsymbol{\alpha}'''(s) = \left(\kappa'(s)\right)^2 + \left(\kappa(s)\right)^4 + \left(\kappa(s)\right)^2\left(\tau(s)\right)^2, \\ \boldsymbol{\alpha}'''(s)\cdot\boldsymbol{\alpha}^{(4)}(s) = \kappa'(s)\kappa''(s) + \kappa(s)\tau(s)\left(\kappa(s)\tau(s)\right)' \end{array} \tag{24}$$

and the vector cross product

$$\boldsymbol{\alpha}''(s)\times\boldsymbol{\alpha}'''(s) \;=\; \left(\kappa(s)\right)^2\tau(s)\mathbf{t}(s) + \left(\kappa(s)\right)^3\mathbf{b}(s). \tag{25}$$

As consequence of (23) and (25), we get

$$\begin{array}{rcl} \det\left(\boldsymbol{\alpha}''(s), \boldsymbol{\alpha}'''(s), \boldsymbol{\alpha}^{(4)}(s)\right) & = & \left(\boldsymbol{\alpha}''(s)\times\boldsymbol{\alpha}'''(s)\right)\cdot\boldsymbol{\alpha}^{(4)}(s) \\ & = & \left(\kappa(s)\right)^3\left(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)\right) \\ & = & \left(\kappa(s)\right)^5\left(\dfrac{\tau(s)}{\kappa(s)}\right)'. \end{array} \tag{26}$$

Npw we can examine unit-speed curves in $\mathbb{E}^4$ which are closely related to $\boldsymbol{\alpha}$.

Theorem 3.1. *Let (22) be a parametrization of a unit-speed Frenet curve $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ of class C^4, and let $\kappa(s)$ and $\tau(s)$ be the curvature and the torsion of $\boldsymbol{\alpha}$. Suppose that the curve $\boldsymbol{\beta}_1 : I \longrightarrow \mathbb{E}^4$ is defined by*

$$\boldsymbol{\beta}_1(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}x(s) \\ \frac{1}{\sqrt{2}}y(s) \\ \frac{1}{\sqrt{2}}z(s) \\ \frac{s}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\alpha}(s) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{s}{\sqrt{2}} \end{pmatrix}, \quad s \in I. \tag{27}$$

Then:

(i) $\boldsymbol{\beta}_1$ is also a unit-speed curve.

(ii) $\boldsymbol{\beta}_1$ is a Frenet curve in $\mathbb{E}^4$ with curvatures

$$K_1(s) = \frac{\kappa(s)}{\sqrt{2}},\ K_2(s) = \frac{\kappa(s)}{\sqrt{2}}\sqrt{1 + 2\left(\frac{\tau(s)}{\kappa(s)}\right)^2},\ K_3(s) = -\frac{\sqrt{2}\left(\frac{\tau(s)}{\kappa(s)}\right)'}{1 + 2\left(\frac{\tau(s)}{\kappa(s)}\right)^2} \tag{28}$$

if and only if $\boldsymbol{\alpha}$ is a non-helical curve, i.e., $\left(\frac{\tau(s)}{\kappa(s)}\right)' \neq 0$ for any s.

Proof. The derivatives of the vector function $\boldsymbol{\beta}_1(s)$ up to fourth order are

$$\begin{aligned} \boldsymbol{\beta}_1'(s) &= \left(\tfrac{1}{\sqrt{2}}x'(s), \tfrac{1}{\sqrt{2}}y'(s), \tfrac{1}{\sqrt{2}}z'(s), \tfrac{1}{\sqrt{2}}\right)^T \\ \boldsymbol{\beta}_1''(s) &= \left(\tfrac{1}{\sqrt{2}}x''(s), \tfrac{1}{\sqrt{2}}y''(s), \tfrac{1}{\sqrt{2}}z''(s), 0\right)^T \\ \boldsymbol{\beta}_1'''(s) &= \left(\tfrac{1}{\sqrt{2}}x'''(s), \tfrac{1}{\sqrt{2}}y'''(s), \tfrac{1}{\sqrt{2}}z'''(s), 0\right)^T \\ \boldsymbol{\beta}_1^{(4)}(s) &= \left(\tfrac{1}{\sqrt{2}}x^{(4)}(s), \tfrac{1}{\sqrt{2}}y^{(4)}(s), \tfrac{1}{\sqrt{2}}z^{(4)}(s), 0\right)^T. \end{aligned} \tag{29}$$

Since the norm $\|\boldsymbol{\beta}_1'(s)\|$ is equal to 1 for any parameter value, then $\boldsymbol{\beta}_1$ is a unit-speed curve. From (26) and (29) it follows that

$$\begin{aligned} &\det\left(\boldsymbol{\beta}_1'(s), \boldsymbol{\beta}_1''(s), \boldsymbol{\beta}_1'''(s), \boldsymbol{\beta}_1^{(4)}(s)\right) \\ &= (-1)^5 \tfrac{1}{4} \det\left(\boldsymbol{\alpha}''(s), \boldsymbol{\alpha}'''(s), \boldsymbol{\alpha}^{(4)}(s)\right) = -\frac{1}{4}(\kappa(s))^5 \left(\frac{\tau(s)}{\kappa(s)}\right)'. \end{aligned} \tag{30}$$

Hence, $\boldsymbol{\beta}_1$ is a Frenet curve whenever the ratio $\frac{\tau(s)}{\kappa(s)}$ is not a constant function. It is easily to see that $\boldsymbol{\beta}_1'(s) \cdot \boldsymbol{\beta}_1'(s) = \boldsymbol{\alpha}'(s) \cdot \boldsymbol{\alpha}'(s) = 1$ and $\boldsymbol{\beta}_1^{(i)}(s) \cdot \boldsymbol{\beta}_1^{(j)}(s) = \frac{1}{2}\boldsymbol{\alpha}^{(i)}(s) \cdot \boldsymbol{\alpha}^{(j)}(s)$ for $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$ and $i + j > 2$. Replacing the dot products $\boldsymbol{\beta}_1^{(i)}(s) \cdot \boldsymbol{\beta}_1^{(j)}(s)$ in (15), (16), (17) with the right hand sides of the above identities and having in mind (24) and (30) we obtain (28). □

Let us consider other ways to determine a $4D$ unit-speed curve from a unit-speed Frenet curve in $\mathbb{E}^3$.

