Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 6 (2020), pp. 749-758

http://www.ripublication.com/gjpam.htm

Some Characterizations of Intra-regular Semigroups in Terms of Interval Valued Fuzzy Ideals

Thiti Gaketem

School of Science, University of Phayao, Phayao 56000, Thailand.

Abstract

In this paper, we characterized intra-regular semigroups in terms of interval valued fuzzy ideals.

AMS Subject Classification: 03G25, 03E72, 08A72, 18B40, 20M17

Key Words and Phrases: intra-regular semigroup, interval valued fuzzy ideal

1. INTRODUCTION

The concept of fuzzy sets was proposed by L. A. Zadeh in 1965 [16] These concepts were applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology etc. In 1971 [12], Rosenfeld was the first who consider in case S is a groupoid. He gave the definition of fuzzy subgroups fuzzy left (right, two-sided) ideals of S. In 1979, Kuroki [9] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. The concept of a quasi-ideal in rings and semigroups was studied by Stienfeld [6]. The notion of quasi-ideals is a generalization of left and right ideals whereas the bi-ideals are generalization of quasi-ideals.

In 1975 [13], Zadeh made an extension of the concept of fuzzy sets by an interval valued fuzzy sets, where the values of the membership functions are intervals of the numbers instead of the numbers. The notion of interval valued fuzzy sets have many applications such as medical science [4], image processing [2] etc. In 1994 Biswas [3] defined the interval valued fuzzy subgroups of the same nature which are of the fuzzy subgroups of Rosenfeld.

Later, in 2006 [11] Narayanan and Manikantan studied and discussed the concept of an interval valued fuzzy left ideal (right ideal, interior ideal, bi-ideal) in semigroup. In 2012 [**?**], Thillaigovindan and Chinnadurai studied interval valued fuzzy quasi-ideals in semigroup.

In 2013 [13], Singaram and Kandasamy characterized regular and intra-regular semigroups in terms of interval valued fuzzy left, (right) ideals as follows.

In this paper, we characterizes an intra-regular semigroup in terms of interval valued fuzzy ideals.

2. PRELIMINARIES

In this topic, some basic definitions are given.

A non-empty subset A of a semigroup S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset A of a semigroup S is called a *left* (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$. An *ideal* of S is a non-empty subset which is both a left ideal and a right ideal of S. A non-empty subset A of S is called a *quasi-ideal* of S if $AS \cap SA \subseteq A$.

For any $a_i \in [0,1]$ for all $i \in I$, define

$$\bigvee_{i\in I} a_i := \sup_{i\in I}\{a_i\} \quad \text{and} \quad \bigwedge_{i\in I} a_i := \inf_{i\in I}\{a_i\}.$$

We see that for any $a, b \in [0,1]$, we have

$$a \vee b = \max\{a,b\} \quad \text{and} \quad a \wedge b = \min\{a,b\}.$$

Now we will introduce a new relation of an interval. Let $D[0,1]$ be the set of all closed subinterval of the interval $[0,1]$, i.e.,

$$D[0,1] = \{\overline{a} = [a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

An interval $\overline{a}$ on $[0,1]$ is a closed subinterval of $[0,1]$, that is $\overline{a} = [a^-, a^+]$ such that $0 \leq a^- \leq a^+ \leq 1$.

We note that $[a,a] := \{a\}$ for all $a \in [0,1]$. For $a = 0$ or 1 we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

Definition 2.1. [13] Let $\overline{a} := [a^-, a^+]$ and $\overline{b} := [b^-, b^+]$ in $D[0,1]$. Define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{a} \preceq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$

(2) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$

(3) $\overline{a} \curlywedge \overline{b} = [(a^- \wedge b^-), (a^+ \wedge b^+)]$

(4) $\overline{a} \curlyvee \overline{b} = [(a^- \vee b^-), (a^+ \vee b^+)]$.

If $\overline{a} \succeq \overline{b}$, we mean $\overline{b} \preceq \overline{a}$.

