Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 6 (2020), pp. 843-850

http://www.ripublication.com/gjpam.htm

Explicit Single–Step Rational Method for Initial Value Problems (IVPs)

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Abstract

In this work, an explicit single–step rational method for solving initial value problems whose solutions possess singularities is constructed. The stability and convergence properties of the constructed method are investigated. Also, the associated local truncation error is presented. Using some standard test problems, the accuracy and efficiency of the method are established by its implementation and the results obtained are compared with those discussed in the literature.

Keywords: Ordinary Differential Equations, First Order, Initial Value Problems, Rational, Singularities.

2000 MSC : 65L05, 65L06, 65L20

1. INTRODUCTION

The assumption that the solution of the initial value problem

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}(\mathbf{x})), \quad x \in [x_0, X], \quad \mathbf{y}(x_0) = \eta \tag{1}$$

is locally representable by a polynomial has been the basis of numerous numerical methods for solving (1). When a given initial value problem or its theoretical solution $y(x)$ is known to posses a singularity, it is however, particularly inappropriate to represent the solution in the neighbourhood of the singularity by a polynomial [2], [1].

Runge–Kutta type methods, Obrechkoff methods and general linear multistep methods usually produce very poor solutions around singularity points as they are based on local representation by polynomials [4], [5], [1], [3]. A pioneer work on quadrature formulas based on rational interpolating functions was developed by the authors in [4]. The rational interpolation schemes proposed in [2] and [5] were seen to be effective in the neighbourhood of the singularity and even beyond. A modification to the work [4] was made by [2], the author in [2] replaced the general rational function of [4] by $F(x) = \frac{P_m(x)}{Q_n(x)}$ where $P_m(x)$ and $Q_n(x)$ are respectively polynomial of degree m and n. Since rational functions are more appropriate for the representation of functions close to singularities than polynomials, the limitation of classical methods in solving problems with singularities can be overcome by constructing methods that use a rational functions as local representation of the theoretical solution [1]. Several method have been constructed using this approach [6], [7], [8], [9] [10]. The works [6],[7], [9], [10], [11], [12], [14] established that solution around singularity point are well approximated by this approach. In this work, an explicit single–step rational method for solving (1) is presented. The local truncation error and absolute stability of the method are also discussed.

2. CONSTRUCTION OF METHOD

In this section, the construction of an explicit single–step rational method for solving (1) is presented. We demand that the theoretical solution $y(x)$ be locally represented by the rational interpolant

$$r(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 \exp(x)}{b_0 + x} \tag{2}$$

satisfying the following:

$$\left.\begin{array}{l} r(x_{n+j}) = y_{n+j}, \quad j = 0, 1, 2 \\ r^{(i)}(x_{n+j}) = y^{(i)}_{n+j}, \quad j = 0, \quad i = 1, 2, 3, 4. \end{array}\right\} \tag{3}$$

Substituting for expressions and simplifying (3) yields

$$y_n = \frac{1}{b_0 + x}\left(a_3x^3 + a_2x^2 + a_1x + a_4e^x + a_0\right) \tag{4}$$

$$y_{n+1} = \frac{1}{b_0 + x_{n+1}}\left(a_4e^{x_{n+1}} + x_{n+1}\left(x_{n+1}\left(a_3x_{n+1} + a_2\right) + a_1\right) + a_0\right) \tag{5}$$

$$\begin{aligned} y_n' &= \frac{1}{\left(b_0 + x_n\right)^2}\left(a_4e^{x_n}\left(b_0 + x_n - 1\right) + x_n\left(a_2\left(2b_0 + x_n\right) + \right.\right.\\ &\quad \left.\left. a_3x_n\left(3b_0 + 2x_n\right)\right) + a_1b_0 - a_0\right) \end{aligned} \tag{6}$$

$$\begin{aligned} y_n^{(2)} &= \frac{1}{\left(b_0 + x_n\right)^3}a_4e^{x_n}\left(\left(2b_0\left(x_n - 1\right) + b_0^2 + \left(x_n - 2\right)x_n + 2\right) + \right.\\ &\quad \left. 2\left(a_3x_n\left(3b_0x_n + 3b_0^2 + x_n^2\right) + a_2b_0^2 - a_1b_0 + a_0\right)\right) \end{aligned} \tag{7}$$

