Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 5 (2020), pp. 675-685

https://dx.doi.org/GJPAM/16.5.2020.675-685

Asymptotic of Second Order Polar Polynomials

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Abstract

In this paper we derive useful results regarding the asymptotic properties of new set of monic polynomials primitives of orthogonal polynomials on the unit circle, called second order polar polynomials.

Keywords: Orthogonal polynomials, Polar polynomials, Asymptotic, Circle.

2010 Mathematics Subject Classification: Primary 05C38, 15A15; Secondary 05A15, 15A18

1. INTRODUCTION

Let μ be a finite positive measure defined on the Borelian σ-algebra of $\mathbb{C}$ and concentrated on the unit circle $T = \{z, |z| = 1\}$. μ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $[-\pi, +\pi]$, i.e

$$d\mu(\theta) = \rho(\theta)d\theta, \;\; \rho \geq 0, \;\; \rho \in L^1([-\pi, +\pi], d\theta). \tag{1}$$

Let us consider $L_n(z) = z^n+$......the n-th monic (i.e its leading coefficient is equal to one) a monic orthogonal polynomial with respect to μ, that is

$$\frac{1}{2\pi}\int_0^{2\pi} L_n(z)\,(\overline{z})^k\, d\mu(\theta) = 0, \qquad k = 0, 1....., n-1, \qquad \left(z = e^{i\theta}\right). \tag{2}$$

Theses monic orthogonal polynomials are related by the following mutually equivalent recursions relations :

$$L_n(z) = zL_{n-1}(z) + L_n(0)\,L^*_{n-1}(z) \quad \text{and} \quad L^*_n(z) = L^*_{n-1}(z) + \overline{L_n(0)}zL_{n-1}(z)$$

where :

$$L^*_n(z) = z^n\overline{L_n}\left(\frac{1}{z}\right), \qquad (z \neq 0).$$

Let α be a fixed complex number, let us consider a monic polynomial $P_n(z)$ of degree n and such that

$$(n+1)\,L_n(z) = ((z-\alpha)\,P_n(z))' = P_n(z) + (z-\alpha)\,P'_n(z) \tag{3}$$

P_n is called the n-th polar polynomial of $L_n(z)$. For a fixed complex number α, let us consider a monic polynomial Q_n of degre n such that

$$\left((z-\alpha)^2 Q_n(z)\right)'' = (n+2)\,(n+1)\,L_n(z) \tag{4}$$

Q_n is called the n-second order polar polynomial of $L_n(z)$. Obviously

$$\left((z-\alpha)^2 Q_n(z)\right)' = (n+2)\,(z-\alpha)\,P_n(z). \tag{5}$$

Note that $\Lambda(z) = (z-\alpha)P_n(z)$,$\Pi(z) = (z-\alpha)^2 Q_n(z)$ are respectively monic polynomials primitives of $(n+1)L_n(z)$,and $(n+2)\,(n+1)\,L_n(z)$ normalized respectively by $\Lambda(\alpha) = 0$ and $\Pi(\alpha) = 0$ and $\Pi'(\alpha) = 0.P_n(z)$ is said to be the polar polynomial of $L_n(z)$ in the sense of Pijeira, (see [1]) for all details concerning polar polynomials of monic orthogonal polynomials with respect to a measure supported on un interval , in the case of $[-1,+1]$, instead of the unit circle $T = \{z, |z| = 1\}$. One usually studies the asymptotics of Q_n in one of four ways, the first is strong asymptotic

$$\lim_{n\longrightarrow\infty} \frac{Q'_n(z)}{nL_n(z)}$$

the second is root asymptotics :

$$\lim_{n\longrightarrow\infty} |Q_n(z)|^{\frac{1}{n}}$$

the third is ratio asymptotics :

$$\lim_{n\to\infty} \frac{Q_{n+1}(z)}{Q_n(z)}$$

and the fourth is Szegö asymptotics :

$$\lim_{n\to\infty} \frac{Q_n(z)}{(\varphi(z))^n}, \qquad \varphi(z) \text{ is analytic on } \mathbb{C} \setminus \text{ch}(\mu)$$

where ch(μ) denotes the convex hull of the support of the measure μ.