Lemma 2.2. *[5] Let $\overline{a}, \overline{b}, \overline{c} \in D[0,1]$. Then the following properties hold:*

(1) $\overline{a} \curlywedge \overline{a} = \overline{a}$ *and* $\overline{a} \curlyvee \overline{a} = \overline{a}$*,*

(2) $\overline{a} \curlywedge \overline{b} = \overline{b} \curlywedge \overline{a}$ *and* $\overline{a} \curlyvee \overline{b} = \overline{b} \curlyvee \overline{a}$*,*

(3) $(\overline{a} \curlywedge \overline{b}) \curlywedge \overline{c} = \overline{a} \curlywedge (\overline{b} \curlywedge \overline{c})$ *and* $(\overline{a} \curlyvee \overline{b}) \curlyvee \overline{c} = \overline{a} \curlyvee (\overline{b} \curlyvee \overline{c})$*,*

(4) $(\overline{a} \curlywedge \overline{b}) \curlyvee \overline{c} = (\overline{a} \curlyvee \overline{c}) \curlywedge (\overline{b} \curlyvee \overline{c})$ *and* $(\overline{a} \curlyvee \overline{b}) \curlywedge \overline{c} = (\overline{a} \curlywedge \overline{c}) \curlyvee (\overline{b} \curlywedge \overline{c})$*,*

(5) *If* $\overline{a} \preceq \overline{b}$*, then* $\overline{a} \curlywedge \overline{c} \preceq \overline{b} \curlywedge \overline{c}$ *and* $\overline{a} \curlyvee \overline{c} \preceq \overline{b} \curlyvee \overline{c}$*.*

Definition 2.3. [13] For each interval $\overline{a}_i = [a_i^-, a_i^+] \in D[0,1]$, $i \in I$ where I is an index set, we define

$$\underset{i\in I}{\curlywedge} \overline{a}_i = [\underset{i\in I}{\wedge} a_i^-, \underset{i\in I}{\wedge} a_i^+] \quad \text{and} \quad \underset{i\in I}{\curlyvee} \overline{a}_i = [\underset{i\in I}{\vee} a_i^-, \underset{i\in I}{\vee} a_i^+].$$

A fuzzy subset (fuzzy set) of a set X is a function $f : X \to [0,1]$.

Definition 2.4. [13] Let X be a non-empty set. An interval valued fuzzy (IVF) subset $\overline{F} : X \to D[0,1]$ of X is defined by

$$\overline{F} = \{(x, [F^-(x), F^+(x)]) \mid x \in X\},$$

where F^- and F^+ are two fuzzy subsets of X such that $F^-(x) \leq F^+(x)$ for all $x \in X$.

Definition 2.5. [13] Let $A \subseteq X$. An interval valued characteristic function $\overline{\mathcal{C}}_A$ of A is defined to be a function $\overline{\mathcal{C}}_A : X \to D[0,1]$ by

$$\overline{\mathcal{C}}_A(x) = \begin{cases} \overline{1} & \text{if} \quad x \in A \\ \overline{0} & \text{if} \quad x \notin A \end{cases}$$

for all $x \in X$.

Proposition 2.6. *[**?**] Let A and B be a non-empty subset of a semigroup S Then the following statements hold*

(1) $(\overline{\mathcal{C}}_A \curlywedge \overline{\mathcal{C}}_B) = (\overline{\mathcal{C}}_{A\cap B})$.

(2) $(\overline{\mathcal{C}}_A) \circ (\overline{\mathcal{C}}_B) = (\overline{\mathcal{C}}_{AB})$.

Definition 2.7. [5] For two IVF subsets $\overline{F}$ and $\overline{G}$ in a semigroup S. Define

(1) $\overline{F} \sqsubseteq \overline{G} \Leftrightarrow \overline{F}(x) \preceq \overline{G}(x), \quad \forall x \in S$,

(2) $\overline{F} = \overline{G} \Leftrightarrow \overline{F} \sqsubseteq \overline{G}$ and $\overline{G} \sqsubseteq \overline{F}$,

(3) $(\overline{F} \sqcap \overline{G})(x) = \overline{F}(x) \curlywedge \overline{G}(x), \quad \forall x \in S$,

(4) $(\overline{F} \sqcup \overline{G})(x) = \overline{F}(x) \curlyvee \overline{G}(x), \quad \forall x \in S$.