Theorem 3.2. *Let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a unit-speed Frenet curve of class C^4 given by (22). Then, the curve $\boldsymbol{\beta}_2 : I \longrightarrow \mathbb{E}^4$ defined by*

$$\boldsymbol{\beta}_2(s) = (x(s), y(s), \cos(z(s)), \sin(z(s)))^T, \quad s \in I \tag{31}$$

and the curve $\boldsymbol{\beta}_3 : I \longrightarrow \mathbb{E}^4$ defined by

$$\boldsymbol{\beta}_3(s) = \left(x(s), y(s), 2\arctan(z(s)) - z(s), \ln\left(1 + (z(s))^2\right)\right)^T, \quad s \in I \tag{32}$$

are unit-speed curves of class C^4. Moreover:
(a) For the circular helix $\boldsymbol{\alpha}$ given by (10) with constant curvature and torsion (11), the curve $\boldsymbol{\beta}_2$ is either a spherical curve with constant nonzero curvatures or a circle in 2-dimensional affine subspace of $\mathbb{E}^4$, and the curve $\boldsymbol{\beta}_3$ is a Frenet curve in $\mathbb{E}^4$ with curvatures

$$\begin{aligned}
K_1(s) &= \sqrt{\frac{(1-b^2)\left(ab^2s^2+a\right)^2+4b^4}{\left(b^2s^2+1\right)^2}} = \sqrt{(\kappa(s))^2 + \frac{4b^4}{\left(b^2s^2+1\right)^2}} \\
K_2(s) &= \sqrt{\frac{(\kappa(s)\tau(s))^2 p(s)}{a^4\left(b^2s^2+1\right)^2\left((\kappa(s))^2\left(b^2s^2+1\right)^2+4b^4\right)^2}} \\
K_3(s) &= \frac{2b^2\tau(s)\left(1-b^2\right)\left[a^4\left(b^2s^2+1\right)^3+2(\tau(s))^2\left(3b^4s^4-2b^2s^2-5\right)+24b^4\right]}{(K_1(s))^3(K_2(s))^2\left(b^2s^2+1\right)^4}
\end{aligned}, \tag{33}$$

where $p(s) = 4\left(3b^2-2\right)(a\tau(s))^2\left(b^2s^2+1\right)^4 + \left(a^2\kappa(s)\right)^2\left(b^2s^2+1\right)^6$
$+ 16b^4\left(b^2\left(s^2-3\right)+1\right)\left(ab^2s^2+a\right)^2 + 64b^8$.
(b) For any general helix $\boldsymbol{\alpha}$ of class C^4 given by (12), the corresponding curve $\boldsymbol{\beta}_2$ defined by (31) is a Frenet curve in $\mathbb{E}^4$ if and only if

$$\Delta_2(s) = \det\begin{pmatrix} x'''(s) + b^2x'(s) & y'''(s) + b^2y'(s) \\ x^{(4)}(s) + b^2x''(s) & y^{(4)}(s) + b^2y''(s) \end{pmatrix} \tag{34}$$

is nonzero for any parameter value.

Proof. It is clear that both vector functions $\boldsymbol{\beta}_2(s)$ and $\boldsymbol{\beta}_3(s)$ have continuous derivatives up to fourth order. The first derivatives
$\boldsymbol{\beta}_2'(s) = (x'(s), y'(s), -z'(s)\sin(z(s)), z'(s)\cos(z(s)))^T$ and
$\boldsymbol{\beta}_3'(s) = \left(x'(s), y'(s), \dfrac{1-(z(s))^2}{1+(z(s))^2}z'(s), \dfrac{2z(s)}{1+(z(s))^2}z'(s)\right)^T$ are unit vectors for any s.

Therefore, both $\boldsymbol{\beta}_2(s)$ and $\boldsymbol{\beta}_3(s)$ are unit-speed curves.
(a) Suppose that $\boldsymbol{\alpha}$ has a parametrization (10). Then the parametrization of $\boldsymbol{\beta}_2$ is

$$\boldsymbol{\beta}_2(s) = \left(\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), \sqrt{\tfrac{1-b^2}{a^2}}\sin(as), \cos(bs), \sin(bs)\right)^T, \tag{35}$$

and the parametrization of $\boldsymbol{\beta}_3$ is $\boldsymbol{\beta}_3(s)$=

$$\left(\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), \sqrt{\tfrac{1-b^2}{a^2}}\sin(as), 2\arctan(bs) - bs, \ln\left(1 + (bz)^2\right)\right)^T. \tag{36}$$

Obviously, $\boldsymbol{\beta}_2$ lies on the 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{1+a^2-b^2}{a^2}$. By (21), $\boldsymbol{\beta}_2$ has three constant nonzero curvatures whenever $a \neq b$. In the case $a = b$, $\boldsymbol{\beta}_2$ is the intersection of the above sphere and the plane
$\left\{x_1 - \sqrt{\frac{1-b^2}{a^2}}x_3 = x_2 - \sqrt{\frac{1-b^2}{a^2}}x_4 = 0\right\}$. The vector function $\boldsymbol{\beta}_3(s)$ given by (36) has derivatives

$$\boldsymbol{\beta}_3'(s) = \left(-a\sqrt{\tfrac{1-b^2}{a^2}}\sin(as), a\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), \tfrac{b\left(1-b^2s^2\right)}{b^2s^2+1}, \tfrac{2b^2s}{b^2s^2+1}\right)^T$$
$$\boldsymbol{\beta}_3''(s) = \left(-a^2\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), -a^2\sqrt{\tfrac{1-b^2}{a^2}}\sin(as), -\tfrac{4b^3s}{\left(b^2s^2+1\right)^2}, \tfrac{2b^2-2b^4s^2}{\left(b^2s^2+1\right)^2}\right)^T$$
$$\boldsymbol{\beta}_3''' = \left(a^3\sqrt{\tfrac{1-b^2}{a^2}}\sin(as), -a^3\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), \tfrac{4b^3\left(3b^2s^2-1\right)}{\left(b^2s^2+1\right)^3}, \tfrac{4b^4s\left(b^2s^2-3\right)}{\left(b^2s^2+1\right)^3}\right)^T$$
$$\boldsymbol{\beta}_3^{(4)}(s) =$$
$$\left(a^4\sqrt{\tfrac{1-b^2}{a^2}}\cos(as), a^4\sqrt{\tfrac{1-b^2}{a^2}}\sin(as), -\tfrac{48b^5s\left(b^2s^2-1\right)}{\left(b^2s^2+1\right)^4}, -\tfrac{12\left(b^8s^4-6b^6s^2+b^4\right)}{\left(b^2s^2+1\right)^4}\right)^T.$$

