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Modified Numerical Technique for the Solution of Fractional Delay Differential Equations via Bernoulli Wavelets

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Abstract

In this paper, modified and accurate numerical technique based on Bernoulli wavelets operational matrix for the solution of delay differential equations of fractional order is presented. The fundamental definitions of Bernoulli wavelets and their properties are utilized to construct the operational matrix of fractional integration which transforms the given problem in to system of algebraic equations with unknown coefficients. The numbers of examples are included to exhibit the efficiency, accuracy and wide range of applicability of the present scheme.

Keywords: Fractional delay differential equations; Caputo fractional derivative; Fractional integration operational matrix; Numerical solution.

1. INTRODUCTION

Recently, fractional calculus is considered as an important and developing concept.It may be also considered as an old topic since it was introduced in 1695 for the derivative of order $\left(\frac{1}{2}\right)$. The mathematicians L'Hospital and Leibnitz have worked on fractional calculus in the beginning. The equations containing fractional order are mainly occurring in engineering branches, physics, medicine, economics, signal processing, solid mechanics, fluid dynamics traffic model, etc. Eventually, the prime importance is given to the fractional equations occurred from various models.

In the available literature, many authors have been worked on analytic solutions for the differential equations of fractional order [1, 2].But it is also noted that obtaining analytic solution is difficult and for most of the fractional differential equations, analytic solutions are not known. It is because of this, developing numerical methods for these equations is essential.

Delay differential equation is the extended concept of ordinary differential equations and it consists of the systems whose present course of actions depends on their previous data. From last two decades, many researchers devoted to the numerical solution of the delay differential equations. Some of them are Chebyshev polynomials [3], Laguerre polynomials [4], Hermite polynomials [5], Adomian decomposition method [6], Haar wavelets [7], variational iteration method [8], etc.

Fractional delay differential equations are the habitual class of delay differential equations. We found only few numerical methods for solving these equations which contain fractional order. Some of the available numerical techniques are based on Hermite wavelets [9], Bernoulli wavelets [10], Taylor wavelets [11] and Legendre functions [12].In this article, we use the Bernoulli wavelets for the numerical solution of fractional order delay differential equations of various kinds in modified form as compare to [10].

In the last few years, wavelet theory has received more importance in many fields like signal processing, numerical analysis and optimal control because of its powerful characteristic properties. In this present work, we have used Bernoulli wavelets for the numerical solution delay differential equations of fractional order.

In this paper, we consider the fractional delay differential equations of the form:

$$D^{\alpha}y(t)=f\left(t,y(t),y(t-\tau)\right) \quad , \quad 0\le t\le 1, \alpha>0 \,\&\, 0<\tau<1 \tag{1}$$

$$\text{with } \begin{cases} y^{(j)}(0)=\mu_j \quad j=0,\ldots,\lceil\alpha\rceil \\ \quad y(t)=\varphi(t) \quad t<0 \end{cases} \tag{2}$$

Where, y is unknown function, $f \,\&\, \varphi$ are known functions. α , τ and the initial values λ_j are given. Here $\lceil\alpha\rceil$ is the smallest integer which is larger than or equal to $|\alpha|$. In this work, fractional derivative of the unknown function $D^{\alpha}y(t)$ in the given problem is approximated by linear combinations of the Bernoulli wavelets and then truncating them at suitable levels. Finally, the problem is converted in to a system of algebraic equations by using the Bernoulli wavelets operational matrix of the fractional integration.

The paper is organized as follows: In section 2, the basic definitions of fractional calculus, Bernoulli wavelets and function approximations are given. Operational matrix for the fractional order integration using Bernoulli wavelets introduced in section 3. In section 4, the numerical method of solution is described. Error analysis of the approximate solution is given in section 5. The numbers of standard numerical examples are included to demonstrate the accuracy of the proposed technique. Lastly, conclusion and scope is presented in section 7.

2. PRELIMINARIES OF FRACTIONAL CALCULUS AND BERNOULLI WAVELETS

In this part, we briefly recall the basic definitions of fractional calculus and prelimaries of Bernoulli wavelets.

2.1 Fractional Calculus

Definition : The fractional integral operator of order $m \geq 0$ is defined in Riemann-Liouville sense as [12]

$$I^m \times V(x) = \begin{cases} \frac{1}{\Gamma m} \int_{s=0}^{x} \frac{v(s)}{(x-s)^{1-m}} ds, & m > 0, t > 0 \\ v(x), & m = 0 \end{cases} \tag{3}$$

The following properties hold good for the Riemann-Liouville fractional integral

$(i) I^m [Av(x) + Bv(x)] = AI^m v(x) + BI^m v(x)$, where A and B are the constants.