$$\begin{aligned} y_n^{(3)} &= \frac{1}{\left(b_0 + x_n\right)^4}\left(-6a_0 + 6b_0\left(b_0\left(a_3b_0 - a_2\right) + a_1\right) + \right.\\ &\quad a_4e^{x_n}\left(3b_0^2\left(x_n - 1\right) + 3b_0\left(\left(x_n - 2\right)x_n + 2\right) + \right.\\ &\quad \left.\left. b_0^3 + x_n\left(\left(x_n - 3\right)x_n + 6\right) - 6\right)\right) \end{aligned} \tag{8}$$

$$\begin{aligned} y_n^{(4)} &= \frac{1}{\left(b_0 + x_n\right)^5}\left(24a_0 - 24b_0\left(b_0\left(a_3b_0 - a_2\right) + a_1\right) + \right.\\ &\quad e^{x_n}a_4\left(24 + b_0^4 + 4b_0^3\left(x_n - 1\right) + 6b_0^2\left(\left(x_n - 2\right)x_n + 2\right) + \right.\\ &\quad 4b_0\left(x_n\left(\left(x_n - 3\right)x_n + 6\right) - 6\right) + x_n\left(x_n\left(\left(x_n - 4\right)x_n + 12\right) - 24\right)\left.\right) \end{aligned} \tag{9}$$

$$\begin{aligned} y_n^{(5)} &= \frac{1}{\left(b_0 + x_n\right)^6}\left(-120a_0 + 120b_0\left(b_0\left(a_3b_0 - a_2\right) + a_1\right) + \right.\\ &\quad e^{x_n}a_4\left(-120 + b_0^5 + 5b_0^4\left(x_n - 1\right) + 10b_0^3\left(\left(x_n - 2\right)x_n + 2\right) + \right.\\ &\quad 10b_0^2\left(x_n\left(\left(x_n - 3\right)x_n + 6\right) - 6\right) + x_n\left(x_n\left(x_n\left(\left(x_n - 5\right)x_n + 20\right) - \right.\right.\\ &\quad \left.\left. 60\right) + 120\right) + 5b_0\left(x_n\left(x_n\left(\left(x_n - 4\right)x_n + 12\right) - 24\right) + 24\right)\left.\right)\left.\right) \end{aligned} \tag{10}$$

Eliminating the undetermined coefficients a_0, a_1, a_2, a_3, a_4 and b_0 in (4) results in

$$\begin{aligned} y_{n+1} &= \frac{1}{6\left(-(h+5){y_n}^{(4)} + h{y_n}^{(5)} + 4{y_n}^{(3)}\right)} \times \\ &\quad \left(12h^2{y_n}^{(3)}y_n'' + 4h^3\left({y_n}^{(3)}\right)^2 - 15h^2{y_n}^{(4)}y_n'' - 3h^3{y_n}^{(4)}y_n'' - \right.\\ &\quad 5h^3{y_n}^{(3)}{y_n}^{(4)} + 30\left({y_n}^{(4)}\right)^2 + 30h\left({y_n}^{(4)}\right)^2 + 15h^2\left({y_n}^{(4)}\right)^2 + \\ &\quad 5h^3\left({y_n}^{(4)}\right)^2 + 3h^3{y_n}^{(5)}y_n'' - 24{y_n}^{(3)}{y_n}^{(5)} - 24h{y_n}^{(3)}{y_n}^{(5)} - \\ &\quad 12h^2{y_n}^{(3)}{y_n}^{(5)} - 4h^3{y_n}^{(3)}{y_n}^{(5)} + 6y_n\left(-(h+5){y_n}^{(4)} + \right.\\ &\quad \left. h{y_n}^{(5)} + 4{y_n}^{(3)}\right) + 6h\left(-(h+5){y_n}^{(4)} + h{y_n}^{(5)} + 4{y_n}^{(3)}\right)y_n' - \\ &\quad \left. 6e^h\left(5\left({y_n}^{(4)}\right)^2 - 4{y_n}^{(3)}{y_n}^{(5)}\right)\right) \end{aligned} \tag{11}$$

The resulting method (11) is an explicit single–step rational method. We shall refer to (11) as ***ESRM*** which is the method proposed in this work.

