Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 5 (2020), pp. 657-673

http://www.ripublication.com/gjpam.htm

Integrated Simpson's Collocation Method for Solving Fourth Order Volterra Integro-Differential Equations

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Abstract

This paper compares Integrated Simpson's Collocation Method with the exact solution for solving fourth order Volterra Integro-Differential Equations. The highest derivative that appeared in the problems considered is approximated by the Power series and Chebyshev polynomials of suitable degree. The lower ordered derivatives are then obtained from this approximation and the unknown function is obtained by applying Simpson's rule on the first ordered derivative. These derivatives and the unknown function are then substituted into the given problem and after simplification, the resulting equation is then collocated at some equally spaced interior points in the interval of consideration, this leads to system of linear algebraic equations which are then solved by 'Maple 18' package to obtain the values of the unknown constants that are contained in the assumed solution. These values are then substituted back into the unknown function to obtain the approximate solution required. From the computational view-point, the Integrated Simpson's Collocation Method is easy to apply and of high accuracy. It is observed from the results presented in Tables 1-3, that the two basis functions produce similar results and that the method yields the desired accuracy when compared with the exact solutions.

Keywords: Integrated Simpson's Method, Collocation method, Simpson's rule, Power series, Chebyshev polynomials, Integro-Differential Equations.

1.0 INTRODUCTION.

Integro-Differential Equations with initial/boundary conditions are very important in the various domains of sciences, engineering and social sciences. It is the type of equation where both the differential and the integral operators appeared. Volterra Integro-Differential Equations were formed from converting initial value problems with prescribed initial values, while Fredholm Integro-Differential Equations were derived from boundary value problems with the given boundary conditions [1]. In most cases, it is very complicated to achieve analytic solutions of this class of equations [2]. Of all the methods that have been proposed to numerically solve differential equations, collocation methods are seen to be the simplest way to discretize differential equations.

In recent time, many numerical analysts have examined the numerical solutions of high order Integro-Differential Equations. Some of the methods applied include: Legendre Multi-wavelet Direct method [3]; Modified Homotopy-Perturbation Method [4]; Variational Iteration Method [5, 6, 7]; Wavelet-Galerkin compression techniques [8, 9] and Adomian Decomposition Method [10]. Other methods applied to solve Integro-Differential Equations are: Taylor Polynomial solution technique [11]; Rationalized Haar functions method [12]; Petrov-Galerkin method [13]; finite difference method [14]; Block-Pulse Functions [15]; Lagrange Interpolation Method [16]; Sine-Cosine Wavelets Method [17]; Combined Laplace transform-Adomian decomposition method [1]; Chebyshev Polynomial Approach [18]; Compact finite difference method [19] and Finite volume element methods [20].

2.0 INTEGRATED SIMPSON'S COLLOCATION METHOD (ISCM)

Consider the following general kth order Volterra Integro-Differential Equation (VIDE)

$$\sum_{i=0}^{k} P_i y^{(i)}(x) + \int_a^x K(x,t)y(t)dt = f(x) \tag{1}$$

Equation (1) is subjected to the conditions

$$y^{(i)}(a) = \alpha_j, \quad i = 0,\ 1,\ 2, ..., n-1 \tag{2}$$

and

$$y^{(i)}(b) = \beta_j, \quad i = n,\ (n+1),\ (n+2), ..., (k-1) \tag{3}$$

where, P_i $(i \geq 0)$ are constants, $K(x,t)$ and $f(x)$ are given smooth functions in $[a,b]$, $y^{(i)}(x)$ denotes the ith derivative of $y(x)$, α_j $(0 \leq j \leq n-1)$ and β_j $(n \leq j \leq k-1)$ are real finite constants and $y(x)$ is the unknown function to be determined.

To illustrate the basic concept of the Integrated Simpson's Collocation Method (ISCM), the following general nonlinear system is considered:

$$L[y(x)] + N[y(x)] + M[y(x)] = f(x) \tag{4}$$

where, L, M are linear operators, N is the nonlinear operator, which is assumed to be analytic and $f(x)$ is a given smooth function.

For nonlinear problem, the Taylor's series linearization scheme is applied to obtain a linear approximation at $t_0 = 0$ [21].

2.1 Integrated Simpson's Collocation Method by Power Series (ISCMPS)

In order to apply this method to solve (1), (2) and (3), the following power series approximation is assumed as the solution.

$$y^{iv}(x) = \sum_{j=o}^{N} a_j x^j \tag{5}$$

and

$$y'''(x) = \int \sum_{j=o}^{N} a_j x^j dx + c_1 \tag{6}$$

$$y''(x) = \int \int \sum_{j=o}^{N} a_j x^j dxdx + c_1 x + c_2 \tag{7}$$

$$y'(x) = \int \int \int \sum_{j=o}^{N} a_j x^j dxdxdx + \frac{1}{2} c_1 x^2 + c_2 x + c_3 \tag{8}$$

Obtaining $y(x)$ from (8), the unknown function is re-written as

$$y(x) = \int y'(x)dx + c_4 \tag{9}$$

where, c_4 is the constant of integration.

