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On Positive Periodic Solutions of a Class of First Order Neutral Differential Equations

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Abstract

In this work, sufficient conditions are established for positive periodic solutions of first order nonlinear neutral delay differential equations of the form

$$[u(t)-p(t)f(u(t-\alpha))-q(t)g(u(t-\beta))]' = -r(t)u(t)+h(t,u(t-\alpha),u(t-\beta))$$

by using Krasnoselskii's fixed point theorem.

Mathematics subject classification (2010): 34K13, 34A34

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1. INTRODUCTION

It is known that a growing population is likely to consume more or less food than a matured one depending on individual species. If we consider human population, then we let $u(t)$ be the matured population at the age $t > 0$. Accordingly, if we categorise the human tenure of living span into three phases, then we assume that $u(t-\alpha)$ and $u(t-\beta)$ are the other two of the ages $t-\alpha$ and $t-\beta$ respectively. Therefore, the total food consumption(which may be nonlinear in modality) of a complete tenure of human population span comprising of three categories can be formulated by means of a nonlinear neutral differential equations of the type

$$[u(t)-p(t)f(u(t-\alpha))-q(t)g(u(t-\beta))]' = -r(t)u(t)+h(t,u(t-\alpha),u(t-\beta)), \quad (1.1)$$

where $p, q, r \in C(\mathbb{R},\mathbb{R})$, $f, g \in C'(\mathbb{R},\mathbb{R})$ satisfying the properties $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0, x \in \mathbb{R}$, $h \in C(\mathbb{R}^3,\mathbb{R})$ such that p, q, r, f, g, h are $T-$ periodic functions, $T > 0$ and $0 < \alpha, \beta < T$.

As long as the food consumption is concerned, a modality of a system may exhibit periodic nature involving the solution of the neutral differential equation. Hence, the objective of this work is to present the existence of $T-$periodic solutions of (1.1) by using Krasnoselskii's fixed point theorem. Indeed, our motivation came from the ecological and population models (see for e.g. [2], [3], [8]) which are dealing with neutral functional differential equations of the type:

$$[u(t) - pu(t-\alpha(t))]' = -r(t)u(t) + b(t)e^{-\tau(t)u(t-\alpha(t)}, \tag{1.2}$$

$$[u(t) - pu(t-\alpha(t))]' = -r(t)u(t) + b(t)u(t-\alpha(t))e^{-\tau(t)u(t-\alpha(t)} \tag{1.3}$$

and

$$[u(t) - pu(t-\alpha(t))]' = -r(t)u(t) + b(t)\frac{u(t-\alpha(t))}{1+u^n(t-\alpha(t))},\ n > 0. \tag{1.4}$$

The present work not only emphasises the study of periodic solutions of (1.2), (1.3) and (1.4) in comparison with (1.1), but also it improves and generalizes the corresponding results of [6] and [7], where the authors have discussed the special case $p(t) \equiv 0$ and $q(t) \equiv 0$ of (1.1). Meanwhile, the work [5] came to notice in which the authors have studied the special case $q(t) \equiv 0$ and $p(t) = p(constant)$ of (1.1) such that either $p \in (-1, 0)$ or $p \in (0, 1)$.

By a solution of (1.1) we understand a function $u \in C([-\rho, \infty), \mathbb{R})$ such that $(u(t) - p(t)f(u(t-\alpha)) - q(t)g(u(t-\beta))$ is continuously differentiable and (1.1) is satisfied for $t \geq 0$, where $\rho = \max\{\alpha, \beta\}$, and $\sup\{|u(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$.

Lemma 1.1. *[1](Krasnoselskii's Fixed Point Theorem)*
Let X be a Banach space and S be a bounded closed subset of X. Consider two map T_1 and T_2 of S into X such that $T_1x + T_2y \in S$ for every pair $x, y \in S$. If T_1 is a contraction and T_2 is completely continuous, then the equation $T_1x + T_2y$ has a fixed point in S.