3. LOCAL TRUNCATION ERROR AND ABSOLUTE STABILITY OF CONSTRUCTED METHOD

In this section, we consider the associated local truncation error (lte) and the absolute stability properties of the proposed method.

3.1. Local Truncation Error

Local Truncation Error: The local truncation error T_{n+k} at x_{n+k} is defined as

$$T_{n+k} \quad = \quad y(x_n + kh) - y_{n+k} \tag{12}$$

where, $y(x_n)$ is the theoretical solution. From the above, the local truncation error of the proposed ***ESRM*** method is obtained as

$$\begin{aligned} T_{n+1} \quad = \quad & \frac{1}{720\left(4y^{(3)}(x) - 5y^{(4)}(x)\right)} \times \\ & h^6\left(5y^{(4)}(x)^2 + 6y^{(5)}(x)^2 - 4y^{(3)}(x)\left(y^{(5)}(x) - y^{(6)}(x)\right) - \right. \\ & \left. y^{(4)}(x)\left(5y^{(6)}(x) + 6y^{(5)}(x)\right)\right) \end{aligned} \tag{13}$$

3.2. Order of a Ordinary Differential Equation

Order of the proposed* NE2M *method: A numerical method is said to be of order *p* if *p* is the largest integer for which $T_{n+k} = \mathcal{O}(h^{p+1})$ for every *n* and $p \geq 1$. Using the above, the order of the proposed ***ESRM*** method is obtained as $p = 6$

3.3. Consistency

To establish the consistency of the proposed ***ESRM*** method, it is sufficient to show that

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} \quad = \quad 0. \tag{14}$$

Our proposed method (11) is consistent as it satisfies (14).

3.4. Stability

The stability of the proposed method (11) is established by implementing it on the standard test problem

$$y' = \lambda y, \quad Re(\lambda) \quad < \quad 0 \tag{15}$$

and the stability function $R(z) = \frac{y_{n+1}}{y_n}$ is obtained as

$$R(z) = \frac{y_{n+1}}{y_n} = \frac{z^5}{120} + \frac{z^4}{24} + \frac{z^3}{6} + \frac{z^2}{2} + z + 1; \qquad z = \lambda h \tag{16}$$

and the region of absolute stability is seen in Figure 1.

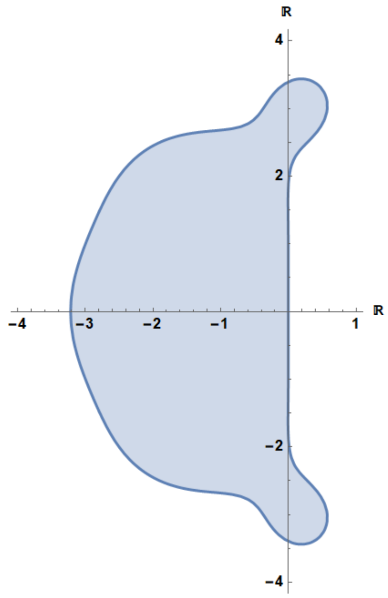

Figure 1: Region of absolute stability of (11)

4. NUMERICAL EXAMPLES

The first problem considered in this work is the nonlinear initial value problem

$$y' = 1 + y^2; \qquad y(0) = 1 \tag{17}$$

whose theoretical solution is given as

$$y(x) = \tan\left(x + \frac{\pi}{4}\right). \tag{18}$$

For this problem, the absolute errors of the results obtained by the method proposed in this work are first compared with those of Non-linear One-Step methods for initial value problems of [7] and the derivative–free methods proposed in [11] as shown in Figure 2. A comparison of the maximum absolute error obtained by the proposed methods against those produced by the methods of the authors in [4, 6, 7, 9, 13] is a also presented in Figure 3.