The Simpson's rule is applied to evaluate the integral part of (9), to obtain

$$\begin{aligned} y_{N,n}(x) = \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + ... + f(x_{2n-1})) \\ +2(f(x_2) + f(x_4) + ... + f(x_{2n-2})) + f(x_{2n})] + c_4 \end{aligned} \tag{10}$$

where, $x_0 = 0$, $x_n = x_0 + nh$ such that $(n \geq 2)$ is even, $f(x_n) = y'(x_n)$ and $h = \frac{b-a}{n}$.
Equation (1) is re-written as

$$P_0 y(x) + P_1 y'(x) + P_2 y''(x) + ... + P_k y^{(k)}(x) + \int_a^x K(x,t) y_{N,n}(t)dt = f(x) \tag{11}$$

Thus, (5) - (10) are substituted into (11), by selecting $k = 4$ to obtain

$$P_0 y_{N,n}(x)+P_1[\int\int\int\sum_{j=o}^{N} a_j x^j dxdxdx+\frac{1}{2}c_1x^2+c_2x+c_3]+P_2[\int\int\sum_{j=o}^{N} a_j x^j dxdx+c_1x+c_2]$$

$$+P_3[\int\sum_{j=o}^{N} a_j x^j dx + c_1] + P_4\sum_{j=o}^{N} a_j x^j + \int_a^x K(x,t)y_{N,n}(t)dt = f(x) \tag{12}$$

Expanding and simplifying (12), leads to

$$\begin{aligned} &P_0[a_0W_0^{(4)}(x) + a_1W_1^{(4)}(x) + a_2W_2^{(4)}(x) + ... + a_NW_N^{(4)}(x)] \\ &+P_1[a_0W_0^{(3)}(x) + a_1W_1^{(3)}(x) + a_2W_2^{(3)}(x) + ... + a_NW_N^{(3)}(x)] \\ &+P_2[a_0W_0^{(2)}(x) + a_1W_1^{(2)}(x) + a_2W_2^{(2)}(x) + ... + a_NW_N^{(2)}(x)] \\ &+P_3[a_0W_0^{(1)}(x) + a_1W_1^{(1)}(x) + a_2W_2^{(1)}(x) + ... + a_NW_N^{(1)}(x)] \\ &+P_4[a_0W_0^{(0)}(x) + a_1W_1^{(0)}(x) + a_2W_2^{(0)}(x) + ... + a_NW_N^{(0)}(x)] \\ &+C(X) + G_1(X,t) = f(x) \end{aligned} \tag{13}$$

Here,

$$W_N^{(4)}(x)\int\int\int\int x^N dxdxdxdx \tag{14}$$

$$W_0^{(4)}(x)\int\int\int\int dxdxdxdx \tag{15}$$

$$G_1(X,t) = \int_a^x K(x,t)y_{N,n}(t)dt \tag{16}$$

and $C(X)$ is the sum of all the expressions containing $c_i \;:\; i = 1,\ 2,\ 3,\ 4$ and $P_i \;: i = 0,\ 1,\ 2,\ 3,\ 4$.

After evaluating the terms involving integrals in equation (13), and with further simplification, the left-over is then collocated at the point $x = x_k$, to obtain

$$\begin{aligned} &P_0[a_0W_0^{(4)}(x_k) + a_1W_1^{(4)}(x_k) + a_2W_2^{(4)}(x_k) + ... + a_NW_N^{(4)}(x_k)] \\ &+P_1[a_0W_0^{(3)}(x_k) + a_1W_1^{(3)}(x_k) + a_2W_2^{(3)}(x_k) + ... + a_NW_N^{(3)}(x_k)] \\ &+P_2[a_0W_0^{(2)}(x_k) + a_1W_1^{(2)}(x_k) + a_2W_2^{(2)}(x_k) + ... + a_NW_N^{(2)}(x_k)] \\ &+P_3[a_0W_0^{(1)}(x_k) + a_1W_1^{(1)}(x_k) + a_2W_2^{(1)}(x_k) + ... + a_NW_N^{(1)}(x_k)] \\ &+P_4[a_0W_0^{(0)}(x_k) + a_1W_1^{(0)}(x_k) + a_2W_2^{(0)}(x_k) + ... + a_NW_N^{(0)}(x_k)] \\ &+C(X_k) + G_1(X_k,t) = f(x_k) \end{aligned} \tag{17}$$

where,

$$x_k = a + \frac{(b-a)k}{N+2}, \;\; k = 1,\ 2,\ 3, ..., N+1 \tag{18}$$

Putting equation (18) into (17), leads to (N+1) algebraic equations with (N+5) unknown constants. Four extra equations are obtained using the initial/boundary conditions given in equations (2) and (3). Altogether, there are (N+5) algebraic equations with (N+5) unknown constants. This system of (N+5) algebraic linear equations is put in vector form as $A\underline{X} = \underline{b}$ and then solved using Gaussian 'Maple 18' software to obtain the unknown constants a_j $(j \geq 0)$ and c_i's. These values are then substituted into the assumed solution to obtain the approximate solution.