$(ii) I^m x^\alpha = \frac{\Gamma \alpha + 1}{\Gamma \alpha + m + 1} x^{m+\alpha}, \alpha > -1.$

Definition :The frcational derivative operator of order m in the Caputo's sense id defined as

$$D^m v(x) = \frac{1}{\Gamma k - m} \int_{s=0}^{x} \frac{v^{[k]}(s)}{(x-s)^{m+1-k}} ds, k - 1 < m \leq k, k \in N \tag{4}$$

The following properties hold good for the Caputo derivative

$(i) D^m I^m v(x) = v(x)$

$(ii) I^m D^m v(x) = v(x) - \sum_{i=0}^{k-1} v^{(i)}(0) \frac{x^i}{i!}$

2.2 Bernoulli wavelets

Wavelets constitute a family of functions constructed by the dilation variable a and translation variable b of a signal function is known as 'mother wavelet'. If dilation parameter and translation parameter vary continuously, then the family of continuous wavelets given by the following equation

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left[\frac{t-b}{a}\right] \text{ where, } a \in R, b \in R \text{ and } a \neq 0 \tag{5}$$

When these parameters $a\&b$ restrict to the discrete values like $a = a_0^{-(k)}$, $b = nb_0 a_0^{-(k)}$ with $a_0 > 1, b_0 > 0$, n & k are positive constants. Then the family of discrete wavelets are given by

$$\psi_{k,n}[t] = |a_0|^{\frac{k}{2}} \psi\left[a_0^{\,k} t - nb_0\right]$$

where the wavelet basis for $L^2(R)$ is formed $\psi_{k,n}(t)$.

The Bernoulli polynomials $\beta_p(x)$ having order p and defined in $0 \le x \le 1$ as [10]

$$\beta_p[x] = \sum_{i=0}^{m} \binom{p}{i} \beta_{p-i} x^i \tag{6}$$

with $\beta_i = \beta_i[0], i = 0,1,2,3,..., p$ are the Bernoulli numbers. The starting few polynomials are as follows:

$$\beta_0(x) = 1, \beta_1(x) = x - \frac{1}{2}, \beta_2(x) = x^2 - x + \frac{1}{6}, \beta_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2} \text{ and so on.}$$

It is also noted that Bernoulli polynomials hold the following property

$$\int_{x=0}^{1} \beta_q(x)\beta_p(x)dx = (-1)^{q-1} \frac{p!q!}{(p+q)!} \beta_{p+q}, p \ge 1, q \ge 1 \tag{7}$$

If $x \in [0,1)$ then the Bernoulli wavelets denoted by $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ are defined as

$$\psi_{n,m}[x] = \begin{cases} 2^{\frac{k}{2}} \tilde{\beta}_p\left(2^k x - \hat{q}\right), a_1 \le x \le b_1 \\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{8}$$

where $a_1 = \dfrac{\hat{q}-1}{2^k}$ and $b_1 = \dfrac{\hat{q}+1}{2^k}$

with
$$\tilde{\beta}_p(x) = \begin{cases} 1, p = 0 \\ \dfrac{1}{\sqrt{\dfrac{(-1)^{p-1}[p!]^2}{(2p)!} \beta_{2p}}}, p > 0 \end{cases} \tag{9}$$

where the parameters k, p, q are varies as : $k = 2,3,4,..., \hat{q}$ $q = 1,2,3,...,2^{k-1}$ & $p = 0,1,2,3,..., M-1$. Here p is the Bernoulli polynomial order , M is the fixed positive integer.

When $a = 2^{-k}$ and $b = \hat{q}2^{-k}$ then the coefficient parameter $\sqrt{\frac{1}{\frac{(-1)^{p-1}[p!]^2}{(2p)!}\beta_{2p}}}$ holds the normality property.

2.3 Function approximation

Let f be a function such that $f \in L^2$ defined in $0 \le x \le 1$. By using Bernoulli wavelets f can be expressed as

$$f(x) \cong \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \tag{10}$$

It can also written by truncated series of the function f as

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = c^T \psi(x) \tag{11}$$

where $m = 0,1,2,3,...,M-1$ and $n = 1,2,3,...,2^{k-1}$

The above equation () can be re-written as

$$f(x) \cong \sum_{i=1}^{m} c_{i,p} \psi_{i,p}(x) = c^T \psi(x) \tag{12}$$

Where C is the coefficient matrix and is

$$C = \left[c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1}, c_{2^{k-1},0}, c_{2^{k-1},1}, ..., c_{2^{k-1},M-1} \right]^T \tag{13}$$

Also,

$$\psi(x) = \left[\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), ..., \psi_{2,M-1}(x), \psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), ..., \psi_{2^{k-1},M-1}(x) \right]^T \tag{14}$$

The coefficient vector C can be determined as follows:

$$f_p = \left\langle f(x) . \psi_p(x) \right\rangle = \int_{x=0}^{1} f(x) \psi_p(x) dx$$