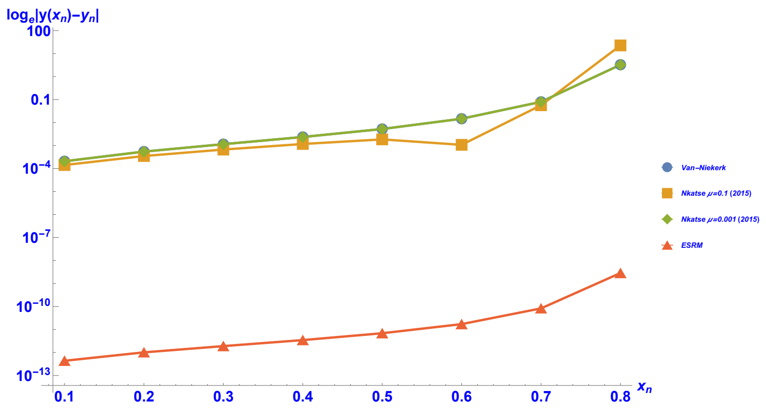

Figure 2: Logarithm of absolute errors for the solutions of Problem 1 with step–size h = 0.01

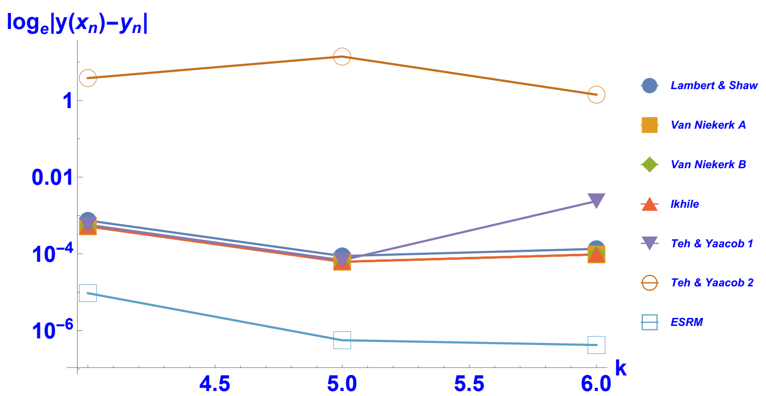

Figure 3: Log plot of maximum absolute errors for Problem 1 as a function of the step–size $h = \frac{0.8}{2^k}, k = 4(1)6$

4.1. Problem 2

The second test problem considered is given as

$$y' = y^2; \qquad y(0) = 1. \tag{19}$$

The exact Solution is

$$y(x) = \frac{1}{1-x}. \tag{20}$$

The logarithm of absolute errors for the solutions obtained are compared with other methods discussed in [12] as given in Figure 4

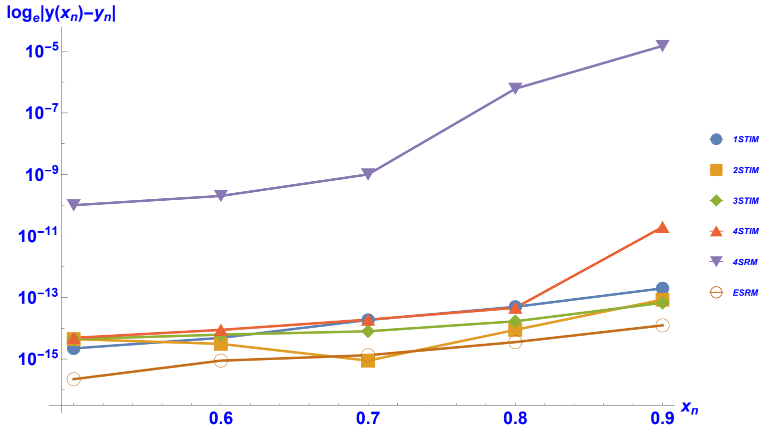

Figure 4: Logarithm of absolute errors for the solutions of (19) with step–size h = 0.01

5. CONCLUSION

The explicit single–step rational method constructed in this work is consistent and absolutely stable. Its region of absolute stability is larger than those of the methods discussed in the literature. The method gave more accurate result on the standard test problems compared with other methods discussed. Hence, the method is suitable for solving problems whose solution possesses singularity.

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