Consider the real-valued function

$$\begin{aligned}\Delta_3(s) &= \det\left(\boldsymbol{\beta}_3'(s), \boldsymbol{\beta}_3''(s), \boldsymbol{\beta}_3'''(s), \boldsymbol{\beta}_3^{(4)}(s)\right)\\ &= \frac{2ab^3\left(1-b^2\right)\left[a^4\left(b^2s^2+1\right)^3 + 2a^2b^2\left(3b^4s^4 - 2b^2s^2 - 5\right) + 24b^4\right]}{\left(b^2s^2+1\right)^4}.\end{aligned}$$

Since the equation $\Delta_3(s) = 0$ has no real roots, $\boldsymbol{\beta}_3$ is a Frenet curve. Substituting obtained expressions of the derivatives and the determinant into (15), (16) and (17) yields (33).
(b) Suppose that $\boldsymbol{\alpha}$ has a parametrization (12). Then the parametrization of $\boldsymbol{\beta}_2$ is

$$\boldsymbol{\beta}_2(s) = (x(s), y(s), \cos(bs), \sin(bs))^T \quad s \in I. \tag{37}$$

The first four derivatives of the above vector function are

$$\begin{aligned}
\boldsymbol{\beta}_2'(s) &= (x'(s), y'(s), -b\sin(bs), b\cos(bs))^T\\
\boldsymbol{\beta}_2''(s) &= \left(x''(s), y''(s), -b^2\cos(bs), -b^2\sin(bs)\right)^T\\
\boldsymbol{\beta}_2'''(s) &= \left(x'''(s), y'''(s), b^3\sin(bs), -b^3\cos(bs)\right)^T\\
\boldsymbol{\beta}_2^{(4)}(s) &= \left(x^{(4)}(s), y^{(4)}(s), b^4\cos(bs), b^4\sin(bs)\right)^T.
\end{aligned}$$

Consider the matrices $A = \begin{pmatrix} x'(s) & y'(s) \\ x''(s) & y''(s) \end{pmatrix}$,
$B = \begin{pmatrix} -b\sin(bs) & b\cos(bs) \\ -b^2\cos(bs) & -b^2\sin(bs) \end{pmatrix}$, $C = \begin{pmatrix} x'''(s) & y'''(s) \\ x^{(4)}(s) & y^{(4)}(s) \end{pmatrix}$,
$D = \begin{pmatrix} b^3\sin(bs) & -b^3\cos(bs) \\ b^4\cos(bs) & b^4\sin(bs) \end{pmatrix}$. Then, $\boldsymbol{\beta}_2$ given by (37) is a Frenet curve if and only if

$$\begin{aligned}
\det\left(\boldsymbol{\beta}_2'(s), \boldsymbol{\beta}_2''(s), \boldsymbol{\beta}_2'''(s), \boldsymbol{\beta}_2^{(4)}(s)\right) &= \det\left(\boldsymbol{\beta}_2'(s), \boldsymbol{\beta}_2''(s), \boldsymbol{\beta}_2'''(s), \boldsymbol{\beta}_2^{(4)}(s)\right)^T\\
&= \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0.
\end{aligned}$$

Since D is an invertible matrix, the well-known formula for the determinant of a 2×2 block matrix (see [18]) yields $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} =$
$\det(D)\det\left(A - BD^{-1}C\right) = b^3 \det\begin{pmatrix} x'''(s) + b^2x'(s) & y'''(s) + b^2y'(s) \\ x^{(4)}(s) + b^2x''(s) & y^{(4)}(s) + b^2y'' \end{pmatrix}$.
This completes the proof of the assertion (b). □

Corollary 3.3. *For a circular helix $\boldsymbol{\alpha}$ given by (10) with $a = b$, the curve $\boldsymbol{\beta}_3$ defined by (36) is a Frenet curve in $\mathbb{E}^4$, but the curve $\boldsymbol{\beta}_2$ defined by (35) is not a Frenet curve in $\mathbb{E}^4$.*

Let us consider a set of general helices in $\mathbb{E}^3$ which are not circular helices.

Proposition 3.4. *Suppose that $\boldsymbol{\alpha} : (-\infty, \infty) \longrightarrow \mathbb{E}^3$ is a unit speed Frenet curve parameterized by*

$$\boldsymbol{\alpha}(s) = \left(\sqrt{\frac{1-b^2}{a^2}}(2\arctan(as) - as), \sqrt{\frac{1-b^2}{a^2}}\ln\left(1 + (as)^2\right), bs\right)^T, \tag{38}$$

where a and b are real constants satisfying $a \neq 0$, $\quad b \in \{(-1,0) \cup (0,1)\}$. Then, the curve $\boldsymbol{\alpha}$ is a general helix with a non-constant curvature and a non-constant torsion.

Moreover, by (31), the curve $\boldsymbol{\alpha}$ determines a parameterized curve

$$\begin{aligned}\boldsymbol{\beta}_2(s) = & \left(\sqrt{\tfrac{1-b^2}{a^2}}(2\arctan(as) - as), \sqrt{\tfrac{1-b^2}{a^2}}\ln\left(1 + (as)^2\right), \cos(bs), \sin(bs)\right)^T, \\ & s \in (-\infty, \infty)\end{aligned} \tag{39}$$

which is a Frenet curve in $\mathbb{E}^4$.