2. EXISTENCE OF PERIODIC SOLUTION

Let $X = \{u(t) : u \in C(\mathbb{R}, \mathbb{R}), u(t) = u(t+T), t \in \mathbb{R}\}$ be the space of real continuous functions with the norm

$$\|u\| = \sup_{t\in[0,T]} |u(t)|.$$

Then X is a Banach space with the $\|\cdot\|$. We put

$$H(t,u_\alpha,u_\beta)=\frac{h(t,u_\alpha,u_\beta)}{r(t)}-p(t)f(u_\alpha)-q(t)g(u_\beta),$$

where $h(t,u_\alpha,u_\beta)=h(t,u(t-\alpha),u(t-\beta))$.

Lemma 2.1. *$x(t)$ is a $T-$periodic solution of (1.1) if and only if $x(t)$ is a $T-$periodic solution of*

$$\begin{aligned}u(t)=\int_t^{t+T}&G(t,s)[h(s,u(s-\alpha),u(s-\beta))\\&-r(s)p(s)f(u(s-\alpha))-r(s)q(s)g(u(s-\beta))]ds\\&+p(t)f(u(t-\alpha))+q(t)g(u(t-\beta)),\end{aligned}\tag{2.5}$$

where

$$G(t,s)=\frac{exp(\int_t^s r(v)dv)}{exp(\int_0^T r(v)dv)-1}.$$

Remark 2.2. *It is easy to see that $\int_t^{t+T}G(t,s)r(s)ds=1$.*

Proof of Lemma Let $x(t)$ be a $T-$periodic solution of (1.1). Then

$$[x(t)-p(t)f(x(t-\alpha))-q(t)g(x(t-\beta))]'=-r(t)x(t)+h(t,x(t-\alpha),x(t-\beta)).\tag{2.6}$$

Setting

$$z(t)=x(t)-p(t)f(x(t-\alpha))-q(t)g(x(t-\beta))$$

in (2.6), it follows that

$$z'(t)=-r(t)z(t)+h(t,x(t-\alpha),x(t-\beta))-r(t)p(t)f(x(t-\alpha))-r(t)q(t)g(x(t-\beta)),$$

that is,

$$\begin{aligned}&[z'(t)+r(t)z(t)]exp\left(\int_t^s r(v)dv\right)\\&=[h(t,x(t-\alpha),x(t-\beta))-r(t)p(t)f(x(t-\alpha))-r(t)q(t)g(x(t-\beta))exp\left(\int_t^s r(v)dv\right).\end{aligned}\tag{2.7}$$

Integrating (2.7) from t to $t+T$ and then simplifying, we get

$$z(t)=\int_t^{t+T}G(t,s)[h(s,x(s-\alpha),x(s-\beta))-r(s)p(s)f(x(s-\alpha))-r(s)q(s)g(x(s-\beta))]ds,$$

that is, $x(t)$ is a $T-$periodic solution of (2.5).

Conversely, let $x(t)$ be a $T-$periodic solution of (2.5). Then

$$x(t)-p(t)f(x(t-\alpha))-q(t)g(x(t-\beta))$$
$$=\int_t^{t+T}G(t,s)[h(s,x(s-\alpha),x(s-\beta))-r(s)p(s)f(x(s-\alpha))-r(s)q(s)g(x(s-\beta))]ds$$

implies that

$$\frac{d}{dt}[x(t)-p(t)f(x(t-\alpha))-q(t)g(x(t-\beta))]$$
$$=\frac{d}{dt}\int_t^{t+T}G(t,s)[h(s,x(s-\alpha),x(s-\beta))-r(s)p(s)f(x(s-\alpha))-r(s)q(s)g(x(s-\beta))]ds$$
$$=\int_t^{t+T}\frac{\partial}{\partial t}G(t,s)[h(s,x(s-\alpha),x(s-\beta))-r(s)p(s)f(x(s-\alpha))-r(s)q(s)g(x(s-\beta))]ds$$
$$+h(t,x(t-\alpha),x(t-\beta))-r(t)p(t)f(x(t-\alpha))-r(t)q(t)g(x(t-\beta))$$
$$=-\int_t^{t+T}r(t)G(t,s)[h(s,x(s-\alpha),x(s-\beta))-r(s)p(s)f(x(s-\alpha))-r(s)q(s)g(x(s-\beta))]ds$$
$$+h(t,x(t-\alpha),x(t-\beta))-r(t)p(t)f(x(t-\alpha))-r(t)q(t)g(x(t-\beta))$$
$$=-r(t)x(t)+h(t,x(t-\alpha),x(t-\beta))$$

and thus, $x(t)$ is a $T-$periodic solution of (1.1).