2.2 Integrated Simpson's Collocation Method by Chebyshev Polynomials (ISCMCP)

In order to apply this method to solve (1), (2) and (3), the Chebyshev polynomial approximation assumed is of the form

$$y^{iv}(x) = \sum_{j=o}^{N} a_j T_j(x) \tag{19}$$

where, $T_j(x)$ is the jth degree Chebyshev polynomial defined by

$$T_j(x) = Cos[jCos^{-1}(\frac{2x-a-b}{b-a})] : \; a \leq x \leq b, \; j \geq 0 \tag{20}$$

and this is satisfied by the recurrence relation

$$T_{j+1}(x) = 2(\frac{2x-a-b}{b-a})T_j(x) - T_{j-1}(x) : \; a \leq x \leq b, \; j \geq 1 \tag{21}$$

Equation (19) is integrated successively to obtain

$$y'''(x) = \int \sum_{j=o}^{N} a_j T_j(x) dx + c_1 \tag{22}$$

$$y''(x) = \int \int \sum_{j=o}^{N} a_j T_j(x) dx dx + c_1 x + c_2 \tag{23}$$

$$y'(x) = \int \int \int \sum_{j=o}^{N} a_j T_j(x) dx dx dx + \frac{1}{2} c_1 x^2 + c_2 x + c_3 \tag{24}$$

To obtain the unknown function $y(x)$ in its segmented form, $y_{N,j}(x)$, (24) is re-written as

$$y_{N,j}(x) = \int y'(x) dx + c_4 \tag{25}$$

where, c_4 is the constant of integration.

The integral part of (25) is evaluated by Simpson's rule, hence, (25) is written in the form

$$y_{N,j}(x) = \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + ... + f(x_{2n-1})) + 2(f(x_2) + f(x_4) + ... + f(x_{2n-2})) + f(x_{2n})] + c_4 \quad (26)$$

where, $x_0 = 0$, $x_j = x_0 + jh$ such that $(j \geq 2)$ is even, $f(x_j) = y'(x_j)$ and $h = \frac{b-a}{j}$.
Equations (19), (22), (23), (24) and (26) are substituted into (11), by selecting $k = 4$ to obtain

$$P_0 y_{N,j}(x) + P_1[\int\int\int\sum_{j=o}^{N} a_j T_j(x)dxdxdx + \frac{1}{2}c_1x^2 + c_2x + c_3] + P_2[\int\int\sum_{j=o}^{N} a_j T_j(x)dxdx + c_1x + c_2]$$

$$+P_3[\int\sum_{j=o}^{N} a_j T_j(x)dx + c_1] + P_4\sum_{j=o}^{N} a_j T_j(x) + \int_a^x K(x,t)y_{N,j}(t)dt = f(x) \quad (27)$$

Expanding and simplifying (27), leads to

$$P_0[a_0T_0^{(4)}(x) + a_1T_1^{(4)}(x) + a_2T_2^{(4)}(x) + ... + a_NT_N^{(4)}(x)]$$
$$+P_1[a_0T_0^{(3)}(x) + a_1T_1^{(3)}(x) + a_2T_2^{(3)}(x) + ... + a_NT_N^{(3)}(x)]$$
$$+P_2[a_0T_0^{(2)}(x) + a_1T_1^{(2)}(x) + a_2T_2^{(2)}(x) + ... + a_NT_N^{(2)}(x)]$$
$$+P_3[a_0T_0^{(1)}(x) + a_1T_1^{(1)}(x) + a_2T_2^{(1)}(x) + ... + a_NT_N^{(1)}(x)]$$
$$+P_4[a_0T_0^{(0)}(x) + a_1T_1^{(0)}(x) + a_2T_2^{(0)}(x) + ... + a_NT_N^{(0)}(x)]$$
$$+C(X_2) + G_2(X,t) = f(x) \quad (28)$$