A direct consequence of (2) and (4) is that

$$\frac{1}{2\pi}\int_{0}^{2\pi}\left((z-\alpha)^2 Q_n(z)\right)''(\overline{z})^k\,d\mu(\theta)=0,\quad k=0,1...n-1\left(\ z=e^{i\theta}\right).$$

This type of orthogonality relations generated by differential operators was intoduced initially by Aptekarev, Lagomazino, Marcelan, where the existence and uniqueness conditions for more general differential expressions were studied in detail. A similar study has been done by A. Fundora, H. Pijeira and W. Urbina [1], in the case of $[-1,+1]$. More information on the history and applications of this concept of polar may be found in [1, 10, 12].

1.1. Equilibrium measure

In order to obtain the asymptotic behaviour of the 2-polar polynomials on the unit circle. We need some definitions and general results and lemmas that we will discuss in what follows

Lemma 1.1. *([1, 6, 14, 17]) Let Φ_n be a polynomial of degree n with simple zeros $z_{n,1}, z_{n,2}, ...z_{n,n}$. The normalised counting measure of the zeros of Φ_n is defined by*

$$\nu_n(\Phi_n)=\frac{1}{n}\sum_{k=1}^{n}\delta_{z_{n,k}} \tag{6}$$

where $\delta_{z_{n,k}}$ is the Dirac measure with mass one at the point $z_{n,k}$, we get

$$\lim_{n\longrightarrow\infty}\frac{\Phi_n'(z)}{n\Phi_n(z)}=\lim_{n\longrightarrow\infty}\int\frac{d\nu_n(x)}{z-x} \tag{7}$$

where $\lim\limits_{n\longrightarrow\infty}\nu_n(x)=\omega_\Delta(x)$. in the sense of the weak $*$ topology ν_n is in this case the counting measure of the zeros of Φ_n, where ω_Δ is the equilibrium measure of Δ.For $\Delta=[-1\ +1]$, the equilibrium measure is arcsin *measure given by : if B is **Borel** set in $[-1\ +1]$,*

$$\mu_\Delta(B)=\int_B\frac{\arcsin' x dx}{\pi}=\frac{1}{\pi}\int_B\frac{dx}{\sqrt{1-x^2}}.$$

The normalised counting measure $\nu_n(P_n)$ converges to equilibrium measure arcsin *measure in tne sense of the weak *topolgy. Then*

$$\lim_{n\longrightarrow\infty}\frac{\Phi_n'(z)}{n\Phi_n(z)}=\int_{-1}^{1}\frac{d\omega_\Delta(x)}{z-x}=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{z-x}\frac{dx}{\sqrt{1-x^2}}.$$

Remark 1.2. In such a case(equilibrium measure ω_Δ) . If $\nu_n \overset{*}{\longrightarrow} \omega_\Delta$,as $n \to \infty$,then

$$\left\langle \nu_n(x), \frac{1}{z-x} \right\rangle \longrightarrow \left\langle \omega_\Delta(x), \frac{1}{z-x} \right\rangle.$$

Lemma 1.3. *([1, 13]) Let $\Delta \subset \mathbb{C}$ be a compact set with empty interior and unbounded connected component with positive logarithmic capacity . If $\{\Phi_n(z)\}_{n=0}^{\infty}$ is a sequence of monic polynomials,* $\deg \Phi_n(z) = n$, *such that*

$$\limsup_{n \longrightarrow \infty} (\|\Phi_n\|_\Delta^{\frac{1}{n}}) \le Cap(\Delta)$$

then $\nu_n(\Phi_n) \overset{*}{\longrightarrow} \omega_\Delta$ here ω_Δ is the equilibrium measure of Δ.

