Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 6 (2020), pp. 771-782

http://www.ripublication.com/gjpam.htm

# Spectral theory of $\mathcal{G}$ -idempotent matrices

**B.Vasudevan[1], N. Anis Fathima[2]**

*[1]Assistant Professor of Mathematics, Thiru.Vi.Ka. Government Arts College, Tiruvarur, Tamilnadu, India.*

*[2]Research Scholar (PT), Department of Mathematics, Govt. Arts College (A), Kumbakonam, Tamilnadu, India.*

*(Affiliated to Bharathidasan University, Tiruchirappalli, Tamilnadu, India.)*

**Abstract**

In this paper, we study the spectrum and $\mathcal{G}$ - spectrum of $\mathcal{G}$ - idempotent matrices. Relations between $\mathcal{G}$ - eigen values of a matrix M and eigen values of the matrix GM are obtained. Spectral characterisations of $\mathcal{G}$-idempotent matrices are investigated.

**Keywords:** Idempotent matrix, $\mathcal{G}$ - idempotent matrix, Spectrum of a matrix, Minkowski Metric Matrix

**AMS Classification (2010):**15A57, 15A15, 15A12

## 1. Introduction

Let the space of $n \times n$ complex matrices be denoted by $\mathbb{C}^{n\times n}$. Let $\mathbb{C}^n$ be the space of complex $n$-tuples. Let $u = (u_0, u_1, u_2, \ldots, u_{n-1}) \in \mathbb{C}^n$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, \ldots, -u_{n-1})$. Then the Minkowski metric matrix $G$ is given by $G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$ and $G^2 = I_n$. Minkowski inner product on $\mathbb{C}^n$ is defined by $(u, v) = \langle u, Gv \rangle$, where $\langle .,. \rangle$ is the conventional Hilbert Space inner product. A space with Minkowski inner product is called a *Minkowski space*, which has been studied by physicists in optics. With respect to the Minkowski inner product the adjoint of a matrix $M \in \mathbb{C}^{n\times n}$ is given by $M^{\sim} = GM^*G$, where $M^*$ is the usual Hermitian adjoint.

A complex matrix $M \in \mathbb{C}^{n\times n}$, that satisfies the relation $M^2 = M$ is called idempotent matrix. Idempotent matrix plays an important role in functional analysis especially spectral theory of transformations and projections. For the properties of idempotent matrices and its generalizations one may refer [1, 2, 4, 5, 7, 9]. In [8], B. Vasudevan and N. Anis Fathima introduced a new generalization of idempotent matrices, namely $\mathcal{g}$ -idempotent matrix in the Minkowski Space.

In this paper, we define $\mathcal{g}$- eigen values of a matrix. The $\mathcal{g}$ - spectral resolution of a $\mathcal{g}$-idempotent matrix is determined. Relations among the multiplicity of eigen values of a $\mathcal{g}$-idempotent matrix M and the matrix functions such as *trM*, *det M* and *rank M* are discussed.

This paper is organized as follows. In Section 2, we define $\mathcal{g}$-eigen value of a matrix as a special case of generalized eigen value problem $Mx = \lambda Nx$. The $\mathcal{g}$ -spectrum of a matrix is discussed. Section 3 deals with the spectral characterisation of $\mathcal{g}$ -idempotent matrices. The $\mathcal{g}$-spectral properties of $\mathcal{g}$-idempotent matrix is analysed in Section 4.

## 2. $\mathcal{g}$ – Eigen value and $\mathcal{g}$ -Eigen vector

The definition of $\mathcal{g}$-*idempotent* matrix has been introduced in [8]. In this section, using the generalized eigen value problem, we define $\mathcal{g}$-eigen value of a matrix and proved that every matrix $M$ satisfies the $\mathcal{g}$- characteristic equation of $GM$. Also $\mathcal{g} -$ similarity of $\mathcal{g}$-idempotent matrix is discussed.

**Definition 2.1**

A complex matrix $M \in \mathbb{C}^{n\times n}$ is said to be $\mathcal{g}$-*idempotent*, if

$M = GM^2G = M^{[2]}$,

where $G$ is the Minkowski metric matrix, $G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$.

**Definition 2.2**

A $\mathcal{g}$ - *eigen value* of a matrix $M$ is defined as the root of the equation $|\lambda G - M| = 0$. i.e., $\det(\lambda G - M) = 0$.