Here,

$$T_N^{(4)}(x)\int\int\int\int x^N dxdxdxdx \quad (29)$$

$$T_0^{(4)}(x)\int\int\int\int dxdxdxdx \quad (30)$$

$$G_2(X,t) = \int_a^x K(x,t)y_{N,j}(t)dt \quad (31)$$

and $C(X_2)$ is the sum of all the expressions containing $c_i\ : i = 1,\ 2,\ 3,\ 4$ and $P_i\ : i = 0,\ 1,\ 2,\ 3,\ 4$.
After evaluating the terms involving integrals in (28), and with further simplification, the left-over is then collocated at the point $x = x_k$, to obtain

$$P_0[a_0T_0^{(4)}(x_k) + a_1T_1^{(4)}(x_k) + a_2T_2^{(4)}(x_k) + ... + a_NT_N^{(4)}(x_k)]$$

$$+P_1[a_0T_0^{(3)}(x_k) + a_1T_1^{(3)}(x_k) + a_2T_2^{(3)}(x_k) + ... + a_NT_N^{(3)}(x_k)]$$
$$+P_2[a_0T_0^{(2)}(x_k) + a_1T_1^{(2)}(x_k) + a_2T_2^{(2)}(x_k) + ... + a_NT_N^{(2)}(x_k)]$$
$$+P_3[a_0T_0^{(1)}(x_k) + a_1T_1^{(1)}(x_k) + a_2T_2^{(1)}(x_k) + ... + a_NT_N^{(1)}(x_k)]$$
$$+P_4[a_0T_0^{(0)}(x_k) + a_1T_1^{(0)}(x_k) + a_2T_2^{(0)}(x_k) + ... + a_NT_N^{(0)}(x_k)]$$
$$+C(X_{2k}) + G_2(X_k, t) = f(x_k) \tag{32}$$

where,

$$x_k = a + \frac{(b-a)k}{N+2}, \;\; k = 1,\ 2,\ 3, ..., N+1 \tag{33}$$

Putting equation (33) into (32), leads to (N+1) algebraic equations with (N+5) unknown constants. Four extra equations are obtained using the initial/boundary conditions given in equations (2) and (3). Altogether, there are (N+5) algebraic equations with (N+5) unknown constants. This system of (N+5) algebraic linear equations is put in vector form as $A\underline{X} = \underline{b}$ and then solved using Gaussian 'Maple 18' software to obtain the unknown constants a_j $(j \geq 0)$ and c_i's. These values are then substituted into the assumed solution to obtain the approximate solution.

3.0 ERROR ANALYSIS

For the integrated Simpson's method with two subintervals, the error in the method is given by

$$E^s_{N,n}(y_{N,j}) = \int_{x_0}^{x_{0+2h}} y_{N,j}(x)dx - \frac{h}{3}[y_{N,j}(x_0) + 4y_{N,j}(x_{0+h}) + y_{N,j}(x_{0+2h})]$$
$$= -\frac{h^5}{180} y^{(iv)}_{N,j}(c) \tag{34}$$

where, N is the degree of the approximating polynomial and $x_0 \leq c \leq x_{0+2h}$.
Simpson's $\frac{1}{3}$ rule is an extension of the trapezoidal rule where the integrand is approximated by a second order polynomial. Given

$$I^s_n(y_{N,j}) = \int_{x_0}^{x_n} y_{N,j}(x)dx \tag{35}$$

Equation (35) is evaluated as

$$I^s_n(y_{N,j}) = nh[y_{N,j}(x_0) + \frac{n}{2}\Delta y_{N,j}(x_0) + \frac{n(2n-3)}{12}\Delta^2 y_{N,j}(x_0) + \frac{n(n-2)^2}{12}\Delta^3 y_{N,j}(x_0) + ...] \tag{36}$$

The formula given by (36) is known as Newton-Cotes closed quadrature formula from which different integration formulae are derived by substituting $n = 1,\ 2,\ 3, ...$ The

formulae are termed closed because they make use of the end points of the interval of integration.
Therefore, substituting $n = 2$ in (36) and considering the curve $y = f(x)$ through the points (x_0, y_0), (x_1, y_1) and (x_2, y_2) as parabola, then (36) becomes

$$I_1^s(y_{N,j}) = \int_{x_0}^{x_2} y_{N,j}(x)dx = 2h[y_{N,j}(x_0) + \Delta y_{N,j}(x_0) + \frac{1}{6}\Delta^2 y_{N,j}(x_0)]$$

$$= \frac{h}{3}[y_{N,j}(x_0) + 4y_{N,j}(x_1) + y_{N,j}(x_2)] \quad (37)$$

Similarly,

$$I_2^s(y_{N,j}) = \int_{x_2}^{x_4} y_{N,j}(x)dx = \frac{h}{3}[y_{N,j}(x_2) + 4y_{N,j}(x_3) + y_{N,j}(x_4)] \quad (38)$$

$$I_3^s(y_{N,j}) = \int_{x_4}^{x_6} y_{N,j}(x)dx = \frac{h}{3}[y_{N,j}(x_4) + 4y_{N,j}(x_5) + y_{N,j}(x_6)] \quad (39)$$

and so on.