Lemma 1.4. *([1, 13, 18]) Let $\{\Phi_n(z)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then for all $k \in \mathbb{Z}_+$*

$$\limsup_{n \longrightarrow \infty} \sqrt[k]{\frac{\left\|\Phi_n^{(k)}\right\|_\Delta}{\|\Phi_n\|_\Delta}} \le 1. \tag{8}$$

There is an interesting fact about Taylor's coefficients of polynomials and their zeros set and its local which based in the following Szegö's theorem (see [14, 15])

1.2. Asymptotics of derivatives of monic orthogonal polynomials.

There is a major theory for asymptotics of this type, with key initial advances due to Mate, Nevai, Rakhmanov, Totik and many later works (see [2]). In particular, Rakhmanov' theorem asserts that if $\mu > 0$ a.e. on $[0, 2\pi]$, then we have this ratio asymptotic. Lubinsky ([8]) proved the following result asymptotic properties for monic orthogonal polynomials on the unit circle have been studied under different hypotheses. The Szegö's condition is satisfied. For instance,

$$\int_0^{2\pi} \log \mu' > -\infty. \tag{9}$$

Asymptotics for derivatives of orthogonal polynomials have been established under various hypotheses by P.N evai and D.S.Lubinsky (see [8]).

Theorem 1.5. *([2, 3, 4]) Assuming the Szegö's condition and by the P. Nevai's condition hold, let $\{\varphi_n\}_{n=0,1,2....}$ denote the orthonormal polynomials for μ, so that*

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(z) \overline{\varphi_m(z)} d\mu(\theta) = \delta_{mn} \qquad \left(z = e^{i\theta}\right).$$

Let $m \geq 1$. Then

$$\lim_{n\to\infty} \frac{z^m \varphi_n^{(m)}(z)}{n^m \varphi_n(z)} = 1 \qquad \left(z = e^{i\theta}\right)$$

Theorem 1.6. *([8]) Lubinsky proved that for each $m \geq 1$: if*

$$\lim_{n\to\infty} \frac{L_{n+1}(z)}{zL_n(z)} = 1 \tag{10}$$

holds uniformly for compact subsets of $|z| = 1$. Let $m \geq 1$. Then

$$\lim_{n\to\infty} \frac{L_{n+1}^{(m)}(z)}{zL_n^{(m)}(z)} = 1 \tag{11}$$

uniformly in $\{z, |z| \geq 1\}$.

Lemma 1.7. *([14, 15])(Szegő's theorem). Given the polynomials*

$$f_n(z) = \sum_{k=0}^{n} \alpha_{nk} C_k^n z^k, \ \alpha_{nn} \neq 0 \quad and \quad g_n(z) = \sum_{k=0}^{n} \beta_{nk} C_k^n z^k, \qquad \beta_{nn} \neq 0$$

and

$$h_n(z) = \sum_{k=0}^{n} \alpha_{nk} \beta_{nk} C_k^n z^k.$$

If all the zeros of $f_n(z)$ lie in a closed disk $\overline{D}$ and $\lambda_{n,1}, \lambda_{n,2}, ..\lambda_{n,n}$ are the zeros of $g_n(z)$. Then every zero of $h_n(z)$ has the form $\lambda_{n,k}\gamma_{n,k}$, where $\gamma_{n,k} \in \overline{D}$.

2. ASYMPTOTIC PROPRIETES OF THE SECOND ORDER POLAR POLYNOMIALS

For the polar monic orthogonal polynomials and 2-polar polynomials and their zeros set and its local, there is an interesting fact about its Taylor's coefficients.

2.1. Localization of zeros

Our next propose is to prove that all the zeros of the polynomials of the two sequences $(P_n(z))_{n=0,1,2.....}$ and $(Q_n(z))_{n=0,1,2.....}$ are contained in a disc which radius is independent of n.First, let us rewrite the polynomials $Q_n(z)$ and $P_n(z)$ in terms of $(z - \alpha)$, that is

$$L_n(z) = \sum_{k=0}^{n} c_{n,k}(z-\alpha)^k, P_n(z) = \sum_{k=0}^{n} b_{n,k}(z-\alpha)^k, Q_n(z) = \sum_{k=0}^{n} a_{n,k}(z-\alpha)^k. \tag{12}$$

It is well-known that the zeros of $L_n(z)$,(see ,Simon,[16]) are contained in the unit closed disk $\overline{D} = \{z, |z| \leq 1\}$.