The polynomial $\det(\lambda G - M)$ is called $\mathcal{g}$- *characteristic polynomial* of $M$.

**Definition 2.3**

A non-zero vector $x\ (\neq 0)$ in $\mathbb{C}^n$is said to be a $\mathcal{g}$ - *eigen vector* of a complex matrix $M$ associated with a $\mathcal{g}$ - eigen value $\lambda$, if it satisfies $Mx = \lambda Gx$.

**Example 2.4**

Let $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M = \frac{-1}{4}\begin{pmatrix} 2 & -1 \\ 12 & 2 \end{pmatrix}$. Then $\mathcal{g}$ - eigen values of $M$ are the roots of the equation $\det(\lambda G - M) = 0$.

$$\begin{vmatrix} \lambda + \frac{1}{2} & -\frac{1}{4} \\ 3 & -\lambda + \frac{1}{2} \end{vmatrix} = 0 \Longrightarrow \lambda = \pm 1$$

The $\mathcal{g}$ - eigen vector of $M$ corresponding to the $\mathcal{g}$ - eigen value 1 is $\begin{pmatrix} \frac{1}{2} \\ 3 \end{pmatrix}$ and the $\mathcal{g}$ - eigen vector of $M$ corresponding to the $\mathcal{g}$ - eigen value -1 is $\begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}$

**Theorem 2.5**

If $M \in \mathbb{C}^{n\times n}$, then

i. $(\lambda, x)$ is a ($\mathcal{g}$ - eigen value, $\mathcal{g}$ - eigen vector) pair for $M$ if and only if it is an (eigen value, eigen vector) pair for $GM$.

ii. Every matrix $M$ satisfies the $\mathcal{g}$- characteristic equation of $GM$.

iii. Any set of $\mathcal{g}$ - eigen vectors corresponding to distinct $\mathcal{g}$ - eigen values of a matrix must be linearly independent.

**Proof**

(i) $(\lambda, x)$ is a ($\mathcal{g}$ - eigen value, $\mathcal{g}$ - eigen vector) pair for $M$

$\Leftrightarrow Mx = \lambda Gx$

$\Leftrightarrow GMx = \lambda x$

$\Leftrightarrow (\lambda, x)$ is a (eigen value, eigen vector) pair for $GM$

(ii) Since every square matrix satisfies its characteristic equation (*Cayley-Hamiton theorem*), we have $\det(\lambda I - M) = 0$. Now

$\det(\lambda G - GM) = \det[G(\lambda I - M)] = \det(G)\det(\lambda I - M) = 0.$

Therefore the matrix $M$ satisfies the $\mathcal{g}$- characteristic equation of $GM$.

(iii) By (i), any set of $\mathcal{g}$ - eigen vectors corresponding to distinct $\mathcal{g}$ - eigen values of a matrix $M$ is the set of eigen vectors correspond to distinct eigen values of $GM$. Hence they are linearly independent.

**Remark 2.6**

If $\sigma(M)$ and $\sigma_{\mathcal{g}}(M)$ denote the spectrum and $\mathcal{g}$-spectrum of $M$ respectively then it is true that $\sigma_{\mathcal{g}}(M) = \sigma(GM)$

**Definition 2.7**

Two matrices $M$ and $N$ in $\mathbb{C}^{n\times n}$ are said to be $\mathcal{g}$ - similar if there exists a non singular matrix $P \in \mathbb{C}^{n\times n}$ such that $N = GP^{-1}GMP$. Equivalently, $M$ is $\mathcal{g}$ - similar to $N$ if and only if $GM$ is similar to $GN$.

**Example 2.8**

Let $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$. Then $N = \frac{1}{2}\begin{pmatrix} -1 + i\sqrt{3} & 0 \\ 0 & 1 + i\sqrt{3} \end{pmatrix}$ is $\mathcal{g}$ - similar to $M$. For a non singular matrix $P = \begin{pmatrix} i & i \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}$, we have $N = GP^{-1}GMP$.