Definition 2.8. [13] Let $\overline{F}$ and $\overline{G}$ be two IVF subsets in a semigroup S. Then the product $\overline{F} \circ \overline{G}$ is defined as follows : for all $x \in S$,

$$(\overline{F} \circ \overline{G})(x) = \begin{cases} \underset{(y,z)\in F_x}{\curlyvee} \{\overline{F}(y) \curlywedge \overline{G}(z)\} & \text{if} \quad F_x \neq \emptyset, \\ \overline{0} & \text{if} \quad F_x = \emptyset, \end{cases}$$

$F_x := \{(y,z) \in S \times S \mid x = yz\}$.

Since semigroup S is associative, the product is associative [15].

Definition 2.9. [11] Let S be a semigroup. An IVF subset $\overline{F}$ of S is said to be an *IVF subsemigroup* of S if $\overline{F}(xy) \succeq \overline{F}(x) \curlywedge \overline{F}(y)$ for all $x, y \in S$.

Definition 2.10. [11] Let S be a semigroup. An IVF subset $\overline{F}$ of S is said to be an *IVF left* (right) ideal of S if $\overline{F}(xy) \succeq \overline{F}(y)(\overline{F}(xy) \succeq \overline{F}(x))$ for all $x, y \in S$. An IVF subset $\overline{F}$ of S is called an IVF two-sided ideal of S if it is both an IVF left ideal and an IVF right ideal of S.

Definition 2.11. [11] Let S be a semigroup. An IVF subset $\overline{F}$ of S is called an *IVF generalized bi-ideal* ideal of S if $\overline{F}(xyz) \succeq \overline{F}(x) \curlywedge \overline{F}(z)$ for all $x, y, z \in S$.

Definition 2.12. [?] An IVF subsemigroup $\overline{F}$ of a semigorup S is called an *IVF bi-ideal* ideal of S if $\overline{F}(xyz) \succeq F(x) \curlywedge \overline{F}(z)$ for all $x, y, z \in S$.

Definition 2.13. [13] An IVF subsemigroup $\overline{F}$ of a semigorup S is called an *IVF interior* ideal of S if $\overline{F}(xay) \succeq \overline{F}(a)$ for all $a, x, y \in S$.

Definition 2.14. [?] Let S be a semigroup. An IVF subset $\overline{F}$ of S is called an *IVF quasi-ideal* of S if
$(\overline{S} \circ \overline{F})(x) \curlywedge (\overline{F} \circ \overline{S})(x) \preceq \overline{F}(x)$, for all $x \in S$ where $\overline{S}$ is an IVF subset of S mapping every element of S on $\overline{1}$.

Theorem 2.15. *Let S be a semigorup. Then A is a left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of S if and only if interval valued characteristic function $\overline{\mathcal{C}}_A$ is an IVF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of S.*

Proof. Suppose that A is a left ideal of S and let $x, y \in S$.

Case(1): $y \in A$. Then $xy \in A$. Thus, $\overline{\mathcal{C}}_A(y) = \overline{\mathcal{C}}_A(xy) = \overline{1}$. It implies that, $\overline{\mathcal{C}}_A(xy) \succeq \overline{\mathcal{C}}_A(y)$.

Case(2): $y \notin A$. Then, $\overline{\mathcal{C}}_A(y) = \overline{0}$. Thus, $\overline{\mathcal{C}}_A(xy) \succeq \overline{0} = \overline{\mathcal{C}}_A(y)$.

From case (1)-(2) we have $\overline{\mathcal{C}}_A(xy) \succeq \overline{\mathcal{C}}_A(y)$. This implies that $\overline{\mathcal{C}}_A$ is an IVF left ideal of S. Similarly we can prove the other cases also.