From the equation (12), we have

$$f_p = \sum_{i=1}^{n} c_{i,p} \int_{x=0}^{1} \psi_i(x) \psi_p(x) dx = \sum_{i=1}^{m} c_{i,p} k_{i,p} \text{ where } p = 1,2,3,...,m$$

with $\quad k_{i,p} = \int_{x=0}^{1} \psi_i(x) \psi_p(x) dx, p = 1,2,3,...,\hat{m}$

Therefore, $f_p = c^T \left[k_{1,p}, k_{2,p}, ..., k_{m,p} \right]^T \; p = 1,2,3,...,m$

$$\Rightarrow [F]^T = [C]^T [k] \tag{15}$$

where $\quad [F]^T = \left[f_{1,0}, f_{1,1}, f_{1,2}, ..., f_{1,m} \right]^T$ and $k = \left[k_{i,p} \right]$

where k is a matrix of order $m \times n$ and is

$$k = \langle \psi(x), \psi(x) \rangle = \int_{x=0}^{1} \psi(x).\psi^T(x)\,dx$$

Thus, the coefficient vector c^T in (11) is given by the following equation as

$$c^T = [F]^T [k]^{-1} \tag{16}$$

3. FRACTIONAL INTEGRATION OPERATIONAL MATRIX

We obtain a operational matrix of fractional integration by using Bernoulli wavelets.The following property and theorem are helpful in constructing the opeartional matrix.

Property: Any component $\psi(x)$ of

$\psi(x) = \left[\psi_{1,0}(x), \psi_{1,1}(x), \ldots, \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), \ldots, \psi_{2,M-1}(x), \psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), \ldots, \psi_{2^{k-1},M-1}(x)\right]^T$

can be expressed by using Bernoulli polynomials as

$$\psi_{n,m}(x) = 2^{\frac{k}{2}} \begin{cases} \eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}(x), \; p = 0 \\ \tilde{\beta}\left(2^k x - (2q-1)\right) \eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}, \; p \neq 0 \end{cases} \tag{17}$$

where $m = 0,1,2,3,\ldots,M-1$, $n = 1,2,3,\ldots,2^{k-1}$, $\eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}(x)$ denotes the characteristic function and is given by $\eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}(x) = \begin{cases} 1, \frac{q-1}{2^{k-1}} \le x \le \frac{q}{2^{k-1}} \\ 0, \quad \text{otherwise} \end{cases}$

Theorem: Let $\psi_{n,m}(x)$ be the k^{th} term of the equation (14) and $I^{(u)}\psi_{n,m}(x)$ be the fractional integral having order u of $\psi_{n,m}(x)$. Then

$$I^{(u)}\psi_{n,m}(x) = 2^{k/2} \begin{cases} \sum_{l=(q-1)M+1}^{qM} A_l^q \psi_{n,m}(x), \; p = 0 \\ \sum_{l=(q-1)M+1}^{qM} B_l^{(p,q)} \psi_{n,m}(x) \; p \neq 0 \end{cases} \tag{18}$$

where $\left[A_{(q-1)M+1}^q, A_{(q-1)M+2}^q, \ldots, A_{qM}^q\right] = \left\langle I^{(u)} \eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}(x).\psi_q(x) \right\rangle \left[K_q\right]^{-1}$,

$$B_l^{(p,q)} = \sum_{i=0}^{p} \sum_{j=0}^{i} r_{jl} \frac{\binom{p}{i} \beta_{p-i}}{\sqrt{\frac{(-1)^{p-1}(p!)^2}{(2p!)} \beta_{2p}}} \qquad \text{and}$$

$$\left[r_{j,(q-1)M+1}, r_{j,(q-1)M+2}, \ldots, r_{j,(q)M} \right] = \left\langle I^{(u)} \left(x^j \eta_{\left[\frac{q-1}{2^{k-1}}, \frac{q}{2^{k-1}}\right]}(x) \right), \psi_q(x) \right\rangle \left[K_q \right]^{-1}, \ j = 0,1,2,\ldots,i$$

also $\psi_q(x) = \left[\psi_{(q-1)M+1}, \psi_{(q-1)M+2}, \ldots, \psi_{(q)M} \right]^T$, $\left[k_q \right] = \langle \psi_q(x), \psi_q(x) \rangle$.