**Theorem 2.9**

Let $M$ be a $\mathcal{g}$-idempotent matrix.

i. Let $N$ be a matrix similar to $M$and $N = P^{-1}MP$ then $N$ is $\mathcal{g}$-idempotent if $G$ commutes with $P$ i.e., $GP = PG$.
ii. Let $S$ be a matrix $\mathcal{g}$ - similar to $M$ and $S = GQ^{-1}GMQ$ then $S$ is $\mathcal{g}$-idempotent if $G$ commutes with $Q$ i.e., $GQ = QG$

**Proof**

Since $M$ is a $\mathcal{g}$-idempotent matrix, we have $GM^2G = M$.

(i) $GN^2G = GP^{-1}MPP^{-1}MPG = GP^{-1}M^2PG = P^{-1}GM^2GP = P^{-1}MP = N$ . Hence $N$ is $\mathcal{g}$-idempotent.

(ii) $S = GQ^{-1}GMQ = Q^{-1}GGMQ = Q^{-1}MQ$. Therefore $S$ is similar to $M$.

Hence by (i), $S$ is $\mathcal{g}$-idempotent.

**Theorem 2.10**

Let $M$ and $N$ be two $\mathcal{g}$-idempotent matrices. If $N$ is $\mathcal{g}$ - similar to $M$, then $N^3$ is similar to $M^3$.

**Proof**

Since $N$ is $\mathcal{G}$ - similar to $M$, we have $N = GP^{-1}GMP$, for some non singular matrix $P$. So,

$$GN = P^{-1}GMP$$

$$(GN)^2 = P^{-1}GMPP^{-1}GMP$$

$$(GN)^2 = P^{-1}(GM)^2P$$

$$N^3 = P^{-1}M^3P \qquad \text{(by Remark 2.5 of [8])}$$

Hence $N^3$ is similar to $M^3$.

## 3. Spectral Characterizations of $\mathcal{G}$ -Idempotent Matrices

This section deals with the spectral resolution of $\mathcal{G}$-idempotent matrices.

**Theorem 3.1**

Let $M$ be a $\mathcal{G}$-idempotent matrix. Then the eigen values of $M$ are zero or cube roots of unity.

**Proof**

Let $\lambda$ be an eigen value of a $\mathcal{G}$-idempotent matrix $M$. Then

$$Mx = \lambda x \Rightarrow M^2x = \lambda Mx \Rightarrow M^2x = \lambda^2 x \Rightarrow M^4x = \lambda^2 M^2 x$$

$$\Rightarrow Mx = \lambda^4 x \Rightarrow \lambda x = \lambda^4 x$$

$$\Rightarrow (\lambda^4 - \lambda)x = 0$$

$$\Rightarrow \lambda(\lambda^3 - 1)x = 0.$$

Since $x \neq 0$, we have $\lambda(\lambda^3 - 1) = 0 \Rightarrow \lambda = 0$ or $\lambda^3 = 1$

$$\Rightarrow \lambda = 0 \text{ or } 1, \omega, \omega^2 \text{ where } \omega = \frac{-1+i\sqrt{3}}{2}$$

**Example 3.2**

Let $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$. Then $M$ is $\mathcal{G}$-idempotent matrix. The eigen values of $M$ are $\omega$ and $\omega^2$.

**Theorem 3.3**

If $M \in \mathbb{C}^{n\times n}$, is a $g$-idempotent matrix then it is diagonalizable and $\sigma(M) \subseteq \{0, 1, \omega, \omega^2\}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. Moreover, there exist unique disjoint oblique projectors $P_i$ for $i \in \{0, 1, 2, 3\}$ such that

$$M = \sum_{j=1}^{3} \omega^j P_j \tag{3.1}$$

and

$$I = \sum_{i=0}^{3} P_i \tag{3.2}$$

**Proof**

Since $M$ is $g$-idempotent matrix, $M^4 = M$ (*by theorem 2.11 of* [8]) and so the polynomial $q(t) = t^4 - t$ is a multiple of $q_M(t)$ of $M$ and every root of $q_M(t)$ has multiplicity 1 (cf.[3], pp11). Hence the matrix $M$ is diagonalizable.