In general,

$$I_n^s(y_{N,j}) = \int_{x_{2n-2}}^{x_{2n}} y_{N,j}(x)dx = \frac{h}{3}[y_{N,j}(x_{2n-2}) + 4y_{N,j}(x_{2n-1}) + y_{N,j}(x_{2n})] \quad (40)$$

Adding the integrals (37) - (40), the general Simpson's rule is obtained as

$$I_n^s(y_{N,j}) = \int_{x_0}^{x_2} y_{N,j}(x)dx + \int_{x_2}^{x_4} y_{N,j}(x)dx + ... + \int_{x_{2n-2}}^{x_{2n}} y_{N,j}(x)dx \quad (41)$$

$$I_n^s(y_{N,j}) \approx \frac{h}{3}[y_{N,j}(x_0)+4y_{N,j}(x_1)+y_{N,j}(x_2)]+\frac{h}{3}[y_{N,j}(x_2)+4y_{N,j}(x_3)+y_{N,j}(x_4)]+...$$

$$+\frac{h}{3}[y_{N,j}(x_{2n-2}) + 4y_{N,j}(x_{2n-1}) + y_{N,j}(x_{2n})] \quad (42)$$

where, $nh = x_n - x_0$, $x_i = x_0 + ih$, $i = 0,\ 1,\ 2, ..., n.$
Then, the error

$$E_{N,n}^s(y_{N,j}) \equiv \int_{x_0}^{x_{2n}} y_{N,j}(x)dx - I_n^s(y_{N,j}) \quad (43)$$

can be obtained by adding together the errors over the n-subintervals. Since,

$$\int_{x_0}^{x_{2n}} y_{N,j}(x)dx - \frac{h}{3}[y_{N,j}(x_0) + 4y_{N,j}(x_{0+h}) + y_{N,j}(x_{0+2h})] = -\frac{h^5}{180}y_{N,j}^{(iv)}(c) \quad (44)$$

then on a general subinterval, $[x_{2i-2},\ x_{2i}]$ the error is

$$\int_{x_{2i-2}}^{x_{2i}} y_{N,j}(x)dx - \frac{h}{3}[y_{N,j}(x_{2i-2}) + 4y_{N,j}(x_{2i-1}) + y_{N,j}(x_{2i})] = -\frac{h^5}{180}y_{N,j}^{(iv)}(c_i) \quad (45)$$

with $x_{2i-2} \leq c_i \leq x_{2i}$
Then, the errors in all the n-subintervals put together, is given by

$$E^s_{N,n}(y_{N,j}) = -\frac{h^5}{180}y^{(iv)}_{N,j}(c_1) - \frac{h^5}{180}y^{(iv)}_{N,j}(c_2) - ... - \frac{h^5}{180}y^{(iv)}_{N,j}(c_n) \tag{46}$$

$$= -\frac{h^5}{180}[\frac{y^{(iv)}_{N,j}(c_1) + y^{(iv)}_{N,j}(c_2) + ... + y^{(iv)}_{N,j}(c_n)}{n}]$$

$$= -\frac{h^5 n}{180}\phi_n \tag{47}$$

where

$$\phi_n = \frac{y^{(iv)}_{N,j}(c_1) + y^{(iv)}_{N,j}(c_2) + ... + y^{(iv)}_{N,j}(c_n)}{n}$$

such that

$$\min_{x_{2n-2}\leq x\leq x_{2n}} y^{(iv)}_{N,j}(x) \leq \phi_n \leq \max_{x_{2n-2}\leq x\leq x_{2n}} y^{(iv)}_{N,j}(x) \tag{48}$$

with the assumption that $y^{(iv)}_{N,j}(x)$ is a continuous function, then there is a number G_n in $[x_{2n-2},\ x_{2n}]$ for which

$$y^{(iv)}_{N,j}(G_n) = \phi_n \tag{49}$$

Therefore,

$$E^s_{N,n}(y_{N,j}) = -\frac{h^5 n}{180}\phi_n = -\frac{h^5 n}{180}y^{(iv)}_{N,j}(G_n)$$

$$= -\frac{h^4(x_{2n-2} - x_{2n})}{180}y^{(iv)}_{N,j}(G_n) \tag{50}$$

since

$$h = \frac{x_{2n-2} - x_{2n}}{n}$$

3.1 An Error Estimate

Equation (46) is re-written as

$$E^s_{N,n}(y_{N,j}) = -\frac{h^4}{180}[y^{(iv)}_{N,j}(c_1)h + y^{(iv)}_{N,j}(c_2)h + ... + y^{(iv)}_{N,j}(c_n)h] \tag{51}$$