Proof: [10]

Corollary: Let $\psi(x)$ denote the Bernoulli wavelets vector as defined in (14). Then,

$$I^{(u)} \psi(x) = P^{(u)} \psi(x) \tag{19}$$

where $P^{(u)}$ is the matrix of order $\hat{m} \times \hat{m}$ and is called Bernoulli wavelet operational matrix of integration of order u . It is given by

$$P^{(u)} = \begin{bmatrix} H_1 & 0 & \cdots & 0 & 0 \\ 0 & H_2 & 0 & \cdots & 0 \\ 0 & 0 & H_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & H_{2^{k-1}} \end{bmatrix} \tag{20}$$

Here $H_\mu, \mu = 1,2,3,\ldots,2^{k-1}$ is $\hat{M} \times \hat{M}$ matrix and the $(w,z)^{\text{th}}$ component is

$$\left[H_\mu \right]_{w,z} = 2^{\frac{k}{2}} \begin{cases} Q_z^\mu, & w = 1 \\ \sum_{i=0}^{w-1} \sum_{j=0}^{i} d_{jz} e_{(w-1),i} \binom{i}{j} 2^{kj} [-1]^{i-j} (2\mu - 1)^{i-j}, & w \in \left[2, \hat{M} \right] \end{cases} \tag{21}$$

Proof: The instant consequence of the above theorem.

To understand the actual importance of the above corollary, we have the following examples:

(i) For $M=3, k=2$ and $u=1$, the operational matrix using Bernoulli wavelets can be expressed a

$$P^{(1)}=\begin{bmatrix} \frac{3}{8} & \frac{90211}{1250000} & -\frac{69389}{25000}\times10^{-17} & 0 & 0 & 0 \\ -\frac{469097}{500000} & \frac{166533}{100000}\times10^{-16} & \frac{322749}{10000000} & 0 & 0 & 0 \\ \frac{341}{100} & -\frac{129099}{1000000} & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{8} & \frac{90211}{1250000} & \frac{194289}{25000}\times10^{-16} \\ 0 & 0 & 0 & -\frac{32927}{6250} & \frac{105471}{50000}\times10^{-15} & \frac{322749}{10000000} \\ 0 & 0 & 0 & \frac{117533}{2500} & -\frac{129099}{1000000} & -\frac{1}{8} \end{bmatrix} \quad (22)$$

(ii) For $M=3, k=2$ and $u=\frac{1}{2}$, the operational matrix using Bernoulli wavelets can be expressed as

$$P^{\left(\frac{1}{2}\right)}=\begin{bmatrix} \frac{1383}{2000} & \frac{148069}{2500000} & -\frac{849583}{100000000} & 0 & 0 & 0 \\ -\frac{117139}{100000} & \frac{391343}{1000000} & \frac{416931}{5000000} & 0 & 0 & 0 \\ \frac{392507}{100000} & -\frac{28037}{25000} & -\frac{150251}{1000000} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3296}{3125} & \frac{433813}{10000000} & -\frac{13041}{12500000} \\ 0 & 0 & 0 & -\frac{423529}{100000} & \frac{53041}{100000} & \frac{338039}{10000000} \\ 0 & 0 & 0 & \frac{340343}{10000} & -\frac{251573}{100000} & \frac{11571}{50000} \end{bmatrix} \quad (23)$$

4. METHOD OF SOLUTION

In this section, an accurate numerical scheme for the solution of fractional delay differential equations is described as follows:

For the positive integer k, the function $D^{(\alpha)}y(t)$ can be expressed over $0 \le t < 1$ as

$$D^{(\alpha)}y(t) \cong \sum_{m=0}^{\hat{M}-1}\sum_{n=1}^{2^{k-1}} C_{n,m}\psi_{n,m}(t) \approx C^{[T]}\psi_{k,\hat{M}}(t) \tag{24}$$

where the coefficient vector $C^{[T]}$ and $\psi_{k,\hat{M}}(t)$ are given respectively in equations (13) and (14).

Now, the integral operator $I^{(\alpha)}$ is applied o both sides of the above equation, then we have

$$y(t) \cong I^{(\alpha)}\left[C^{[T]}\psi_{k,\hat{M}}(t)\right] \approx C^{[T]}I^{(\alpha)}\psi_{k,\hat{M}}(t) = C^{[T]}P^{(u)}\psi_{k,\hat{M}}(t) + \sum_{j=0}^{\lceil\alpha\rceil}\frac{\mu_j}{j!}t^j \tag{25}$$

By using the properties of Caputo derivatives along with initial conditions

$y^{(j)}(0) = \mu_j$ for $j = 0,1,2,3,\ldots,\lceil\alpha\rceil$, we get

$$y(t) = \begin{cases} C^{[T]}I^{(\alpha)}\psi_{k,\hat{M}}(t) + \sum_{j=0}^{\lceil\alpha\rceil}\frac{\mu_j}{j!}t^j, & \text{when } 0 \le t < 1 \\ \varphi(t), & \text{when } t < 0 \end{cases} \tag{26}$$

Thus,

$$y(t-\tau) = \begin{cases} C^{[T]}I^{(\alpha)}\psi_{k,\hat{M}}(t-\tau) + \sum_{j=0}^{\lceil\alpha\rceil}\frac{\mu_j}{j!}(t-\tau)^j, & \text{when } \tau \le t < 1 \\ \varphi(t), & \text{when } t < \tau \end{cases} \tag{27}$$

Substituting the values of the above equations (24), (26) and (27) in the given fractional delay differential equation, we get algebraic equation with unknown constant coefficients. Then, we collocate resultant algebraic equation using following Newton-Cotes nodes as

$$t_i = \frac{2i-1}{2^k\hat{M}}, i = 1,2,3,\ldots,2^k\hat{M}$$

Now, we obtained a system of algebraic equations of $2^{k-1}\hat{M}$ numbers with unknown constant coefficients $C_{n,m}$. By solving these systems of equations using Newton's method, we get the approximate solution of the given problem. It is also noted that with the increase in the of values of $\hat{M}$, the result will be more convergent towards exact solution.