Moreover, $\sigma(M) \subseteq \{0, 1, \omega, \omega^2\}$ (*by theorem 3.1 above*)

Let us define $P_i$'s by the formula,

$$P_0 = \frac{f_0(M)}{f_0(0)}, \text{where } f_0(\lambda) = \prod_{i=1}^{3} (\lambda - \omega^i) \text{ and}$$

$$P_j = \frac{f_j(M)}{f_j(\omega^j)}, \text{where } f_j(\lambda) = \prod_{\substack{i=1,\\ i\neq j}}^{3} \lambda(\lambda - \omega^i) \text{ for } j = 1, 2, 3$$

Since $1 + \omega + \omega^2 = 0$, we have

$$P_0 = I - M^3,$$

$$P_1 = \frac{1}{3}(M^3 + \omega M^2 + \omega^2 M)$$

$$P_2 = \frac{1}{3}(M^3 + \omega^2 M^2 + \omega M)$$

$$P_3 = \frac{1}{3}(M^3 + M^2 + M)$$

When $\omega^j \notin \sigma(M)\ for\ j \in \{1, 2, 3\}$, we see that $P_j = 0$. Similarly when $0 \notin \sigma(M)$, we see that $P_0 = 0$.

By spectral theorem, we see that the non-zero $P_i$'s so obtained are disjoint oblique projectors to satisfy (3.1) and (3.2).

<u>Proof for Uniqueness</u>

Let us assume that, if possible, let $Q_i$'s be non-zero disjoint oblique projectors such that

$$M = \sum_{i=1}^{m} \alpha_i Q_i, \tag{3.3}$$

for complex numbers $\alpha_i$ and

$$I = \sum_{i=1}^{m} Q_i \tag{3.4}$$

**Claim**: (3.3) and (3.4) are identical with (3.1) and (3.2)

First we prove that $\alpha_i$'s are eigen values of $M$.

Since $Q_i \neq 0$, there exists a non-zero vector $x$ in the range of $Q_i$ such that $Q_i x = x$ and $Q_j x = 0$ for $j \neq i$.

$$Mx = \left(\sum_{i=1}^{m} \alpha_i Q_i\right) x$$

$$Mx = \alpha_i x.$$

Thus $\alpha_i$ is an eigen value of $M$.

Conversely, if $\lambda$ is an eigen value of $M$, then $Mx = \lambda x$

$$\left(\sum_{i=1}^{m} \alpha_i Q_i\right) x = \lambda I x = \lambda \left(\sum_{i=1}^{m} Q_i\right) x$$

$$\sum_{i=1}^{m} (\lambda - \alpha_i) Q_i x = 0 \tag{3.5}$$

Since $Q_i$'s are disjoint, we can find at least one $x \neq 0$ among the non-zero vectors for which (3.5) is linearly independent. Hence $\lambda = \alpha_i$ for some $i$ and the set of $\alpha_i$'s equals the set of eigen values of $M$. Also by changing the order of terms suitably, we can have

$$M = \sum_{i=1}^{3} \omega^i Q_i.$$

Since the expression for $P_i$ is unique in terms of $M$, we have $Q_i = P_i$ for $i = \{0, 1, 2, 3\}$.

Hence the decompositions (3.1) and (3.2) are unique.

**Example 3.4**

Consider the matrices $M$ and $G$given in *example 3.2*. The eigen values of $M$ are $\omega$ and $\omega^2$. The oblique projectors of $M$ are found to be

$$P_0 = 0,$$

$$P_1 = \begin{pmatrix} \frac{1}{2} & -\frac{i\sqrt{3}}{6} \\ \frac{i\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{i\sqrt{3}}{6} \\ -\frac{i\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \text{ and}$$

$$P_3 = 0.$$

We can easily verify that the above projectors are disjoint, that is $P_i P_j = 0$ for $i \neq j$, and the equations (3.1) and (3.2) are satisfied by the projectors $P_j$'s.

**Remark 3.5**

Theorem 3.3 tells that every $\mathcal{G}$-idempotent matrices are diagonalizable. Let $m_i(i = 0, 1, 2)$ denote the multiplicity of eigen values $0, 1$ and $(\omega, \omega^2)$ respectively of a $\mathcal{G}$-idempotent matrix $M$. Since conjugate roots occur in pairs, $m_2$ denotes the multiplicity of $\omega$ as well as $\omega^2$. By spectral theorem, a $\mathcal{G}$-idempotent matrix $M$ can be reduced to the following form (diagonal).

$$\begin{pmatrix} 0 & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & 0 & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & \omega & & & & & \\ & & & & & & & \ddots & & & & \\ & & & & & & & & \omega & & & \\ & & & & & & & & & \omega^2 & & \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & & \omega^2 \end{pmatrix}$$

That is, we can find a matrix $X$ such that

$$X^{-1}MX = diag(0 - m_0\text{times}, 1 - m_1\text{times}, \omega - m_2\text{times}, \omega^2 - m_2\text{times})$$

**Example 3.6**

Let $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$. The eigen values of $M$ are $\omega$ and $\omega^2$. We can find an $X$ such $X^{-1}MX = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$.