If

$$\lim_{n\longrightarrow\infty}[y^{(iv)}_{N,j}(c_1)h + y^{(iv)}_{N,j}(c_2)h + ... + y^{(iv)}_{N,j}(c_n)h] = \int_{x_{2n-2}}^{x_{2n}} y^{(iv)}_{N,j}(x)dx \tag{52}$$

Then

$$\int_{x_{2n-2}}^{x_{2n}} y^{(iv)}_{N,j}(x)dx = y'''_{N,j}(x_n) - y'''_{N,j}(x_{2n-2}) \tag{53}$$

where

$$y^{(iv)}_{N,j}(c_1)h + y^{(iv)}_{N,j}(c_2)h + ... + y^{(iv)}_{N,j}(c_n)h$$

is a Riemann sum for the integral given by (53).
Thus,

$$y^{(iv)}_{N,j}(c_1)h + y^{(iv)}_{N,j}(c_2)h + ... + y^{(iv)}_{N,j}(c_n)h \approx y'''_{N,j}(x_n) - y'''_{N,j}(x_{2n-2}) \quad (54)$$

for $n >>$ (for large values of n).
Substituting (54) into (51), the asymptotic error estimate in the method is given by

$$E^s_{N,n}(y_{N,j}) \approx -\frac{h^4}{180}[y'''_{N,j}(x_n) - y'''_{N,j}(x_{2n-2})] \equiv \bar{E}^s_{N,n}(y_{N,j}) \quad (55)$$

3.2 Numerical Examples
In this section, the Integrated Simpson's Collocation Method by the two bases functions considered in this paper is demonstrated on some linear and nonlinear fourth order Volterra Integro-Differential Equations (VIDEs).
Remark: The error used is defined as

$$\text{Error} = \max_{a \le x \le b} |y(x) - y_{N,j}(x)| \ ; \ N = 1,\ 2,\ 3, ...$$

Example 1: Consider the fourth order linear integro-differential equation

$$y^{iv} - y(x) + \int_0^x y(t)dt = x + (x+3)e^x \ \ : \ 0 \le x \le 1$$

subject to the boundary conditions $y(0) = 1$, $y''(0) = 2$, $y(1) = 1 + e$, $y''(1) = 3e$.
The exact solution of this problem is

$$y(x) = 1 + xe^x$$

[See Afrouzi *et al* (2011)]
Example 2: Consider the fourth order nonlinear integro-differential equation

$$y^{iv} - \int_0^x e^{-t}y^2(t)dt = 1 \ \ : \ 0 \le x \le 1$$

subject to the boundary conditions $y(0) = 1$, $y''(0) = 1$, $y(1) = e$, $y''(1) = e$.
The exact solution of this problem is

$$y(x) = e^x$$

[See Noor and Mohyud-Din (2008)]
Example 3: Consider the fourth order linear integro-differential equation

$$y^{iv}(x) - 3\int_0^x y^3(t)dt = e^{-x} + e^{-3x} - 1 \ \ : \ 0 \le x \le 1$$

subject to the initial conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 1$, $y'''(0) = -1$.
The exact solution of this problem is

$$y(x) = e^{-x}$$

[See Afrouzi *et al* (2011)]

TABLES OF RESULTS

Table 1: Numerical Results for Example 1

x	Exact Solution	Error by Afrouzi *et al* [MHPM (2011)]	Integrated Simpson's Collocation Method: $n = 4$			
			Power Series		Chebyshev Polynomials	
			N=4	Error	N=4	Error
0.0	1.00000	0.20000E-8	1.00000	0.00000	1.00000	0.00000
0.1	1.11053	1.69536E-2	1.11064	2.00E-5	1.11055	1.80E-5
0.2	1.24428	3.24091E-2	1.24476	4.00E-5	1.24432	3.80E-5
0.3	1.40496	4.50108E-2	1.40606	7.00E-5	1.40503	6.50E-5
0.4	1.59673	5.36085E-2	1.59869	8.00E-5	1.59680	7.00E-5
0.5	1.82436	5.73187E-2	1.82738	1.00E-4	1.82439	3.00E-5
0.6	2.09327	5.55948E-2	2.09748	1.10E-4	2.09330	2.90E-5
0.7	2.40963	4.83072E-2	2.41506	1.20E-4	2.40965	2.40E-5
0.8	2.78043	3.58331E-2	2.78701	1.10E-4	2.78045	2.10E-5
0.9	3.21364	1.91596E-2	3.22115	7.00E-5	3.21365	1.50E-5
1.0	3.71828	0.10000E-9	3.71836	0.00000	3.71828	0.00000