Lemma 2.1. *The coefficients $a_{n,k}$ of Q_n and $b_{n,k}$ of P_n and $c_{n,k}$ of L_n in (12) are related by*

$$b_{n,k} = \frac{n+1}{k+1} c_{n,k}, \; a_{n,k} = \frac{n+2}{k+2} b_{n,k}, \tag{13}$$

i.e

$$a_{n,k} = \frac{(n+2)(n+1)}{(k+2)(k+1)} c_{n,k}.$$

Proof. Replacing (12) respectively in (3) and in (4) and (5). □

Theorem 2.2. *All the zeros of P_n are contained in the closed disk $\overline{D}_1$, where*

$$\overline{D}_1 = \{z \in \mathbb{C} : |z| \leq 2 + 3|\alpha|\}. \tag{14}$$

All the zeros of Q_n are contained in the closed disk $\overline{D}_2$, where

$$\overline{D}_2 = \{z \in \mathbb{C} : |z| \leq 2 + 5|\alpha|\}. \tag{15}$$

Proof. Let us write,$w = z - \alpha$, hence

$$f_n(w) = \sum_{k=0}^{n} c_{n,k} w^k = L_n(z) \quad \text{and} \quad h_n(w) = \sum_{k=0}^{n} \frac{n+1}{k+1} c_{n,k} w^k = P_n(z).$$

By Szegő's theorem,we get

$$g_n(w) = \sum_{k=0}^{n} \frac{n+1}{k+1} C_k^n w^k$$

we have

$$g_n(w) = \frac{(1+w)^{n+1} - 1}{w} = \frac{(z - \alpha + 1)^{n+1} - 1}{z - \alpha}, \qquad (z \neq \alpha).$$

If $z_{n,0}$ is a zero of L_n,it is well known that $|z_{n,0}| \leq 1$, hence $w_{n,0} = z_{n,0} - \alpha$ is a zero of $f_n(w)$, lie in the closed disk $\overline{D} = \{|w + \alpha| \leq 1\}$. On the other hand,if $w_{n,1}$ is a zero of $g_n(w)$ then:$|w_{n,1} + 1| = 1$ .Finally, by Lemma 4, if $w_{n,2}$ is a zero of $h_n(w)$,we have that $|w_{n,2}| = |w_{n,0} w_{n,1}| \leq 2(1 + |\alpha|)$ .If $P_n(z_{n,2}) = 0$, then $z_{n,2} = w_{n,2} + \alpha$. Thus property (14) is confirmed.

Now in order to make (15), setting $w = z - \alpha$, hence

$$F_n(w) = \sum_{k=0}^{n} b_{n,k} w^k = P_n(z) \qquad H_n(w) = \sum_{k=0}^{n} \frac{n+2}{k+2} b_{n,k} w^k = Q_n(z)$$

and

$$G_n(w) = \sum_{k=0}^{n} \frac{n+2}{k+2} C_k^n w^k$$

hence,

$$\left(w^2G_n(w)\right)' = \sum_{k=0}^{n}(n+2)\,C_k^n w^{k+1} = (n+2)\,w\,(1+w)^n\,.$$

Integrating by parts both sides from 0 to w, we get

$$w^2G_n(w) = \frac{n+2}{n+1}w\,(1+w)^{n+1} - \frac{1}{n+1}(1+w)^{n+2}$$

i.e

$$G_n(w) = \frac{(1+w)^{n+1}\,((n+1)\,w-1)}{(n+1)\,w^2} \qquad ,(w\neq 0)$$

therefore,

$$G_n(w) = \frac{(z-\alpha+1)^{n+1}\,((n+1)\,z-(n+1)\,\alpha-1)}{(n+1)\,(z-\alpha)^2}, \qquad (z\neq\alpha)\,.$$