Proof. The curve $\boldsymbol{\alpha}$ has a curvature $\frac{2\sqrt{a^2(1-b^2)}}{a^2s^2+1}$ and a torsion $\frac{2ab}{a^2s^2+1}$. This proves the first assertion. The proof of the second assertion is similar to the proof of Theorem 3.2 (a). □

Corollary 3.5. *For a general helix $\boldsymbol{\alpha}$ given by (38) with $a = b$, which is not a circular helix, the curve $\boldsymbol{\beta}_2$ defined by (39) is a Frenet curve in $\mathbb{E}^4$, but the new curve defined by (32) $\boldsymbol{\beta}_3(s) = \Bigg(\sqrt{\frac{1-a^2}{a^2}}(2\arctan(as) - as),$*

$\sqrt{\frac{1-a^2}{a^2}}\ln\left(1 + (as)^2\right), 2\arctan(as) - as, \ln\left(1 + (as)^2\right)\Bigg)^T,$

$a \in \{(-1, 0) \cup (0, 1)\}$ is not a Frenet curve in $\mathbb{E}^4$.

By Theorems 3.1 and 3.2 we may define a new curve in $\mathbb{E}^3$ from a given unit-speed Frenet curve $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \longrightarrow \mathbb{E}^3$ of class C^4. To do this, we propose a construction consisting of three stages: first, the given unit-speed curve $\boldsymbol{\alpha}$ generates a unique unit-speed Frenet curve $\boldsymbol{\beta}$ in $\mathbb{E}^4$, then the obtained curve $\boldsymbol{\beta}$ determines an unique focal curve $\boldsymbol{\beta}^f$ in $\mathbb{E}^4$, finally, the orthogonal projection of this focal curve in the hyperplane $\Omega = \{x_4 = 0\} \subset \mathbb{E}^4$ specifies a new curve in $\mathbb{E}^3$. The first stage, which constitutes the key novelty behind our proposal, relies on the existence of a short way to get a fourth-dimensional unit -speed curve from a three-dimensional unit -speed curve. This short way is realized by combining Theorem 3.1 and Theorem 3.2. The following algorithm gives a complete description of our construction.

Algorithm 3.6. *Obtaining a new $3D$ curve associated to a given $3D$ unit-speed Frenet curve of class C^4.*

Input: *The unit-speed Frenet curve $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ of class C^4, which possesses a parametrization (22).*

1. Compute the curvature $\kappa(s)$ and the torsion $\tau(s)$ of $\boldsymbol{\alpha}$.

2. If $\frac{\tau(s)}{\kappa(s)}$ is a non-constant function, then define a curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ with a parametrization (27), i.e. $\boldsymbol{\beta}(s) = \boldsymbol{\beta}_1(s)$, and go to Step 5, else go to Step 3.

3. If $\frac{\tau(s)}{\kappa(s)}$ is a constant function, but both $\kappa(s)$ and $\tau(s)$ are non-constant functions, then

determine the curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ *with a parametrization (31), i.e.* $\boldsymbol{\beta}(s) = \boldsymbol{\beta}_2(s)$, *and go to Step 5, else* $\boldsymbol{\alpha}$ *is a circular helix, i.e. both* $\kappa(s)$ *and* $\tau(s)$ *are constant functions, and then go to the next step.*
4. Specify a curve $\boldsymbol{\beta} : I \longrightarrow \mathbb{E}^4$ *with a parametrization (32), i.e.* $\boldsymbol{\beta}(s) = \boldsymbol{\beta}_3(s)$, *and go to Step 5.*
5. Find the focal curve $\boldsymbol{\beta}^f$ *of the unit-speed Frenet curve* $\boldsymbol{\beta}$.
6. Determine the orthogonal projection $\boldsymbol{\beta}^f_\Omega$ *of the curve* $\boldsymbol{\beta}^f$ *onto the hyperplane* $\Omega = \{x_4 = 0\} \subset \mathbb{E}^4$.
7. Identify Ω *with the Euclidean space* $\mathbb{E}^3$, *and consider the curve* $\boldsymbol{\beta}^f_\Omega$ *as a curve* $\boldsymbol{\gamma} : I \longrightarrow \mathbb{E}^3$.
***Output**: 3D curve* $\boldsymbol{\gamma}$.

Definition 3.7. *The curve* $\boldsymbol{\gamma}$ *from Algorithm 3.6 is called a generalized focal curve of the unit-speed* 3*D curve* $\boldsymbol{\alpha}$.

Algorithm 3.6 splits the set of all unit speed Frenet curves in $\mathbb{E}^3$ into three disjoin classes: the class of non-helical curves, i.e. curves with non-constant ratio of torsion to curvature; the class of general helices with non-constant curvature and torsion; the class of circular helices. Theorems 3.1 and 3.2 guarantee the existence of a unique generalized focal curve for any non-helical Frenet curve and for any circular helix. Proposition 3.4 and Algorithm 3.6 show that any general helix (38) also determines an unique generalized focal curve.

4. EXAMPLES

In this section we examine three illustrative examples for generalized focal curves to 3D curves with a known arc-length parameterizations. A standard method for obtaining an arc-length parametrization of a parameterized curve is presented in Tu [19, Chap. 1, p.10]. In what follows, the index "i" of the curve $\boldsymbol{\alpha}_i : I \longrightarrow \mathbb{E}^3$ appears as a first index of all scalar and vector functions related to $\boldsymbol{\alpha}_i$.

4.1. A generalized focal curve to a 3D curve with a non-constant ratio of torsion to curvature.

Let us explore the unit speed curve in $\mathbb{E}^3$ with a parametrisation

$$\boldsymbol{\alpha}_1(s) = \left(\frac{1}{3}(s+2)^{3/2}, -\frac{(1-s)^{3/2}}{3\sqrt{2}}, -\frac{(3-s)^{3/2}}{3\sqrt{2}}\right)^T, \quad s \in [-2, 1]. \tag{40}$$

Since its curvature is $\kappa_1(s) = \frac{1}{4}\sqrt{\frac{7-4s}{s^3-2s^2-5s+6}}$ and its torsion is $\tau_1(s) = -\frac{15}{4(7-4s)\sqrt{s^3-2s^2-5s+6}}$, the ratio $\frac{\tau_1(s)}{\kappa_1(s)}$ is a well-defined non-constant function in the open interval $(-2, 1)$. Note that both functions $7-4s$ and $s^3-2s^2-5s+6 = (s+2)(1-s)(3-s)$ take only positive values in the same interval.