Conversely, suppose that $\overline{\mathcal{C}}_A$ is an IVF left ideal of S and let $y \in A$. Then, $\overline{\mathcal{C}}_A(y) = \overline{1}$. Since $\overline{\mathcal{C}}_A$ is an IVF left of S, we have $\overline{\mathcal{C}}_A(xy) \succeq \overline{\mathcal{C}}_A(y)$. Thus $xy \in A$. Hence A is a left ideal of S. Similarly we can prove the other cases also. □

Theorem 2.16. *[**?**] Every IVF interior ideal of a semigroup S is an IVF ideal of S.*

Theorem 2.17. *[**?**] Every IVF quasi-ideal of a semigroup S is an IVF bi-ideal of S.*

3. CHARACTERIZATIONS OF INTRA-REGULAR SEMIGROUPS BY THEIR INTERVAL VALUED FUZZY IDEALS.

In this section we characterizes an intra-regular semigroups in terms IVF left ideal (right ideal, bi-ideal, interior ideal) in semigroup.

Lemma 3.1. *Let M, N and O be a non-empty subset of a semigroup S Then the following statements hold*

(1) $((\overline{\mathcal{C}}_M) \curlywedge (\overline{\mathcal{C}}_N) \curlywedge (\overline{\mathcal{C}}_O)) = (\overline{\mathcal{C}}_{M\cap N\cap O})$.

(2) $((\overline{\mathcal{C}}_M) \circ (\overline{\mathcal{C}}_N) \circ (\overline{\mathcal{C}}_0)) = (\overline{\mathcal{C}}_{MNO})$.

Proof. (1) Let $a \in S$. If $a \in M \cap N \cap O$ then $a \in M$, $a \in N$ and $a \in O$. Thus $\overline{\mathcal{C}}_M(a) = \overline{\mathcal{C}}_N(a) = \overline{\mathcal{C}}_O(a) = \overline{1}$.

Consider

$$\begin{aligned}((\overline{\mathcal{C}}_M) \curlywedge (\overline{\mathcal{C}}_N) \curlywedge (\overline{\mathcal{C}}_O))(a) &= \overline{\mathcal{C}}_M(a) \curlywedge \overline{\mathcal{C}}_N(a) \curlywedge \overline{\mathcal{C}}_O(a) \\ &= \overline{1} \curlywedge \overline{1} \curlywedge \overline{1} = \overline{1}.\end{aligned}$$

On other hand

$$(\overline{\mathcal{C}}_{M\cap N\cap O})(a) = \overline{1}$$

Thus $((\overline{\mathcal{C}}_M) \curlywedge (\overline{\mathcal{C}}_N) \curlywedge (\overline{\mathcal{C}}_O)) = (\overline{\mathcal{C}}_{M\cap N\cap O})$.

If $a \notin M \cap N \cap O$, then $a \notin M$ or $a \notin N$ or $a \notin O$. Thus $\overline{\mathcal{C}}_M(a) = \overline{0}$ or $\overline{\mathcal{C}}_N(a) = \overline{0}$ or $\overline{\mathcal{C}}_O(a) = \overline{0}$.

Consider

$$\begin{array}{rcl}(\overline{\mathcal{C}}_M) \curlywedge (\overline{\mathcal{C}}_N) \curlywedge (\overline{\mathcal{C}}_O)(a) & = & \overline{\mathcal{C}}_M(a) \curlywedge \overline{\mathcal{C}}_N(a) \curlywedge \overline{\mathcal{C}}_O(a) \\ & = & \overline{0} \curlywedge \overline{0} \curlywedge \overline{0} = \overline{0}.\end{array}$$

On other hand

$$(\overline{\mathcal{C}}_{M\cap N\cap O})(a) = \overline{0}$$

Thus $((\overline{\mathcal{C}}_M) \curlywedge (\overline{\mathcal{C}}_N) \curlywedge (\overline{\mathcal{C}}_O)) = (\overline{\mathcal{C}}_{M\cap N\cap O})$.