Here $X = \begin{pmatrix} i\sqrt{3} & i\sqrt{3} \\ -3 & 3 \end{pmatrix}$, $X^{-1} = \frac{-1}{6}\begin{pmatrix} i\sqrt{3} & 1 \\ i\sqrt{3} & -1 \end{pmatrix}$.

**Theorem 3.7**

Let $m_i (i = 0, 1, 2)$ denote the multiplicity of eigen values $0, 1$ *and* $(\omega, \omega^2)$ respectively of a $\mathcal{G}$-idempotent matrix $M$ of order $n$. Then

i. $\operatorname{tr} M = m_1 - m_2$

ii. $\det M = 0$ or $1$

iii. $\operatorname{rank} M = 3m_1 - 2\operatorname{tr} M$

**Proof**

By Theorem 3.3, the $\mathcal{G}$-idempotent matrix $M$ is diagonalizable. So we can find a matrix $X$ such that

$$X^{-1}MX = diag(0 - m_0\text{times}, 1 - m_1\text{times}, \omega - m_2\text{times}, \omega^2 - m_2\text{times})$$

It is obvious that

$$n = m_0 + m_1 + 2m_2$$

i. $\operatorname{tr} M = 0 \times m_0 + 1 \times m_1 + (\omega + \omega^2)m_2 = m_1 + (-1)m_2 = m_1 - m_2$

ii. If $m_0 > 0$ then we have $\det M = 0$. Otherwise,

$\det M = 1^{m_1}\omega^{m_2}\omega^{2^{m_2}} = 1(\omega^3)^{m_2} = 1.$

iii. $\operatorname{rank} M = n - m_0 = m_0 + m_1 + 2m_2 - m_0 = m_1 + 2(m_1 - \operatorname{tr} M) = 3m_1 - 2\operatorname{tr} M$

**Example 3.8**

Let $M = \begin{pmatrix} -1/2 & 3i/2 \\ i/2 & -1/2 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $M$ is $\mathcal{G}$ − idempotent. The eigen values of $M$ are $\omega$ and $\omega^2$. That is $m_0 = 0;\ m_1 = 0;\ m_2 = 1$.

$\operatorname{tr} M = m_1 - m_2 = 0 - 1 = -1$

$\det M = 1$

$\operatorname{rank} M = 3m_1 - 2\operatorname{tr} M = 3(0) - 2(-1) = 2$

## 4. $\mathcal{g}-$ Spectral Characterizations of $\mathcal{g}$ -Idempotent Matrices

In this section, $\mathcal{g}$ −spectral resolution of a $\mathcal{g}$ − idempotent matrix is studied.

**Theorem 4.1**

The $\mathcal{g}$ −eigen values of a $\mathcal{g}$ −idempotent matrix $M$ are 0, 1 and -1.

**Proof**

The $\mathcal{g}$ −eigen values of $M$ are given by $\det(\lambda G - M) = 0$ and the $\mathcal{g}$ −eigen vector of $M$ associated with a $\mathcal{g}$ −eigen value $\lambda$ is given by

$$Mx = \lambda Gx \tag{4.1}$$

$$\Rightarrow GMx = \lambda x$$

$$\Rightarrow (GM)^2 x = \lambda GMx$$

$$\Rightarrow M^3 x = \lambda^2 x$$

$$\Rightarrow M^4 x = \lambda^2 Mx$$

$$\Rightarrow Mx = \lambda^2 Mx$$

$$\Rightarrow \lambda Gx = \lambda^3 Gx$$

$$\Rightarrow (\lambda - \lambda^3)Gx = 0$$

Since $Gx \neq 0$, we have $\lambda = 0, 1, -1$.

**Example: 4.2**

Consider the $\mathcal{g}$ −idempotent matrix $M$ given in *example 3.8*. The $\mathcal{g}$ −eigen values of $M$ are 1 and -1.