5. ERROR ANALYSIS

In this part, we estimate the error related to the best Bernoulli wavelet based approximation. To do this, we have the following results.

Lemma: If $f(x) \in L^2$ defined over $a \le x \le b$ is approximated using Bernoulli polynomials having degree at most $\hat{M}-1$ and $f(x) \cong f_{\hat{M}}(x) = \sum_{i=0}^{\hat{M}-1} k_i \beta_i(x)$ in [13] where $\beta_i(x) = 0,1,2,3,...,\hat{M}-1$. Then, $\lim_{\hat{M}\to\infty} \left\| f(x) - f_{\hat{M}}(x) \right\|_2 = 0$. (28)

Theorem: If $f(x) \in L^2$ defined over interval $0 \le x \le 1$ and is approximated by $f_{app}(x)$ as

$$f(x) \cong f_{app}(x) \cong \sum_{i=1}^{\hat{m}} C_i \psi_i(x) = C^T \psi(x) \quad \text{where,}$$

$$C = \left[c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1}, c_{2^{k-1},0}, c_{2^{k-1},1}, ..., c_{2^{k-1},M-1} \right]^T \quad \text{and}$$

$$\psi(x) = \left[\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), ..., \psi_{2,M-1}(x), \psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), ..., \psi_{2^{k-1},M-1}(x) \right]^T$$

Then, $\lim_{\hat{m}\to\infty} \left\| f(x) - f_{\hat{m}}(x) \right\|_2 = 0$. (29)

Proof: For proof see in [10]

6. NUMERICAL EXAMPLES

In this section, we apply the present method as described in previous section for various fractional delay differential equations.

Example 1. Consider the following fractional delay differential equation [9].

$$D^{\alpha} y(t) = -y(t) - y(t-0.3) + f(t), t \in [0,1], \alpha \in (2,3] \tag{30}$$

where $f(t) = e^{-t+0.3}$ with initial conditions $y(0) = 1, y'(0) = -1, y''(0) = 1$, $t < 0$.

The exact solution of the above equation (30) is $y(t) = e^{-t}$ when $\alpha = 3$.In table 1, the values of the exact solution and the numerical solutions obtained by applying the present method and comparison with other existing method have given.

Table 1. Comparison of exact solution, Hermite wavelet method solution and present method $k = 2 \& M = 7, \alpha = 3$ at different values of t for example 1.

t	Exact solution	Hermite wavelet [9]	Present method
0.0	1.0000	1.0000	1.0000
0.2	0.8187	0.8187	0.8179
0.4	0.6703	0.6703	0.6738
0.6	0.5488	0.5488	0.5486
0.8	0.4493	0.4493	0.4482