The focal curve (9) of $\boldsymbol{\alpha}_1$ possesses a parametrization

$$\boldsymbol{\alpha}_1^f(s) = \begin{pmatrix} -\frac{(s+2)^{3/2}}{15}(8s-21) \\ \frac{(1-s)^{3/2}}{3\sqrt{2}}(8s-5) \\ -\frac{(3-s)^{3/2}}{15\sqrt{2}}(24s+17) \end{pmatrix}, \quad s \in [-2, 1]. \tag{41}$$

By Theorem 3.1 we get a 4D unit-speed curve $\boldsymbol{\beta}_1$ with a vector parametric wquation

$$\boldsymbol{\beta}_1(s) = \left(\frac{1}{3\sqrt{2}}(s+2)^{3/2}, -\frac{1}{6}(1-s)^{3/2}, -\frac{1}{6}(3-s)^{3/2}, \frac{s}{\sqrt{2}}\right)^T, \quad s \in (-2, 1)$$

Then we can derive the curvatures of $\boldsymbol{\beta}_1$ using (15), (16), and (17)

$$\begin{gathered} K_{11} = \frac{1}{4\sqrt{2}}\sqrt{\frac{7-4s}{s^3-2s^2-5s+6}}, \; K_{12} = \frac{1}{4\sqrt{2}(7-4s)}\sqrt{\frac{-64s^3+336s^2-588s+793}{s^3-2s^2-5s+6}} \\ K_{13} = \frac{90\sqrt{2(7-4s)}}{-64s^3+336s^2-588s+793} \end{gathered}.$$

The unit normal vectors of $\boldsymbol{\beta}_1$ are

$$\mathbf{N}_{11}(s) = \begin{pmatrix} \sqrt{\frac{(1-s)(3-s)}{7-4s}} \\ -\sqrt{\frac{(s+2)(3-s)}{2(7-4s)}} \\ -\sqrt{\frac{(s+2)(1-s)}{2(7-4s)}} \\ 0 \end{pmatrix}, \mathbf{N}_{12}(s) = \begin{pmatrix} -\frac{\sqrt{s+2}\left(16s^2-56s+79\right)}{2\sqrt{2}\sqrt{7-4s}\sqrt{-64s^3+336s^2-588s+793}} \\ -\frac{\sqrt{1-s}\left(16s^2-56s+199\right)}{4\sqrt{7-4s}\sqrt{-64s^3+336s^2-588s+793}} \\ -\frac{\sqrt{3-s}\left(16s^2-56s-41\right)}{4\sqrt{7-4s}\sqrt{-64s^3+336s^2-588s+793}} \\ \frac{(7-4s)^{3/2}}{\sqrt{2}\sqrt{-64s^3+336s^2-588s+793}} \end{pmatrix},$$

$$\mathbf{N}_{13}(s) = \begin{pmatrix} \frac{2(s+2)^{3/2}}{\sqrt{-64s^3+336s^2-588s+793}} \\ -\frac{5\sqrt{2}(1-s)^{3/2}}{\sqrt{-64s^3+336s^2-588s+793}} \\ \frac{3\sqrt{2}(3-s)^{3/2}}{\sqrt{-64s^3+336s^2-588s+793}} \\ -\frac{15}{\sqrt{-64s^3+336s^2-588s+793}} \end{pmatrix}.$$

By (19) the focal curvatures of $\boldsymbol{\beta}_1$ are

$$c_{11}(s) = 4\sqrt{2}\sqrt{\frac{s^3-2s^2-5s+6}{7-4s}}, \; c_{12}(s) = \frac{16\left(-8s^3+29s^2-28s-11\right)}{\sqrt{(7-4s)\left(-64s^3+336s^2-588s+793\right)}},$$

$$c_{13}(s) = -\frac{1792s^4-12544s^3+32928s^2-78016s+51007}{90\sqrt{2}\sqrt{-64s^3+336s^2-588s+793}}.$$

The focal curve (18) of $\boldsymbol{\beta}_1$ is a curve $\boldsymbol{\beta}_1^f : (-2, 1) \longrightarrow \mathbb{E}^4$ that is parameterized by

$$\boldsymbol{\beta}_1^f(s) = \boldsymbol{\beta}_1(s) + c_{11}(s)\mathbf{N}_{11}(s) + c_{12}(s)\mathbf{N}_{12}(s) + c_{13}(s)\mathbf{N}_{13}(s).$$

According to Algorithm 3.6 and Definition 3.7 the projection of $\boldsymbol{\beta}_1^f$ onto the hyperplane Ω is identified with the generalized focal curve $\boldsymbol{\gamma}_1 : (-2, 1) \longrightarrow \mathbb{E}^3$ of $\boldsymbol{\alpha}_1$. After doing some calculations, we get the parametrization of $\boldsymbol{\gamma}_1$:

$$\boldsymbol{\gamma}_1(s) = \begin{pmatrix} \frac{14\sqrt{2}}{45}(s+2)^{5/2} \\ \frac{14}{9}(1-s)^{5/2} \\ -\frac{14}{15}(3-s)^{5/2} \end{pmatrix}, \quad s \in (-2, 1).$$

The curve $\boldsymbol{\alpha}_1$ with the irrational parametrization (40), its focal curve $\boldsymbol{\alpha}_1^f$ given by (41), and its generalized focal curve $\boldsymbol{\gamma}_1$ are fully plotted in Fig. 1. Comparing the focal curve

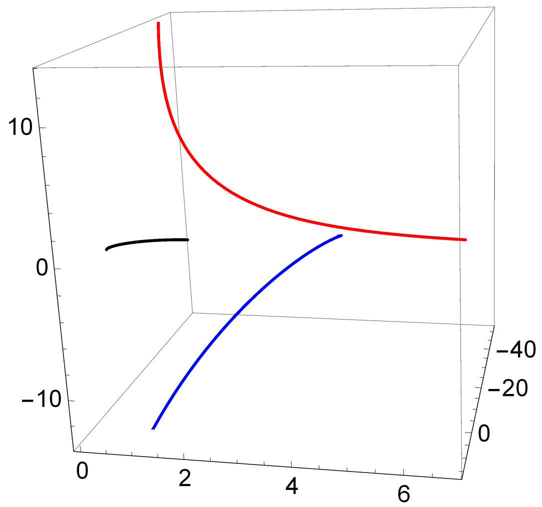

Figure 1: The 3D curve $\boldsymbol{\alpha}_1$ (in black), the focal curve $\boldsymbol{\alpha}_1^f$ (in blue) of $\boldsymbol{\alpha}_1$, and the generalized focal curve $\boldsymbol{\gamma}_1$ (in red) of $\boldsymbol{\alpha}_1$.

of $\boldsymbol{\alpha}_1$ and the generalized focal curve of $\boldsymbol{\alpha}_1$ we can conclude that both $\boldsymbol{\alpha}_1^f$ and $\boldsymbol{\gamma}_1$ are space curves with irrational parameterizations.