Table 2: Numerical Results for Example 2

x	Exact Solution	Noor and Mohyud-Din [VIM (2008)]	Integrated Simpson's Collocation Method: $n = 4$			
			Power Series		Chebyshev Polynomials	
			N=4	Error	N=4	Error
0.0	1.0000000	0.00000	1.0000000	0.00000	1.0000000	0.00000
0.1	1.1051709	1.27E-5	1.1051832	1.23E-5	1.1051825	1.16E-5
0.2	1.2214028	4.36E-5	1.2214346	3.18E-5	1.2214301	2.73E-5
0.3	1.3498588	8.17E-5	1.3499238	6.50E-5	1.3499130	5.42E-5
0.4	1.4918247	1.16E-4	1.4918566	3.19E-5	1.4918454	2.07E-5
0.5	1.6487213	1.38E-4	1.6488353	1.14E-4	1.6488233	1.02E-4
0.6	1.8221188	1.39E-4	1.8222398	1.21E-4	1.8222338	1.15E-4
0.7	2.0137527	1.17E-4	2.0137808	2.81E-5	2.0137657	1.30E-5
0.8	2.2255409	7.47E-5	2.2255941	5.32E-5	2.2255861	4.52E-5
0.9	2.4596031	2.59E-5	2.4596235	2.04E-5	2.4596231	2.00E-5
1.0	2.7182818	0.00000	2.7182818	0.00000	2.7182818	0.00000

Table 3a: Numerical Results for Example 3

x	Exact Solution	Error by Afrouzi *et al* [MHPM (2011)]	Integrated Simpson's Collocation Method: $n = 4$ by Power Series	
			N=4	Error
0.0	1.0000000000	0.000000000000	1.0000000000	0.0000000000000
0.04	0.9607894392	0.212300000E-4	0.9607894382	1.0000000872E-9
0.08	0.9231163464	0.169065600E-3	0.9231163323	1.3999999937E-8
0.12	0.8869204367	0.568149900E-3	0.8869203708	6.5900000012E-8
0.16	0.8521437890	0.134130310E-2	0.8521436030	1.8569999993E-7
0.20	0.8187307531	0.260984490E-2	0.8187303922	3.6090000000E-7
0.24	0.7866278611	0.449388590E-2	0.7866274493	4.1180000010E-7
0.28	0.7557837415	0.711259400E-2	0.7557840126	2.7119999990E-7
0.32	0.7261490371	0.105844384E-1	0.7261522600	3.2230000000E-6
0.36	0.6976763261	0.150274129E-1	0.6976880553	1.1729200000E-5

Table 3b: Numerical Results for Example 3

x	Exact Solution	Error by Afrouzi *et al* [MHPM (2011)]	Integrated Simpson's Collocation Method: $n = 4$ by Chebyshev Polynomials N=4	Error
0.0	1.0000000000	0.000000000000	1.0000000000	0.0000000000000
0.04	0.9607894392	0.212300000E-4	0.9607894391	1.0000000827E-10
0.08	0.9231163464	0.169065600E-3	0.9231163452	1.1999999883E-9
0.12	0.8869204367	0.568149900E-3	0.8869204345	2.1999996000E-9
0.16	0.8521437890	0.134130310E-2	0.8521437859	3.1000000000E-9
0.20	0.8187307531	0.260984490E-2	0.8187307491	3.9999999979E-9
0.24	0.7866278611	0.449388590E-2	0.7866277592	1.0118999999E-7
0.28	0.7557837415	0.711259400E-2	0.7557839416	2.0009999990E-7
0.32	0.7261490371	0.105844384E-1	0.7261496182	5.8110000001E-7
0.36	0.6976763261	0.150274129E-1	0.6976851243	8.7981999999E-6

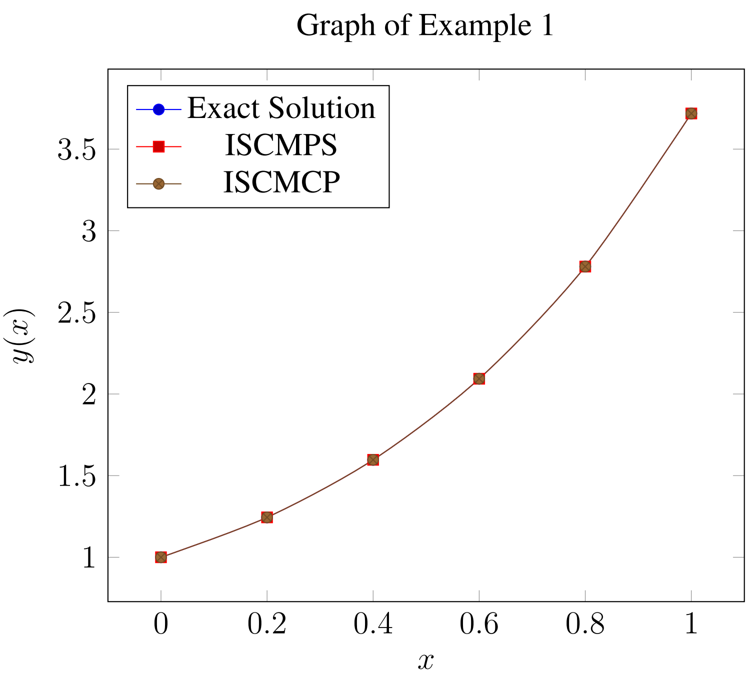

Figure 1: The behaviour of the exact solution compared with the solutions using power series and Chebyshev polynomials as bases functions.