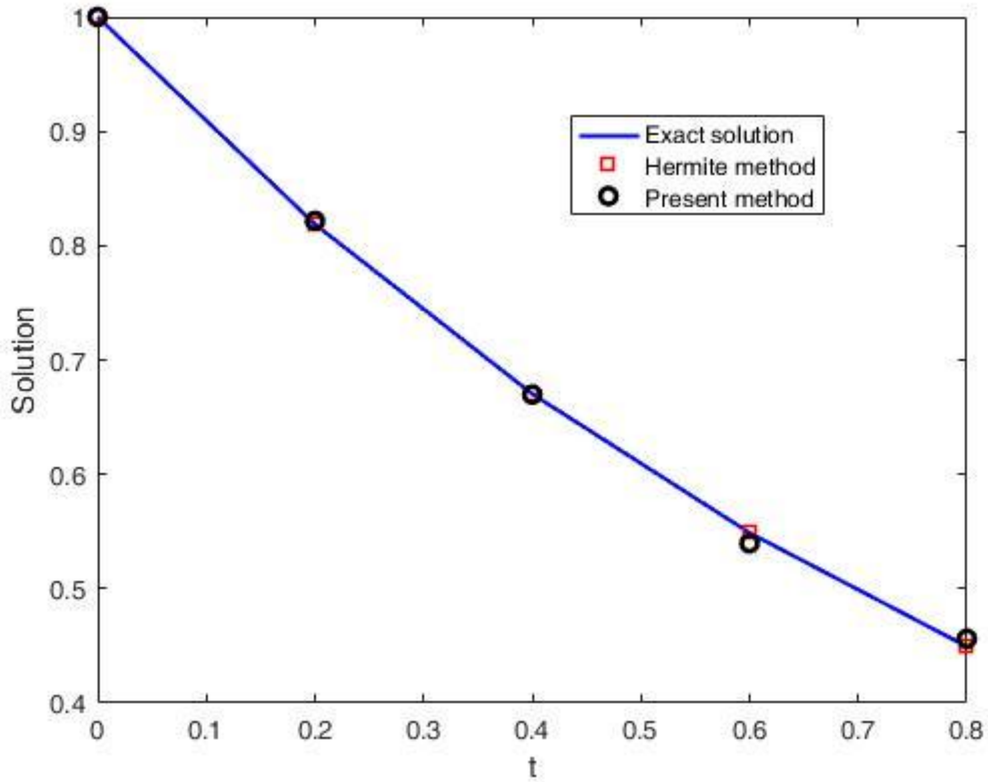

Fig.1 Comparison of the exact solutions, Hermite method and present method at different values of t for example 1.

Example 2. Consider another following fractional delay differential equation [12]

$$D^{(1.5)}y(t) = -y(t) + y\left(\frac{t}{2}\right) + g(t),\ t \in [0,1] \tag{31}$$

where $g(t) = \frac{7}{8}t^3 + \frac{6}{\Gamma\left(\frac{5}{2}\right)}t^{1.5}$ and with the following boundary conditions $y(0) = 0, y(1) = 1$.

We solve this equation by applying the present method and the solutions obtained are very close to the exact solution $y(t) = t^3$. These results are given in the following table 2. Also, we plot the achieved numerical results by the present method with exact solution of the equation (31) in Fig.2. It is clear that numerical solutions approach to the exact solutions at different values of t.

Table 2. Solution by present method at different values of t of example 2.

t	Exact solution	Present method
0.0	0.000000	0.000000
0.1	0.001000	0.009945
0.2	0.008000	0.009895
0.3	0.027000	0.026924
0.4	0.064000	0.059874
0.5	0.125000	0.124975
0.6	0.216000	0.215845
0.7	0.343000	0.342759
0.8	0.512000	0.511984
0.9	0.729000	0.728941
1.0	1.000000	0.998452

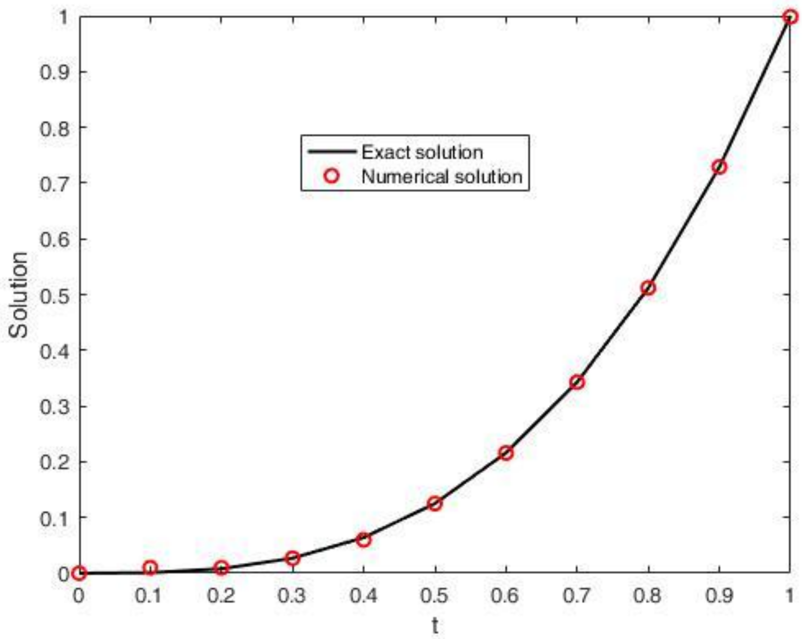

Fig.2 The graphs of the exact solutions and the numerical solutions for example 2 at different values of t.

Example 3**.** Consider the following fractional delay differential equation [10].

$$D^{\alpha}y(t)+y(t)=y(t-\tau)+f(t) \tag{32}$$

where $f(t)=\dfrac{2t^{(2-\alpha)}}{\Gamma(3-\alpha)}-\dfrac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}+2\tau t-\tau^{2}-\tau$ with $t\in[0,1]$, $\alpha\in(0,1]$ and $y(t)=t^{2}-t$ if $t\le 0$.