4.2. A generalized focal curve to a 3D curve with a constant ratio of a non-constant torsion to a non-constant curvature.

Consider the unit-speed space curve with an irrational parametrisation

$$\boldsymbol{\alpha}_2(s) = \left(\frac{1}{3}(1+s)^{3/2}, -\frac{1}{3}(1-s)^{3/2}, \frac{s}{\sqrt{2}}\right)^T, \quad s \in (-1, 1). \tag{42}$$

The curvature and the torsion of $\boldsymbol{\alpha}_2$ are $\kappa_2(s) = \frac{1}{2}\sqrt{\frac{1}{2-2s^2}}$ and $\tau_2(s) = -\frac{1}{2}\sqrt{\frac{1}{2-2s^2}}$, respectively. Consequently, the curve $\boldsymbol{\alpha}_2$ is a general helix with constant ratio $\dfrac{\tau_2(s)}{\kappa_2(s)} = -1$, which is different from a circular helix. The focal curve (9) of $\boldsymbol{\alpha}_2$ is another space curve with an irrational parametrisation

$$\boldsymbol{\alpha}_2^f(s) = \left(\frac{7}{3}(1+s)^{3/2}, -\frac{7}{3}(1-s)^{3/2}, -\frac{7s}{\sqrt{2}}\right)^T, \quad s \in [-1,1].$$

Applying Algorithm 3.6 to $\boldsymbol{\alpha}_2$ we determine the unit speed curve
$\boldsymbol{\beta}_2 : [-1,1] \longrightarrow \mathbb{E}^4$ given by

$$\boldsymbol{\beta}_2(s) = \left(\frac{1}{3}(1+s)^{3/2}, -\frac{1}{3}(1-s)^{3/2}, \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right)\right)^T,$$

and compute its curvatures

$$\begin{aligned} K_{21}(s) &= \tfrac{1}{2}\sqrt{\tfrac{3-2s^2}{2-2s^2}} \\ K_{22}(s) &= \tfrac{1}{2\sqrt{2}}\sqrt{\tfrac{-8s^6+20s^4+2s^2+3}{(1-s^2)(3-2s^2)^2}} \\ K_{23}(s) &= -\tfrac{(2s^2+1)(2s^2+5)}{-8s^6+20s^4+2s^2+3}\sqrt{3-2s^2}. \end{aligned}$$

Obviously, the function $3 + 2s^2 + 20s^4 - 8s^6$ takes only positive values in the interval $[-1,1]$. We can also find the normal vectors of the Frenet frame of $\boldsymbol{\beta}_2$

$$\mathbf{N}_{21}(s) = \begin{pmatrix} \frac{\sqrt{1-s}}{\sqrt{2}\sqrt{3-2s^2}} \\ -\frac{\sqrt{s+1}}{\sqrt{2}\sqrt{3-2s^2}} \\ -\frac{\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{\frac{3-2s^2}{2-2s^2}}} \\ -\frac{\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{\frac{3-2s^2}{2-2s^2}}} \end{pmatrix}, \quad \mathbf{N}_{22}(s) = \begin{pmatrix} \frac{(4s^4-16s^2+8s+3)\sqrt{s+1}}{2\sqrt{(3-2s^2)(-8s^6+20s^4+2s^2+3)}} \\ \frac{(4s^4-16s^2-8s+3)\sqrt{1-s}}{2\sqrt{(3-2s^2)(-8s^6+20s^4+2s^2+3)}} \\ \frac{(4s^4-8s^2+3)\sin\left(\frac{s}{\sqrt{2}}\right)+4\sqrt{2}s\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{2}\sqrt{(3-2s^2)(-8s^6+20s^4+2s^2+3)}} \\ \frac{(-4s^4+8s^2-3)\cos\left(\frac{s}{\sqrt{2}}\right)+4\sqrt{2}s\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{2}\sqrt{(3-2s^2)(-8s^6+20s^4+2s^2+3)}} \end{pmatrix},$$

$$\mathbf{N}_{23}(s) = \begin{pmatrix} -\frac{(2s^3-2s^2-3s+1)\sqrt{s+1}}{\sqrt{-8s^6+20s^4+2s^2+3}} \\ -\frac{(2s^3+2s^2-3s-1)\sqrt{1-s}}{\sqrt{-8s^6+20s^4+2s^2+3}} \\ -\frac{(1-2s^2)\cos\left(\frac{s}{\sqrt{2}}\right)-2\sqrt{2}s\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{-8s^6+20s^4+2s^2+3}} \\ -\frac{(1-2s^2)\sin\left(\frac{s}{\sqrt{2}}\right)+2\sqrt{2}s\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{-8s^6+20s^4+2s^2+3}} \end{pmatrix}.$$

Then, by Steps 5, 6, and 7 of Algorithm 3.6, we get a parametrization of the generalized focal curve $\boldsymbol{\gamma}_2 : [-1, 1] \longrightarrow \mathbb{E}^3$ of $\boldsymbol{\alpha}_2$

$$\boldsymbol{\gamma}_2(s) = \begin{pmatrix} -\frac{4\left(2s^3-8s^2+5s-2\right)}{3\left(1+2s^2\right)\left(5+2s^2\right)}(1+s)^{5/2} \\ -\frac{4\left(2s^3+8s^2+5s+2\right)}{3\left(1+2s^2\right)\left(5+2s^2\right)}(1-s)^{5/2} \\ -\frac{2\sqrt{2}\left(s-2s^3\right)\sin\left(\frac{s}{\sqrt{2}}\right)+4\left(1+4s^2\right)\cos\left(\frac{s}{\sqrt{2}}\right)}{\left(1+2s^2\right)\left(5+2s^2\right)} \end{pmatrix}.$$