If $w_{n,3}$ is a zero of $G_n(w)$ then $|w_{n,3}| \leq 1$.On the other hand, if $w_{n,4}$ is a zero of $F_n(w)$ hence $w_{n,4} = z_{n,4} - \alpha$ where $P_n(z_{n,4}) = 0$, by (31) it is well known that $|z_{n,4}| \leq 2+3|\alpha|$,then we have that $|w_{n,4}+\alpha| \leq 2+3|\alpha|$. By Lemma 4, if $H_n(w_{n,5}) = 0$,we have that $|w_{n,5}| = |w_{n,3}w_{n,4}| \leq 2+4|\alpha|$,thus property (15) is confirmed.The proof of the theorem is completed. □

2.2. Relative asymptotic of the of second order polar polynomials

Relative asymptotic of the polar polynomials $\{P_n\}_{n=0,1,2....}$with respect to the monic orthogonal polynomials $\{L_n\}_{n=0,1,2....}$ have been established under same hypotheses of P.Nevai and D.S.Lubinsky (see [8]).

Theorem 2.3. ([12]) *Let α be a fixed complex number. Let P_n be the polar polynomial of L_n. Assume that*

$$\lim_{n\to\infty}\frac{L_{n+1}(z)}{zL_n(z)} = 1$$

uniformly for compact subsets of $\{|z|>1\}$. Then we have

$$\lim_{n\longrightarrow\infty}\frac{L_n(z)}{P_n(z)} = \frac{z-\alpha}{z} \tag{16}$$

uniformly in $\{|z|>1\}$. And

$$\lim_{n\longrightarrow\infty}\frac{P_n'(z)}{nP_n(z)} = \frac{1}{z} \tag{17}$$

uniformly in $\{|z|>1\}$.In the same way,

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{zP_n(z)} = 1 \tag{18}$$

uniformly in $\{|z|>1\}$.

Proof. see ([12]). □

The main result is the following theorem :

Theorem 2.4. *With the above assumptions,let α be a fixed complex number. Let Q_n be the 2-polar polynomial of L_n. Assume that*

$$\lim_{n\to\infty} \frac{L_{n+1}(z)}{zL_n(z)} = 1$$

then

$$\lim_{n\longrightarrow\infty} \frac{Q_{n+1}(z)}{zQ_n(z)} = 1 \tag{19}$$

uniformly in $\{|z| > 1\}$ *.it holds*

$$\lim_{n\longrightarrow\infty} \frac{Q_n'(z)}{nQ_n(z)} = \frac{1}{z} \tag{20}$$

uniformly in $\{|z| > 1\}$.

Proof. By (5),

$$(z-\alpha)^2 Q_n'(z) + 2(z-\alpha) Q_n(z) = (n+2)(z-\alpha) P_n(z), \qquad (z \neq \alpha). \tag{21}$$

Then

$$\lim_{n\longrightarrow\infty} \frac{P_n(z)}{(z-\alpha) Q_n(z)} = \lim_{n\longrightarrow\infty} \frac{Q_n'(z)}{nQ_n(z)}. \tag{22}$$

If $z \neq \alpha$, we write

$$\frac{d}{dz} \log (z-\alpha)^2 Q_n(z) = \frac{\left((z-\alpha)^2 Q_n(z)\right)'}{(z-\alpha)^2 Q_n(z)} \qquad (z \neq \alpha)$$

hence

$$\frac{d}{dz} \log (z-\alpha)^2 Q_n(z) = (n+2) \frac{P_n(z)}{(z-\alpha) Q_n(z)} \qquad (z \neq \alpha).$$

By Integration in both hand sides of this equality from fixed point $z_1$to z, we get

$$\frac{(z-\alpha)^2 Q_n(z)}{(z_1-\alpha)^2 Q_n(z_1)} = \exp\left((n+2)\int_{z_1}^{z} \frac{P_n(t)}{(t-\alpha) Q_n(t)} dt\right) \qquad (t \neq \alpha)$$

and

$$Q_n(z) = \left(\frac{z_1-\alpha}{z-\alpha}\right)^2 Q_n(z_1) \exp\left((n+2)\int_{z_1}^{z} \frac{P_n(t)}{(t-\alpha) Q_n(t)} dt\right). \tag{23}$$