Theorem 2.3. *Let* $0\le a_1\le p(t)\le a<\infty$ *and* $0\le b_1\le q(t)\le b<\infty$. *Assume that* $H(t,u_\alpha,u_\beta)\ge 0$ *and there exist positive constants* m *and* M *with* $m<M$ *such that*

$$m-a_1f(m)-b_1g(m)\le H(t,u_\alpha,u_\beta)\le M-af(M)-bg(M) \tag{2.8}$$

for all $t\in[0,T],u\in[m,M]$. *For constants* $c,d\in(m,M)$ *if* $a|f'(c)|+b|g'(d)|<1$, *then* (1.1) *has at least one positive* $T-$*periodic solution* $u(t)$ *in* $[m,M]$.

Remark 2.4. *The assumption in* (2.8) *would be feasible if and only if* $m-a_1f(m)-b_1g(m)\le M-af(M)-bg(M)$, *that is, if and only if* $0\le(M-m)-a(f(M)-f(m))-b(g(M)-g(m))$, *that is, if and only if* $(M-m)(1-af'(c)-bg'(d))\ge 0$ *which is true as soon as* $a|f'(c)|+b|g'(d)|\le 1$ *for* $c,d\in(m,M)$.

Proof of Theorem Let $\Omega=\{u\in X: m\le u(t)\le M, 0<m<M\}$. Then Ω is a closed bounded and convex set. Define operators K and S on Ω by

$$\begin{aligned}(Ku)(t)=\int_t^{t+T}&G(t,s)[h(s,u(s-\alpha),u(s-\beta))\\&-r(s)p(s)f(u(s-\alpha))-r(s)q(s)g(u(s-\beta))]ds,\end{aligned} \tag{2.9}$$

$$(Su)(t)=p(t)f(u(t-\alpha))+q(t)g(u(t-\beta)). \tag{2.10}$$

For any $u \in \Omega$, we notice that

$$\begin{aligned}(Ku)(t+T) &= \int_{t+T}^{t+2T} G(t+T,s)[h(s,u(s-\alpha),u(s-\beta)) \\ &- r(s)\{p(s)f(u(s-\alpha)) + q(s)g(u(s-\beta))\}]ds \\ &= \int_{t}^{t+T} G(t+T,y+T)[h(y,u(y-\alpha),u(y-\beta)) \\ &- r(y)\{p(y)f(u(y-\alpha)) + q(y)g(u(y-\beta))\}]dy \\ &= \int_{t}^{t+T} G(t,y)[h(y,u(y-\alpha),u(y-\beta)) \\ &- r(y)\{p(y)f(u(y-\alpha)) + q(y)g(u(y-\beta))\}]dy \\ &= (Ku)(t)\end{aligned}$$

and

$$\begin{aligned}(Su)(t+T) &= p(t+T)f(u(t+T-\alpha)) + q(t+T)g(u(t+T-\beta)) \\ &= p(t)f(u(t-\alpha)) + q(t)g(u(t-\beta)) = (Su)(t),\end{aligned}$$

which then implies that $K(\Omega) \subset X$ and $S(\Omega) \subset X$. Let $u, v \in \Omega$. Therefore,