We solve this equation (32) for $k=2\,\&\,M=3$ and approximate $D^{\alpha}y(t)$ as

$$D^{\alpha}y(t)\approx C_{1,0}\psi_{1,0}(t)+C_{1,1}\psi_{1,1}(t)+C_{1,2}\psi_{1,2}(t)+C_{2,0}\psi_{2,0}(t)+C_{2,1}\psi_{2,1}(t)+C_{2,2}\psi_{2,2}(t)=C^{T}\psi_{2,2}(t)$$

Where $C=\left[C_{1,0},C_{1,1},C_{1,2},C_{2,0},C_{2,1},C_{2,2}\right]^{T}$ is the vector of unknown constants that we need to determine. Then we have

$$y(t)=\begin{cases}C^{T}I^{\alpha}\psi_{2,2}(t), & t\in[0,1]\\ t^{2}-t, & t\le 0\end{cases} \tag{33}$$

and

$$(t-\tau)=\begin{cases}C^{T}I^{\alpha}\psi_{2,2}(t-\tau), & t\in[\tau,\tau+1]\\ (t-\tau)^{2}-(t-\tau), & t\le\tau\end{cases} \tag{34}$$

By substituting equations (33) and (34) to equation (32), we obtain an algebraic equation. By collocating the algebraic equation using $t_i = \frac{2i-1}{12}, i = 1, 2, ..., 6$. We get linear system in the $C_{n,m}$'s and by substituting these values and after simplification, we obtain $y(t) \cong t^2 - t$.

In case, if there is a delay term τ, we obtain approximations of the solutions depending on α & τ .In table 3, we demonstrate the absolute errors for the method by taking $k = 2$ & $M = 3$ or with the number of bases .The values in the table 3 suggest that numerical results are having more accuracy.

Table 3. The absolute errors for $k = 2$ & $M = 3, \alpha = 1$ at different values of τ of example 3.

t	$\tau = 0.1\times10^{-3}$	$\tau = 0.1\times10^{-2}$	$\tau = 0.1\times10^{-1}$
0.2	8.3291×10^{-17}	1.9395×10^{-16}	0.0135×10^{-9}
0.4	2.2237×10^{-16}	3.3296×10^{-16}	1.0962×10^{-16}
0.6	1.4692×10^{-14}	8.5962×10^{-14}	3.1492×10^{-14}
0.8	1.5743×10^{-14}	8.5694×10^{-14}	3.2863×10^{-14}
1.0	0.0025×10^{-6}	0.016×10^{-8}	0.0264×10^{-8}

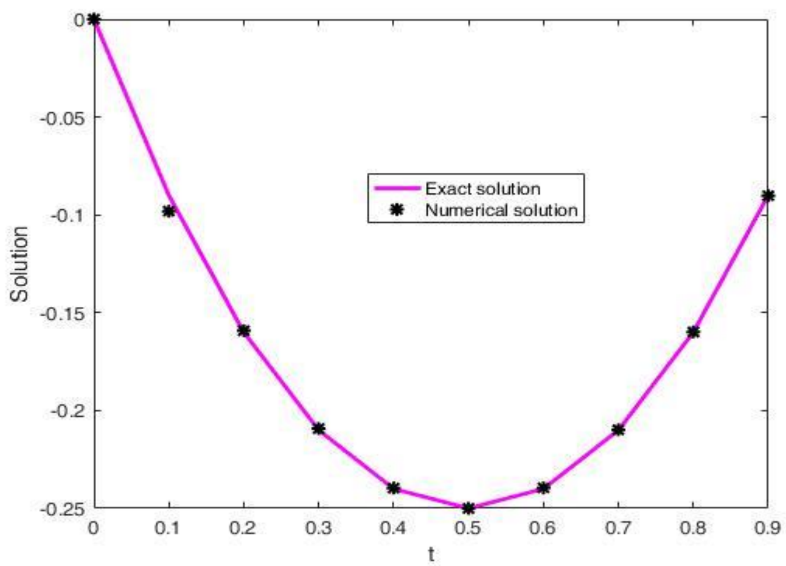

Fig.3 The graphs of the exact solutions and approximate solutions for example 3 at different values of t.

Example 4. Consider the fractional order delay differential equation [14].

$$D^{\alpha} y(t) = y(t-1) + f(t) \quad t \in (0,2] \tag{35}$$

with $y(t) = 1$, $t \le 0$ where $\alpha \in (0,1]$ and the function $f(t)$ is given by

$$f(t) = \begin{cases} -2.1 + 1.05t & t \in (0,1] \\ -1.05 & t \in (1,2] \end{cases}$$

If $\alpha = 1$ then equation (35) has the exact solution as

$$y(t) = \begin{cases} 1 - 1.1t + 0525t^2 & t \in (0,1] \\ -0.25 + 1.575t - 1.075t^2 + 0.175t^3 & t \in (1,2] \end{cases}$$

If $\alpha \ne 1$ then the exact solution of equation (35) is not known .In such case, we consider the residual error

$$D^{\alpha} y(t) - y(t-1) - f(t) = 2^{-\alpha} D^{\alpha} g(t) - g\left(t - \frac{1}{2}\right) - u(2t)$$

We compare the residual errors of numerical solutions from the present method and Legendre multi-wavelet collocation method in [14] with $\alpha = 0.95$.