(2) Let $a \in S$. If $a \in MNO$, then $(\overline{\mathcal{C}}_{NNO}) = \overline{1}$.

On other hand

$$\begin{array}{rcl}(\overline{\mathcal{C}}_M \circ \overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(a) & = & (\overline{\mathcal{C}}_M \circ (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O))(a) \\ & = & (\underset{(i,j)\in F_a}{\curlyvee}(\overline{\mathcal{C}}_M(i) \curlywedge (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(j)) \\ & = & (\underset{(i,j)\in F_a}{\curlyvee}(\overline{1} \curlywedge (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(j)) \\ & = & (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(j)) \\ & = & (\underset{(u,v)\in F_j}{\curlyvee}(\overline{\mathcal{C}}_N(u) \curlywedge \overline{\mathcal{C}}_O)(v)) \\ & = & (\underset{(u,v)\in F_j}{\curlyvee}(\overline{1} \curlywedge \overline{1}) = \overline{1} = (\overline{\mathcal{C}}_{MNO})(a).\end{array}$$

If $a \notin MNO$ then $(\overline{\mathcal{C}}_{MNO})(a) = \overline{0}$.

On other hand

$$\begin{array}{rcl}(\overline{\mathcal{C}}_M \circ \overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(a) & = & (\overline{\mathcal{C}}_M \circ (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O))(a) \\ & = & (\underset{(i,j)\in F_a}{\curlyvee}(\overline{\mathcal{C}}_M(i) \curlywedge (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(j)) \\ & = & (\underset{(i,j)\in F_a}{\curlyvee}(\overline{0} \curlywedge (\overline{\mathcal{C}}_N \circ \overline{\mathcal{C}}_O)(j)) \\ & = & \overline{0} = (\overline{\mathcal{C}}_{MNO})(a).\end{array}$$

Thus $((\overline{\mathcal{C}}_M) \circ (\overline{\mathcal{C}}_N) \circ (\overline{\mathcal{C}}_0)) = (\overline{\mathcal{C}}_{MNO})$.

□

Definition 3.2. [13, p.51] A semigroup S is said to be *intra-regular* if for each element $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$.

Lemma 3.3. *[8, p.102] Let S be a semigroup. Then the following are equivalent:*

(1) *S is intra-regular,*

(2) $I \cap B \cap L \subseteq IBR$, *for every interior ideal* I, *every bi-ideal* B *and every right ideal* R *of* S.

Theorem 3.4. *Let* S *be a semigroup. Then the following equivalent:*

(1) *Let* S *be intra-regular,*

(2) $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, *for every IVF interior ideal* $\overline{F}$, *every IVF quasi-ideal* $\overline{G}$ *and every IVF right ideal* $\overline{H}$ *of* S,

(3) $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, *for every IVF interior ideal* $\overline{F}$, *every IVF bi-ideal* $\overline{G}$ *and every IVF right ideal* $\overline{H}$ *of* S.

Proof. $(1) \Rightarrow (2)$ Let $\overline{F}, \overline{G}$ and $\overline{H}$ be an IVF interior ideal, an IVF quasi-ideal and an IVF right ideal of S respectively. Let $a \in S$. Since S is intra-regular we have there exist $x, y \in S$ such that $a = xa^2y = xaay = x(xaay)(xaay)y = x^2aayxaay^2 = x^2(xaay)ayxaay^2 = (x^2xaay)(ayxa)(ay^2) = (x^3aay)(ayxa)(ay^2)$. Since $\overline{G}$ is an IVF quasi-ideal of a semigroup S we have $\overline{G}$ is an IVF bi-ideal of S by Theorem 2.17. Thus