$$\begin{aligned}&(Ku)(t) + (Sv)(t) \\ &= \int_{t}^{t+T} G(t,s)[h(s,u(s-\alpha),u(s-\beta)) \\ &- r(s)p(s)f(u(s-\alpha)) - r(s)q(s)g(u(s-\beta))]ds + p(t)f(v(t-\alpha)) + q(t)g(v(t-\beta)) \\ &= \int_{t}^{t+T} G(t,s)r(s)H(s,u_\alpha,u_\beta)ds + p(t)f(v(t-\alpha)) + q(t)g(v(t-\beta)) \\ &\leq \int_{t}^{t+T} G(t,s)r(s)[M - af(M) - bg(M)]ds + af(M) + bg(M) = M\end{aligned}$$

and

$$\begin{aligned}&(Ku)(t) + (Sv)(t) \\ &= \int_{t}^{t+T} G(t,s)r(s)H(s,u_\alpha,u_\beta)ds + p(t)f(v(t-\alpha)) + q(t)g(v(t-\beta)) \\ &\geq \int_{t}^{t+T} G(t,s)r(s)[m - a_1f(m) - b_1g(m)]ds + a_1f(m) + b_1g(m) = m\end{aligned}$$

shows that $Ku + Sv \in \Omega$ for all $u, v \in \Omega$.

In order to apply the Krasnoselskii's Fixed Point Theorem, we need to show that K ia a completely continuous operator on Ω and S is a contraction mapping on X. Clearly,

K is continuous. Further,

$$|(Ku)(t)| \leq \int_t^{t+T} G(t,s)r(s)|H(s,u_\alpha,u_\beta)|ds$$
$$\leq \int_t^{t+T} G(t,s)r(s)[M - af(M) - bg(M)]ds = M - af(M) - bg(M)$$

and

$$|(Ku)'(t)| = |\frac{d}{dt}\int_t^{t+T} G(t,s)r(s)H(s,u_\alpha,u_\beta)ds|$$
$$= |-r(t)u(t) + h(t,u(t-\alpha),u(t-\beta)|$$
$$\leq \|r\|M + \|r\|M = 2\|r\|M$$

implies that $K(\Omega)$ is uniformly bounded and equi-continuous. So, K is completely continuous due to Ascoli-Arzela theorem. For $u_1, u_2 \in X$, we have

$$|(Su_1)(t) - (Su_2)(t)| \leq p(t)|f(u_1(t-\alpha)) - f(u_2(t-\alpha))| + q(t)|g(u_1(t-\beta)) - g(u_2(t-\beta))|$$
$$\leq a|f'(c)||u_1(t-\alpha) - u_2(t-\alpha)| + b|g'(d)||u_1(t-\beta) - u_2(t-\beta)|$$
$$\leq [a|f'(c)| + b|g'(d)|]\|u_1 - u_2\|,$$

that is, S is a contraction mapping on X. Hence, $K + S$ has a fixed point by Lemma 1.1. Ultimately, (1.1) has a positive periodic solution $u(t)$ in $[m, M]$ due to Lemma 2.1. This completes the proof of the theorem.

Example 2.5. *Consider*

$$[u(t) - p(t)u^{\frac{1}{5}}(t-\pi) - q(t)u^{\frac{1}{3}}(t-\pi)]' = -(2+\sin t)u(t) + h(t,u(t-\pi),u(t-\pi)), \tag{2.11}$$

where $\frac{1}{3} \leq p(t) = \frac{3+\sin t}{6} \leq \frac{2}{3}$, $\frac{1}{2} \leq q(t) = \frac{5+\sin t}{8} \leq \frac{3}{4}$, $T = 2\pi$ *and*

$$h(t,u(t-\pi),u(t-\pi)) = (2+\sin t)[\frac{1}{2}u(t-\pi) + \frac{1}{2}u(t-\pi)].$$