The tangent vector to this curve

$$\boldsymbol{\gamma}_2'(s) = \begin{pmatrix} -\frac{2s(s+1)^{3/2}\left(2s^2-4s+1\right)\left(4s^4+44s^2+37\right)}{\left(4s^4+12s^2+5\right)^2} \\ \frac{2s(1-s)^{3/2}\left(2s^2+4s+1\right)\left(4s^4+44s^2+37\right)}{\left(4s^4+12s^2+5\right)^2} \\ \frac{2s\left(4s^4+44s^2+37\right)\left(\left(2s^2-1\right)\cos\left(\frac{s}{\sqrt{2}}\right)+2\sqrt{2}s\sin\left(\frac{s}{\sqrt{2}}\right)\right)}{\left(4s^4+12s^2+5\right)^2} \end{pmatrix}$$

has coordinate functions which vanish for $s = 0$. Hence, the curve $\boldsymbol{\gamma}_2$ has a cusp at the point $\boldsymbol{\gamma}_2(0)$. The general helix $\boldsymbol{\alpha}_2$ given by (42) and its generalized focal curve $\boldsymbol{\gamma}_2$ are plotted in Fig. 2. ’

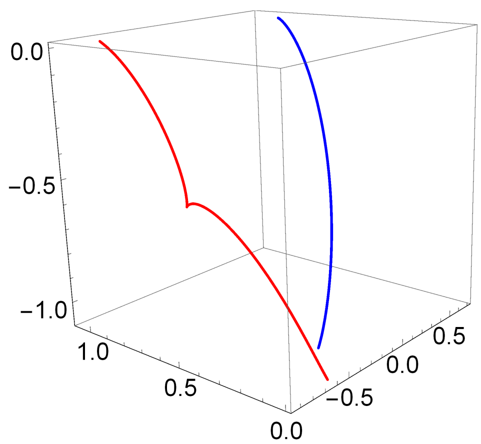

Figure 2: The 3D curve $\boldsymbol{\alpha}_2$ (in blue) and its generalized focal curve $\boldsymbol{\gamma}_2$ (in red).

4.3. A generalized focal curve to a circular helix

Let us examine the unit-speed circular helix with a parametrisation

$$\boldsymbol{\alpha}_3(s) = \left(\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\sin(s), \frac{1}{\sqrt{2}}s\right)^T, \quad s \in (-\infty, +\infty). \tag{43}$$

Firstly, we compute the curvature $\kappa_3 = \frac{1}{\sqrt{2}}$ and the torsion $\tau_3 = \frac{1}{\sqrt{2}}$ of $\boldsymbol{\alpha}_3$. This allows us to get the parametrization of the focal curve $\boldsymbol{\alpha}_3^f : (-\infty, +\infty) \longrightarrow \mathbb{E}^3$ of $\boldsymbol{\alpha}_3$

$$\boldsymbol{\alpha}_3^f(s) = \left(-\frac{1}{\sqrt{2}}\cos(s), -\frac{1}{\sqrt{2}}\sin(s), \frac{1}{\sqrt{2}}s\right)^T, \quad s \in (-\infty, +\infty).$$

Obviously, the space rotation of π radians around the z-axis transforms the circular helix $\boldsymbol{\alpha}_3$ given by (43) into the circular helix $\boldsymbol{\alpha}_3^f$. Secondly, by Theorem 3.2, we observe that

$$\boldsymbol{\beta}_3(s) = \left(\frac{\cos(s)}{\sqrt{2}}, \frac{\sin(s)}{\sqrt{2}}, 2\arctan\left(\frac{s}{\sqrt{2}}\right) - \frac{s}{\sqrt{2}}, \ln\left(\frac{s^2}{2}+1\right)\right)^T,$$

$s \in (-\infty, +\infty)$ is a unit-speed Frenet curve in $\mathbb{E}^4$ with curvatures

$K_{31}(s) = \frac{\sqrt{s^4+4s^2+12}}{\sqrt{2}(s^2+2)}$, $K_{32}(s) = \frac{\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}}{\sqrt{2}(s^6+6s^4+20s^2+24)}$,

$K_{33}(s) = \frac{4(s^2+2)(s^{10}+16s^8+64s^6+176s^4+112s^2+192)}{\sqrt{s^4+4s^2+12}(s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192)}$.

Thirdly, for this curve, we calculate the normal vectors of the Frenet frame

$$\mathbf{N}_{31}(s) = \begin{pmatrix} -\frac{(s^2+2)\cos(s)}{\sqrt{s^4+4s^2+12}} \\ -\frac{(s^2+2)\sin(s)}{\sqrt{s^4+4s^2+12}} \\ -\frac{8s}{(s^2+2)\sqrt{s^4+4s^2+12}} \\ -\frac{2\sqrt{2}(s^2-2)}{(s^2+2)\sqrt{s^4+4s^2+12}} \end{pmatrix},$$

$$\mathbf{N}_{32}(s) = \begin{pmatrix} \frac{(s^8+8s^6+24s^4+32s^2-48)\sin(s)-32s(s^2+2)\cos(s)}{\sqrt{2}\sqrt{s^4+4s^2+12}\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ -\frac{32s(s^2+2)\sin(s)+(s^8+8s^6+24s^4+32s^2-48)\cos(s)}{\sqrt{2}\sqrt{s^4+4s^2+12}\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ -\frac{s^{10}+6s^8-24s^6-144s^4-240s^2+96}{\sqrt{2}(s^2+2)\sqrt{s^4+4s^2+12}\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ \frac{2s(s^8+12s^6+32s^4+16s^2-80)}{(s^2+2)\sqrt{s^4+4s^2+12}\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \end{pmatrix},$$