$$\begin{aligned}
(\overline{F} \circ \overline{G} \circ \overline{H})(a) &= (\overline{F} \circ (\overline{G} \circ \overline{H})(a) = (\bigcurlyvee_{(i,j) \in F_a} (\overline{F}(i) \curlywedge (\overline{G} \circ \overline{H})(j)) \\
&= (\bigcurlyvee_{(i,j) \in F_{(x^3aay)(ayxa)(ay^2)}} (\overline{F}(i) \curlywedge (\overline{G} \circ \overline{H}))(j) \\
&\succeq (\overline{F}(x^3aay) \curlywedge (\overline{G} \circ \overline{H})((ayxa)(ay^2)) \\
&= (\overline{F}(x^3aay) \curlywedge ((\bigcurlyvee_{(i,j) \in F_{(ayxa)(ay^2)}} (\overline{G}(p) \curlywedge \overline{H}(q))) \\
&\succeq (\overline{F}(x^3aay) \curlywedge ((\overline{G}(ayxa) \curlywedge \overline{H}(ay^2)))) \\
&\succeq (\overline{F}(a) \curlywedge (\overline{G}(a) \curlywedge \overline{G}(a) \curlywedge \overline{H}(a))) \\
&= (\overline{F}(a) \curlywedge (\overline{G}(a) \curlywedge \overline{H}(a))) \\
&= (\overline{F}(a) \curlywedge (\overline{G} \curlywedge \overline{H}))(a) = (\overline{F} \curlywedge \overline{G} \curlywedge \overline{H})(a).
\end{aligned}$$

Hence $(\overline{F} \circ \overline{G} \circ \overline{H})(a) \succeq (\overline{F} \curlywedge \overline{G} \curlywedge \overline{H})(a)$. Therefore $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$.

$(2) \Rightarrow (3)$ Every IVF quasi-ideal of a semigroup S is an IVF bi-ideal of S.

$(3) \Rightarrow (1)$ Suppose that $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$. Let I, B and R be an interior ideal, a bi-ideal and a right ideal of S respectively. Then by Theorem 2.15, $\overline{\mathcal{C}}_I, \overline{\mathcal{C}}_B$ and $\overline{\mathcal{C}}_R$ is an IVF interior ideal, an IVF bi-ideal and an IVF right ideal of S respectively. By hypothesis and Lemma 2.6, we have

$$\begin{aligned}
\overline{\beta} \curlyvee \overline{\alpha} &= (\overline{\mathcal{C}}_{I \cap B \cap R})(a) = ((\overline{\mathcal{C}}_I) \curlywedge (\overline{\mathcal{C}}_B) \curlywedge (\overline{\mathcal{C}}_R))(a) \\
&\sqsubseteq ((\overline{\mathcal{C}}_I) \circ (\overline{\mathcal{C}}_B) \circ (\overline{\mathcal{C}}_R))(a) = (\overline{\mathcal{C}}_{IBR})(a).
\end{aligned}$$

Thus $a \in IBR$. Hence $I \cap B \cap R \sqsubseteq IBR$. Therefore by Lemma 3.3, S is intra-regular. $\square$

Lemma 3.5. *[8] Let S be a semigroup. Then the following are equivalent:*

(1) *S is intra-regular,*

(2) *$L \cap B \cap I \subseteq LBI$, for every L is a left ideal, for each B is a bi-ideal and for each I is a interior ideal of S.*

Theorem 3.6. *Let S be a semigroup. Then the following equivalent:*

(1) *S is intra-regular,*

(2) *$\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, for every IVF left ideal $\overline{F}$, every IVF quasi-ideal $\overline{G}$ and every IVF interior ideal $\overline{H}$ of S*

(3) *$\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, for every IVF left ideal $\overline{F}$, every IVF bi-ideal $\overline{G}$ and every IVF interior ideal $\overline{H}$ of S.*