We notice that (2.8) *holds true if and only if*

$$p(t)f(u_\alpha) + q(t)g(u_\beta) + m - a_1f(m) - b_1g(m)$$
$$\leq \frac{h(t,u_\alpha,u_\beta)}{r(t)} \leq M - af(M) - bg(M) + p(t)f(u_\alpha) + q(t)g(u_\beta)$$

which is equivalent to say that

$$m \leq \frac{h(t,u_\alpha,u_\beta)}{r(t)} \leq M \ for \ u \in [m, M].$$

If we choose $m = 5$ *and* $M = 20$*, then it is easy to verify that*

$$3.6851 = m - a_1 f(m) - b_1 g(m) \leq H(t, u_\alpha, u_\beta) \leq M - af(M) - bg(M) = 16.7504$$

and when $f(u) = u^{\frac{1}{5}}$, $g(u) = u^{\frac{1}{3}}$

$$a|f'(c)| + b|g'(d)| = \frac{2}{15} c^{\frac{-4}{5}} + \frac{1}{4} d^{\frac{-2}{3}} < 1 \; for \; c, \; d \; \in [5, 20].$$

Therefore, (2.11) *satisfies all conditions of Theorem 2.3 and hence* (2.11) *has at least one periodic solution in* $[5, 20]$.

Similar to the proof of Theorem 2.3, we can prove the following results:

Theorem 2.6. *Let* $-\infty < a_2 \leq p(t) \leq a_3 \leq 0$ *and* $-\infty < b_2 \leq q(t) \leq b_3 \leq 0$. *Assume that* $H(t, u_\alpha, u_\beta) \geq 0$ *and there exist positive constants* m *and* M *with* $m < M$ *such that*

$$m - a_2 f(M) - b_2 g(M) \leq H(t, u_\alpha, u_\beta) \leq M - a_3 f(m) - b_3 g(m)$$

for all $t \in [0, T], u \in [m, M]$. *For constants* $c, d \in (m, M)$ *if* $|a_2||f'(c)| + |b_2||g'(d)| < 1$, *then* (1.1) *has at least one positive* $T-$*periodic solution* $u(t)$ *in* $[m, M]$.

Theorem 2.7. *Let* $0 \leq a_1 \leq p(t) \leq a < \infty$ *and* $-\infty < b_2 \leq q(t) \leq b_3 \leq 0$. *Assume that* $H(t, u_\alpha, u_\beta) \geq 0$ *and there exist positive constants* m *and* M *with* $m < M$ *such that*

$$m - a_1 f(m) - b_2 g(M) \leq H(t, u_\alpha, u_\beta) \leq M - af(M) - b_3 g(m)$$

for all $t \in [0, T], u \in [m, M]$. *For constants* $c, d \in (m, M)$ *if* $a|f'(c)| + |b_2||g'(d)| < 1$, *then* (1.1) *has at least one positive* $T-$*periodic solution* $u(t)$ *in* $[m, M]$.

Theorem 2.8. *Let* $-\infty < a_2 \leq p(t) \leq a_3 \leq 0$ *and* $0 \leq b_1 \leq q(t) \leq b < \infty$. *Assume that* $H(t, u_\alpha, u_\beta) \geq 0$ *and there exist positive constants* m *and* M *with* $m < M$ *such that*

$$m - a_2 f(M) - b_1 g(m) \leq H(t, u_\alpha, u_\beta) \leq M - a_3 f(m) - bg(M)$$

for all $t \in [0, T], u \in [m, M]$. *For constants* $c, d \in (m, M)$ *if* $|a_2||f'(c)| + b|g'(d)| < 1$, *then* (1.1) *has at least one positive* $T-$*periodic solution* $u(t)$ *in* $[m, M]$.

Remark 2.9. *If we consider the total food consumption(sometimes linear or sometimes nonlinear or may be both in modality) of a complete tenure of human population span comprising of* n*(finite) categories, then a mathematical formulation of the problem can be seen by means of*

$$[u(t) - \sum_{i=1}^{n} p_i(t) f_i(u(t - \alpha_i))]' + r(t)u(t) = h(t, u(t - \alpha_1), u(t - \alpha_2), ..., u(t - \alpha_n)), \tag{2.12}$$

where $h \in C(\mathbb{R}^{n+1}, \mathbb{R})$. *And it would be interesting to study the existence of positive periodic solutions of* (2.12) *by taking the ongoing work into account.*

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