**Theorem: 4.3**

If $M \in \mathbb{C}^{n\times n}$, is a $\mathcal{g}$-idempotent matrix then $\sigma_{\mathcal{g}}(M) \subseteq \{0, 1, -1\}$. Moreover, there exist unique disjoint oblique projectors $Q_j$ for $j \in \{0, 1, -1\}$ such that

$$GM = Q_1 - Q_{-1} \tag{4.2}$$

$$I = Q_0 + Q_1 + Q_{-1} \tag{4.3}$$

**Proof**

Since $\sigma_{\mathcal{g}}(M) = \sigma(GM)$ and by *theorem 4.1*, we have $\sigma_{\mathcal{g}}(M) \subseteq \{0, 1, -1\}$.

Let us define $Q_j$'s by the formula,

$$Q_j = \prod_{\substack{i=0,1,-1 \\ i\neq j}} \frac{GM - iI}{j - i} \text{ for } j = 0, 1, -1$$

Then $Q_0 = I - M^3$; $Q_1 = \frac{1}{2}(M^3 + GM)$; $Q_2 = \frac{1}{2}(M^3 - GM)$.

When $j \notin \sigma_{\mathcal{g}}(M)$for $j \in \{0, 1, -1\}$, we have $Q_j = 0$.

The proof for uniqueness of the disjoint oblique projectors of the non-zero $Q_j$'s satisfying the decompositions (4.2) and (4.3) is analogous to the proof in *theorem 3.3*.

**Example 4.4**

Consider the matrices $M$ and $G$ given in *example 3.8*. The oblique projectors of $GM$ are found to be

$$Q_0 = 0,$$

$$Q_1 = \begin{pmatrix} \frac{1}{4} & \frac{3i}{4} \\ -\frac{i}{4} & \frac{3}{4} \end{pmatrix} \text{ and}$$

$$Q_{-1} = \begin{pmatrix} 3/4 & -3i/4 \\ i/4 & 1/4 \end{pmatrix}.$$

It is easy to verify that the above projectors are disjoint and satisfy the equations (4.2) and (4.3)

**Theorem 4.7**

Let $M$ be a $\mathcal{g}$-idempotent matrix. If $-1$ or $1 \notin \sigma_{\mathcal{g}}(M)$, then $MG = GM$.

**Proof**

Assume that $1 \notin \sigma_{\mathcal{g}}(M)$. Then , by *theorem 4.3*, we have $Q_1 = 0$.

Therefore, $-GM = M^3$. But $M^3 = GM^3G = G(-GM)G = -MG$. Hence $GM = MG$.

Similarly, we can prove $GM = MG$, whenever $-1 \notin \sigma_{\mathcal{g}}(M)$.

## REFERENCES

[1] Baksalary. J.K and Baksalary. O.M, "Idempotency of linear combinations of two idempotent matrices", Lin. Alg. Appl., 321, (2000): 3-7.

[2] Baksalary. O.M. and G¨Otz Trenkler, On K-Potent Matrices, Electronic Journal of Linear Algebra ISSN 1081-3810, Volume 26, pp. 446-470, July 2013

[3] Benitez and Thome.N, "{k}-group periodic matrices", SIAM.J.Matrix Anal.Appl., vol.18., No.1, (2006) : 9 - 25.

[4] Benitez.J and Thome.N, “Idempotency of linear combinations of an idempotent and a – potent matrix that commutes”, Lin.Alg.Appl., 403, (2005) : 414-418.

[5] Erdos. J. A, “On products of idempotent matrices”, Glasgow Math.J., 8, (1967): 118-122.

[6] Horn R.A and Johnson C.R, “Matrix Analysis” Cambridge University press, Cambridge, U.K, 1985.

[7] Pei Yuan Wu, “Sums of idempotent matrices”, Lin. Alg. Appl., 142, (1990): 43-54

[8] Vasudevan B, Anis Fathima N, “Idempotent Matrices in Minkowski Space”, Advances and Applications in Discrete Mathematics, Vol.24, Number 2, 2020, pp 165-177

[9] Yan Wu and Daniel F. Linder, On the Eigenstructures of Functional *K*-Potent Matrices and their Integral Forms, WSEAS Trans. on Maths. ISSN: 1109-2769, Issue 4, Vol. 9, pp. 244-253 April 2010.