Proof. $(1) \Rightarrow (2)$ Let $\overline{F}, \overline{G}$ and $\overline{H}$ be an IVF left ideal, an IVF quasi-ideal and an IVF interior ideal of S respectively. Let $a \in S$. Since S is intra-regular we have there exist $x, y \in S$ such that $a = xa^2y = x(xaay)(xaay)y = x(xaay)(xa(xaay)yy) = (xxa)(ayxa)(xaayyy) = (x^2a)(ayxa)(xaay^3)$. Since $\overline{G}$ is an IVF quasi-ideal of a semigroup S we have $\overline{G}$ is an IVF bi-ideal of S by Theorem 2.17. Thus

$$\begin{aligned}
(\overline{F} \circ \overline{G} \circ \overline{H})(a) &= (\overline{F} \circ (\overline{G} \circ \overline{H}))(a) = (\curlyvee_{(i,j)\in F_a} (\overline{F}(i) \curlywedge (\overline{G} \circ \overline{H}))(j)) \\
&= (\curlyvee_{(i,j)\in F_{(x^2a)(ayxa)(xaay^3)}} (\overline{F}(i) \curlywedge (\overline{G} \circ \overline{H}))(j)) \\
&\succeq (\overline{F}(x^2a) \curlywedge (\overline{G} \circ \overline{H})((ayxa)(xaay^3)) \\
&= (\overline{F}(x^2a) \curlywedge ((\curlyvee_{(i,j)\in F_{(ayxa)(xaay^3)}} (\overline{G}(p) \curlywedge \overline{H}(q)) \\
&\succeq (\overline{F}(x^2a) \curlywedge (((\overline{G}(ayxa) \curlywedge \overline{H}(xaay^3)))) \\
&= (\overline{F}(x^2a) \curlywedge (\overline{G}(ayxa) \curlywedge \overline{H}(xaay^3))) \\
&\succeq (\overline{F}(a) \curlywedge (\overline{G}(a) \curlywedge \overline{G}(a) \curlywedge \overline{H}(a))) \\
&= (\overline{F}(a) \curlywedge (\overline{G}(a) \curlywedge \overline{H}))(a) \\
&= (\overline{F}(a) \curlywedge (\overline{G} \curlywedge \overline{H}))(a) = (\overline{F} \curlywedge \overline{G} \curlywedge \overline{H})(a).
\end{aligned}$$

Hence $(\overline{F} \curlywedge \overline{G} \curlywedge \overline{H})(a) \preceq (\overline{F} \circ \overline{G} \circ \overline{H})(a)$. Therefore $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$.

$(2) \Rightarrow (3)$ Every IVF quasi-ideal of a semigroup S is an IVF bi-ideal of S.

$(3) \Rightarrow (1)$ Suppose that $\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$. Let L, B and I be a left ideal, a bi-ideal and an interior ideal of S respectively. Then by Theorem 2.15, $\overline{\mathcal{C}}_L, \overline{\mathcal{C}}_B$ and $\overline{\mathcal{C}}_I$ is an IVF left ideal, an IVF bi-ideal and an IVF interior ideal of S respectively. By hypothesis and Lemma 2.6, we have

$$\begin{aligned}
\overline{\beta} \curlyvee \overline{\alpha} &= (\overline{\mathcal{C}}_{L\cap B\cap I})(a) = ((\overline{\mathcal{C}}_L) \curlywedge (\overline{\mathcal{C}}_B) \curlywedge (\overline{\mathcal{C}}_I))(a) \\
&\sqsubseteq ((\overline{\mathcal{C}}_L) \circ (\overline{\mathcal{C}}_B) \circ (\overline{\mathcal{C}}_I))(a) = (\overline{\mathcal{C}}_{LBI})(a).
\end{aligned}$$

Thus $a \in LBI$. Hence $L \cap B \cap I \sqsubseteq LBI$. Therefore by Lemma 3.5, S is intra-regular. $\square$

Corollary 3.7. *Let S be a semigroup. Then the following equivalent:*

(1) *S is intra-regular,*

(2) *$\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, for every -IVF left ideal $\overline{F}$, every -IVF quasi-ideal $\overline{G}$ and every -IVF ideal $\overline{H}$ of S,*

(3) *$\overline{F} \curlywedge \overline{G} \curlywedge \overline{H} \sqsubseteq \overline{F} \circ \overline{G} \circ \overline{H}$, for every IVF left ideal $\overline{F}$, every IVF bi-ideal $\overline{G}$ and every IVF interior ideal $\overline{H}$ of S.*

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