$$\mathbf{N}_{33}(s) = \begin{pmatrix} \frac{2\sqrt{2}(2s(s^2+2)\sin(s)+(s^4+4s^2-4)\cos(s))}{\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ \frac{2\sqrt{2}((s^4+4s^2-4)\sin(s)-2s(s^2+2)\cos(s))}{\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ -\frac{2\sqrt{2}s(s^4+6s^2-8)}{\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \\ -\frac{s^6+2s^4-28s^2+8}{\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}} \end{pmatrix},$$

and the focal curvatures $c_{31}(s) = \frac{\sqrt{2}(s^2+2)}{\sqrt{s^4+4s^2+12}}$,

$$c_{32}(s) = \frac{32s(s^2+2)}{\sqrt{s^4+4s^2+12}\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}},$$
$$c_{33}(s) = \frac{(s^4+4s^2-4)(s^{12}+12s^{10}+36s^8-64s^6-208s^4-448s^2-320)}{4(s^2+2)(s^6+12s^4+4s^2+16)\sqrt{s^{12}+12s^{10}+52s^8+160s^6+240s^4-64s^2+192}}.$$

Finally, steps from 5. to 7. of Algorithm 3.6 yield the generalized focal curve

$\boldsymbol{\gamma}_3 : (-\infty, +\infty) \longrightarrow \mathbb{E}^3$ of $\boldsymbol{\alpha}_3$ in a parametric form

$$\boldsymbol{\gamma}_3(s) = \begin{pmatrix} \dfrac{\sqrt{2}\left(s(s^6+6s^4+4s^2-8)\sin(s)-(3s^6+10s^4+28s^2+8)\cos(s)\right)}{(s^2+2)(s^6+12s^4+4s^2+16)} \\ -\dfrac{\sqrt{2}\left((3s^6+10s^4+28s^2+8)\sin(s)+s(s^6+6s^4+4s^2-8)\cos(s)\right)}{(s^2+2)(s^6+12s^4+4s^2+16)} \\ \dfrac{2(s^2+2)(s^6+12s^4+4s^2+16)\tan^{-1}\left(\frac{s}{\sqrt{2}}\right)-\sqrt{2}s(s^8+12s^6+20s^4-16s^2+32)}{(s^2+2)(s^6+12s^4+4s^2+16)} \end{pmatrix},$$

$s \in (-\infty, \infty)$. The last curve has a curvature

$\kappa_{\gamma 3} = \dfrac{(s^2+2)^2(s^6+12s^4+4s^2+16)^3\sqrt{s^6+6s^4+4s^2+8}}{\sqrt{2s^2}(s^{10}+22s^8+72s^6+320s^4+528s^2+416)(s^{10}+13s^8+32s^6-72s^4+48s^2+16)^{3/2}}$ and a torsion

$\tau_{\gamma 3} = -\dfrac{(s^2+2)^2(s^6+12s^4+4s^2+16)^2}{\sqrt{2}(s^6+6s^4+4s^2+8)(s^{10}+22s^8+72s^6+320s^4+528s^2+416)}$. Moreover, $\boldsymbol{\gamma}_3$ has a cusp at the point $\boldsymbol{\gamma}_3(0)$ because all coordinate functions of $\boldsymbol{\gamma}_3'(s)$ vanish for $s = 0$. The mutual position of $\boldsymbol{\alpha}_3$ and $\boldsymbol{\gamma}_3$ is shown in Fig.3.

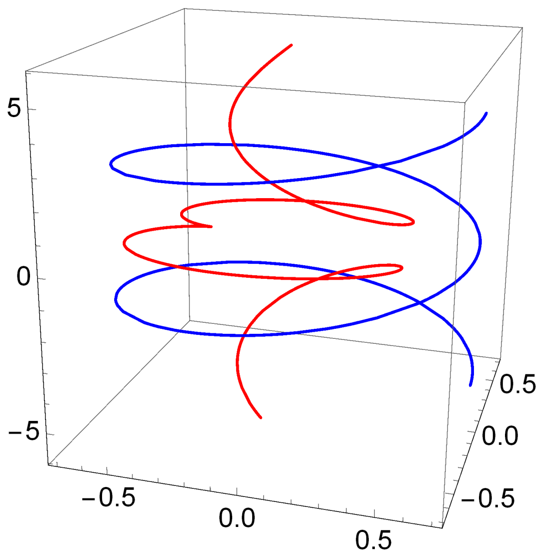

Figure 3: The part of the circular helix α_3 (in blue) defined on the interval $[-2\pi, 2\pi]$ and its generalized focal curve $\boldsymbol{\gamma}_3$ (in red).

5. CONCLUSIONS

It is well-known that any Frenet curve in the Euclidean space $\mathbb{E}^3$ (resp. $\mathbb{E}^4$) possesses a unique focal curve in $\mathbb{E}^3$ (resp. $\mathbb{E}^4$). The construction of the focal curve in $\mathbb{E}^3$ is based on the curvature, the torsion, the principle normal vector, and the binormal vector of the original curve. In this paper, we present a three-stage construction for a new associate curve in $\mathbb{E}^3$ to any $3D$ Frenet curve of class C^4. The new curve is called a generalized focal curve of a Frenet curve in $\mathbb{E}^3$. The first stage of the proposed construction defines a unit-speed Frenet curve in $\mathbb{E}^4$ from a given unit speed Frenet curve in $\mathbb{E}^3$. The second stage of the construction is the finding of the unique $4D$ focal curve of the considered unit-speed Frenet curve in $\mathbb{E}^4$. The final third stage determines a generalized focal curve of the given curve in $\mathbb{E}^3$ as a projection of the obtained $4D$ focal curve into the Euclidean space $\mathbb{E}^3$. By proving Theorems 3.1 and 3.2, and by presenting an algorithm for determining a generalized focal curve we explore the fact that all Frenet curves in $\mathbb{E}^3$ of class C^4 can be naturally divided into three disjoint sets: curves with a non-constant ratio of torsion to curvature; curves with a constant ratio of torsion to curvature, a non-constant curvature, and a non-constant torsion; curves with a constant curvature and a constant torsion. The algorithm for obtaining a generalized focal curve is illustrated by three examples concerning a non-helical curve, a general helix which is not a circular helix, and a circular helix. Determining generalized focal curves of other types of non-helical curves and general helices will be a subject of future research.

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