Taking into account this expression (23) and (22) reads .

$$\lim_{n\longrightarrow\infty}\frac{Q_{n+1}(z)}{Q_n(z)}\left(\frac{Q_{n+1}(z_1)}{Q_n(z_1)}\right)^{-1}=\exp\left(\lim_{n\longrightarrow\infty}\int_{z_1}^{z}\frac{P_n(t)}{(t-\alpha)\,Q_n(t)}dt\right).$$

Thus, we have

$$\lim_{n\longrightarrow\infty}\frac{Q_{n+1}(z)}{Q_n(z)}\left(\frac{Q_{n+1}(z_1)}{Q_n(z_1)}\right)^{-1}=\exp\Lambda(z)$$

where

$$\Lambda(z)=\int_{z_1}^{z}M(t)dt$$

and

$$M(z)=\lim_{n\longrightarrow\infty}\frac{Q_n'(z)}{nQ_n(z)}.$$

Which yields the proof of the existence of $M(z)$. It is easy to conclude that

$$\lim_{n\longrightarrow\infty}\frac{Q_{n+1}(z)}{Q_n(z)}=\lim_{n\longrightarrow\infty}\frac{P_{n+1}(z)}{P_n(z)}=z$$

threfore,

$$\exp\Lambda(z)=\frac{z}{z_1},\qquad (z_1\text{is fixed point})$$

implies

$$M(z)=\Lambda'(z)=\frac{1}{z}.$$

The proof of proprietes (19),(20) may now be completed. □

Remark 2.5. All the zeros of Q_n are contained in the closed disk $\overline{D}_2$, where

$$\overline{D}_3=\{z\in\mathbb{C}:|z|\leq 1+2|\alpha|\}. \tag{24}$$

To make (24), setting $w=z-\alpha$,by (14) :

$$f_n(w)=\sum_{k=0}^{n}c_{n,k}w^k=L_n(z)\,,\ T_n(w)=\sum_{k=0}^{n}\frac{(n+2)(n+1)}{(k+2)(k+1)}c_{n,k}w^k=Q_n(z)$$

and

$$R_n(w)=\sum_{k=0}^{n}\frac{(n+2)(n+1)}{(k+2)(k+1)}C_k^n w^k,\quad w\neq 0$$

hence,

$$\left(w^2R_n(w)\right)''=(n+2)(n+1)\sum_{k=0}^{n}C_k^n w^k=(n+2)(n+1)(1+w)^n$$

$$R_n(w) = \frac{(1+w)^{n+2}}{w^2} = \frac{(z-\alpha+1)^{n+2}}{(z-\alpha)^2}, \quad z \neq \alpha.$$

If $s_{n,3}$ is a zero of $R_n(w)$ then $|s_{n,3}| \leq 1$.On the other hand, if $s_{n,4}$ is a zero of $f_n(w)$ hence $s_{n,4} = z_{n,4} - \alpha$ where $L_n(z_{n,4}) = 0$, it is well known that $|z_{n,4}| \leq 1$,then we have that $|s_{n,4} + \alpha| \leq 1$, by Lemma 4,if $T_n(s_{n,5}) = 0$,since $s_{n,5} = z_{n,5} - \alpha$, where $Q_n(z_{n,5}) = 0$, we have that $|s_{n,5}| = |s_{n,3}s_{n,4}| \leq |s_{n,3}| |s_{n,4} + \alpha - \alpha| \leq 1 + |\alpha|$,the property (15) is confirmed.

Remark 2.6. Since,by (22)

$$\lim_{n\longrightarrow\infty}\left(\frac{P_n(z)}{Q_n(z)}\right)' \lim_{n\to\infty}\frac{P_n(z)}{Q_n(z)}\left(\frac{P_n'(z)}{Q_n(z)} - \frac{Q_n'(z)}{Q_n(z)}\right)$$

imply,

$$\lim_{n\to\infty}\frac{Q_n'(z)}{nQ_n(z)} = \lim_{n\longrightarrow\infty}\frac{P_n'(z)}{nP_n(z)} = \frac{1}{z}.$$

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