Table 4. Comparison of residue errors for numerical solutions by present method and the Legendre multi-wavelet collocation method [LMWM] at $\alpha = 0.95$

t	LMWM[14]		Present Method	
	$M = 7$	$M = 10$	$M = 7$	$M = 10$
0.2	1.92×10^{-4}	1.49×10^{-5}	1.69×10^{-13}	1.09×10^{-12}
0.4	1.36×10^{-5}	1.61×10^{-6}	8.98×10^{-14}	2.58×10^{-11}
0.6	9.72×10^{-6}	1.13×10^{-6}	2.52×10^{-12}	2.68×10^{-10}
0.8	6.10×10^{-5}	4.47×10^{-6}	2.52×10^{-12}	1.79×10^{-9}
1.2	3.11×10^{-5}	1.56×10^{-6}	8.16×10^{-4}	2.42×10^{-4}
1.4	2.85×10^{-6}	2.17×10^{-7}	3.78×10^{-4}	2.48×10^{-5}
1.6	2.40×10^{-6}	1.81×10^{-7}	2.75×10^{-4}	0.78×10^{-5}
1.8	1.72×10^{-5}	8.23×10^{-7}	2.62×10^{-4}	7.68×10^{-5}

Example 5. Consider the nonlinear fractional order delay differential equation [12]

$$D^{\left(\frac{3}{2}\right)}y(t)=y\left(t-\frac{1}{2}\right)+y^3(t)+g(t),\ t\in[0,1] \tag{36}$$

where $g(t)=\frac{2}{\Gamma\left(3/2\right)}t^{\left(\frac{1}{2}\right)}-\left(t-\frac{1}{2}\right)^2-(t)^6$ with boundary conditions:

$y(0)=0, y(1)=1$.

In the above equation the value of α is $\frac{3}{2}$ and solved by applying the present method. The solutions obtained are converging with the exact solution $y(t)=t^2$. Obtained numerical results are tabulated in table 5.

Table 5. Numerical solutions by present method at different values of t of example 5.

t	Exact solution	Numerical solution
0.0	0.000000	0.000000
0.1	0.010000	0.009956
0.2	0.040000	0.039985
0.3	0.090000	0.089547
0.4	0.160000	0.159874
0.5	0.250000	0.248957
0.6	0.360000	0.359842
0.7	0.490000	0.489895
0.8	0.640000	0.639981
0.9	0.810000	0.809748
1.0	1.000000	0.999986

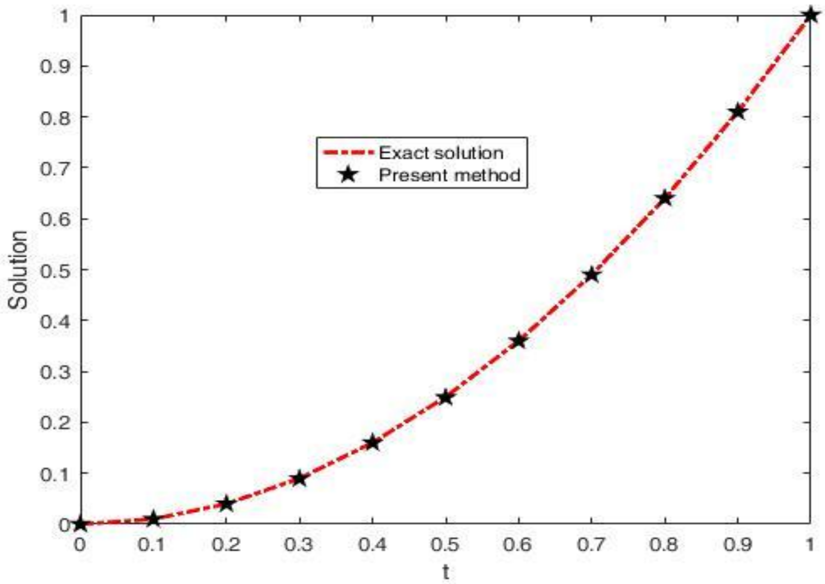

Fig.4 The graphs of the exact solutions and approximate solutions for example 5 at different values of t.

7. CONCLUSION

The main purpose of this research work was to provide an efficient and accurate numerical technique through Bernoulli wavelets. We re-constructed the operational matrix and utilized for solving fractional order delay differential equations. The achieved approximate results from the illustrative examples were compared with exact solutions & some other existing methods and are in good agreement as shown in tables and figures. Thus, we conclude that presented method is very accurate and efficient. Further, this technique can be extended to solve the other types of problems.

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