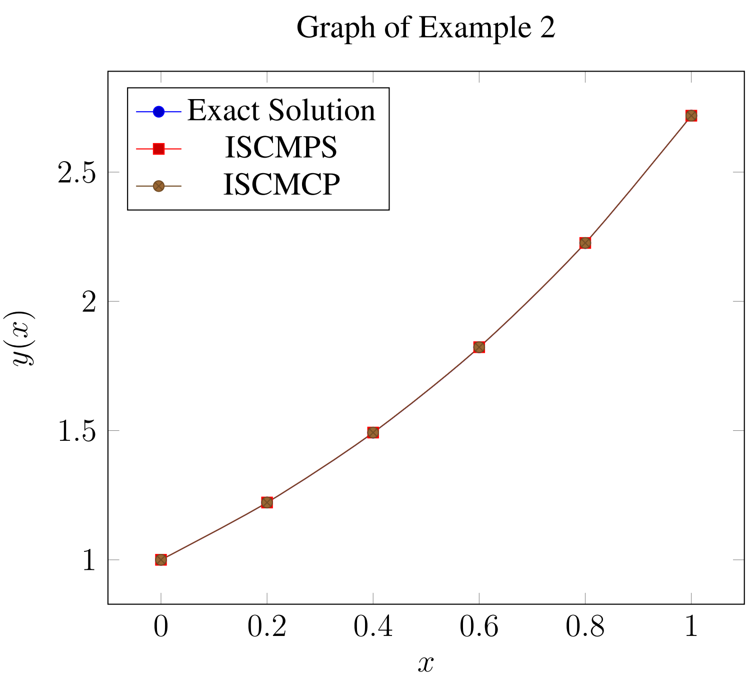

Figure 2: The behaviour of the exact solution compared with the solutions using power series and Chebyshev polynomials as bases functions.

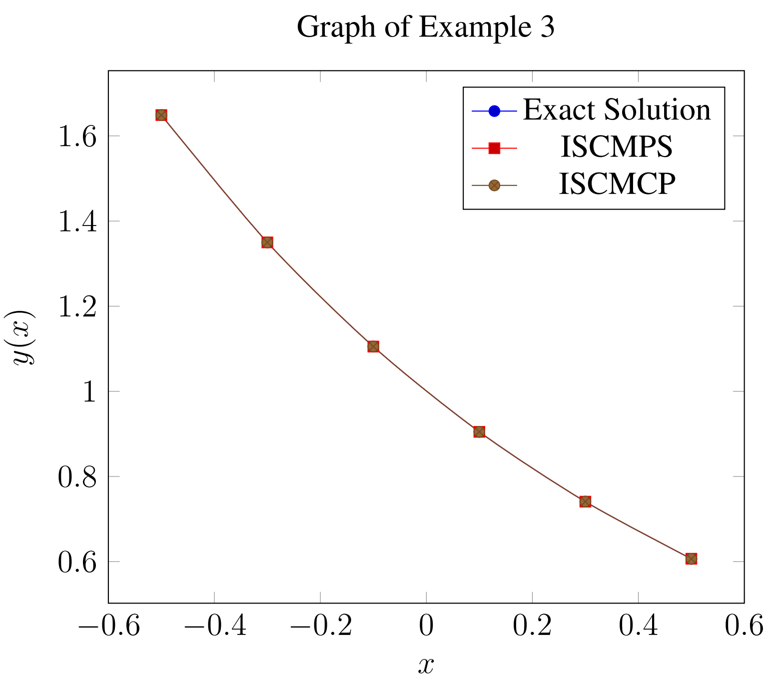

Figure 3: The behaviour of the exact solution compared with the solutions using power series and Chebyshev polynomials as bases functions.

4.0 DISCUSSION OF RESULTS AND CONCLUSION

In this paper, Integrated Simpson's Collocation Method is successfully implemented for the numerical solution of fourth order linear and nonlinear Volterra Integro-Differential Equations. Three test problems are carried out using the two bases functions considered in order to validate and demonstrate the efficiency and accuracy of the method. The results obtained show that the proposed method compared excellently well with the exact solution, where available as shown in tables 1-3. Also, the method performed better than some of the methods existing in literature. Furthermore, the Chebyshev polynomials as basis function produce better results in terms of the absolute errors compared with the power series as basis function. It is recommended that the Integrated Simpson's Collocation Method be applied for the numerical solution of other classes of differential equations.

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