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Bessel Function with Linear Differential Operator

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Abstract

In this paper, a class of analytic functions f defined on the open unit disc satisfying

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right)\right\}\right|^2+\beta>\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right|^2$$

is studied, among other results, inclusion relations and applications involving a certain class of integral operator are also considered.

Keywords: Analytic function, univalent function, starlike function, convex function, subordination, bessel function

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1 INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

in the open unit disc $U = \{z : z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting

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of univalent functions in U. We say that the function f is convex when $f(U)$ is a convex set. Also, we say that a function f is starlike with respect to the origin when $f(U)$ is a starlike set with respect to 0. By K or S^* we denote the subclasses of A consisting of all functions which are convex or starlike respectively, while by $\mathsf{S}^*(\delta)$ we denote the class of starlike functions of order $\delta, \delta \in [0,1)$.

In 1991, Goodman [7] introduced the class UCV of uniformly convex functions. A function $f \in \mathsf{CV}$ is the class UCV if for every circular are $\xi \subset U$, with center in U, the arc $f(\xi)$ is convex. A more useful characterization of class UCV was given by Ma and Minda [11] (see also [17]) as:

$$f \in \mathsf{UCV} \Leftrightarrow f \in \mathsf{A} \text{ and } Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in U).$$

In 1999, Kanas and Wisniowska [8] (see also [9]) introduced the class of k-uniformaly convex functions, $k \geq 0$, denoted by k-UCV and the class k-ST related to k-UCV by Alexandar type relation, i.e.,

$$f \in k-\mathsf{UCV} \Leftrightarrow zf' \in k-\mathsf{ST} \Leftrightarrow f \in \mathsf{A} \text{ and } Re\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in U).$$

In [8] and [9] respectively, their geometric definitions and connections with the conic domains were also considered. For a fixed $k \geq 0$, the class k-UCV is defined purely geometrically as a subclass of univalent functions which map the intersection of U with any disk centered at $\zeta, |\zeta| \leq k$, onto a convex domain. The notion of k-uniform convexity is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center ζ is the origin and the class k-UCV reduces to the class $k\mathsf{V}$. Moreover for $k = 1$ it coincides with the class of uniformly convex functions UCV introduced by Goodman [7] and studied extensively by R ø nning [17] and independently by Ma and Minda [11]. The class k-UCV started much earlier in papers [5, 6] with some additional conditions but without the geometric interpretation.

We say that a function $f \in \mathsf{A}$ is in the $\mathsf{S}^*_{k,\gamma}, k \geq 0, \gamma \in \mathsf{C} \setminus \{0\}$, if and only if

$$Re\left[1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right] > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right|, \quad (z \in U).$$

A lot of authors investigated the properties of the class $\mathcal{S}^*_{k,\gamma}$ and their generalizations in several directions, e.g., see [1, 2, 6, 9, 15, 17, 19]. An analytic function f is said to be subordinate to an analytic function g (written as $f \prec g$) if and only if there exists an analytic function ω with

$$\omega(0) = 0 \ \ and \ \ |\omega(z)| < 1 \ \ for \ \ z \in U$$

such that

$$f(z) = g(\omega(z)) \ for \ z \in U.$$

In particular, if g is univalent in U, we have the following equivalence

$$f \prec g \Leftrightarrow f(0) = g(0) \ \ and \ \ f(U) \subset g(U).$$

The convolution or Hadamard product of two functions of class $\mathcal{A}$ is denoted and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where f has the form (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad z \in U.$$

Let us consider the following second-order linear homogeneous differential equation (see for details [3] and [4]):

$$z^2\omega''(z) + bz\omega'(z) + \left[dz^2 - v^2 + (1-b)v\right]\omega(z) = 0 \quad (v, b, d \in \mathbb{C}). \tag{1.2}$$

The function $\omega_{v,b,d}(z)$, which is called the generalized Bessel function of the first kind of order v, it is defined as a particular solution of (1.2). The function $\omega_{v,b,d}(z)$ has the familiar representation as

$$\omega_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma\left(v + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{1.3}$$

Here Γ stands for the Euler gamma function. The series (1.3) permits the study of Bessel, modified Bessel, and spherical Bessel function altogether. It is worth mentioning that, in particular:

i) For $b = d = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order u

defined by

$$\mathsf{J}_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{1.4}$$

ii) For $b = 1$ and $d = -1$ in (1.3), we obtain the modified Bessel function of the first kind of order v defined by

$$\mathsf{I}_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{1.5}$$

iii) For $b = 2$ and $d = 1$ in (1.3), the function $\omega_{v,b,d}(z)$ reduces to $\frac{\sqrt{2}}{\sqrt{\pi}} \mathsf{S}_v(z)$, where S_v is the spherical Bessel function of the first kind of order v, defined by

$$\mathsf{S}_v(z) = \frac{\sqrt{\pi}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(v+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{1.6}$$

Now, consider the function $u_{v,b,d}(z) : \mathbb{C} \to \mathbb{C}$, defined by the transformation

$$u_{v,b,d}(z) = 2^v \Gamma\left(v + \frac{b+1}{2}\right) z^{\frac{-v}{2}} \omega_{v,b,d}(\sqrt{z}). \tag{1.7}$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler Γ function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

and $(\lambda)_0 = 1$, we obtain for the function $u_{v,b,d}(z)$ the following representation

$$u_{v,b,d}(z) = \sum_{n \geq 0} \frac{\left(\frac{-d}{4}\right)^n}{\left(v + \frac{b+1}{2}\right)_n} \frac{z^n}{n!},$$

where $k = v + \frac{b+1}{2} \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second order linear differential equation

$$4z^2u''(z)+2(2v+b+1)zu'(z)+dzu(z)=0.$$

Now, we introduce the function $\varphi_{v,b,d}(z)$ defined in terms of generalized Bessel function $\omega_{v,b,d}(z)$, defined by

$$\begin{aligned}\varphi_{v,b,d}(z) &= zu_{v,b,d}(z)\\ &= 2^v\Gamma\left(v+\frac{b+1}{2}\right)z^{1-\frac{v}{2}}\,\omega_{v,b,d}(\sqrt{z})\\ &= z+\sum_{n=1}^{\infty}\frac{(-d)^n z^{n+1}}{4^n n!(k)_n},\quad where\ k=\left(v+\frac{b+1}{2}\right)\\ &= g(k,d,z)\end{aligned}$$

Motivated by Selvaraj and Karthikeyan [14], we define the following $D_\lambda^m(k,d)f(z):U\to U$ by

$$D_\lambda(k,d)f(z) = f(z)*g(k,d,z) \tag{1.8}$$

$$D_\lambda^1(k,d)f(z)=(1-\lambda)\big(f(z)*g(k,d,z)\big)+\lambda z(f(z)*g(k,d,z))' \tag{1.9}$$

$$\vdots$$

$$D_\lambda^m(k,d)f(z) = D_\lambda^1\big(D_\lambda^{m-1}(k,d)f(z)\big) \tag{1.10}$$

If $f\in A$, then from (1.9) and (1.10) we may easily deduce that

$$D_\lambda^m(k,d)f(z) = z+\sum_{n=2}^{\infty}\frac{\big(1+(n-1)\lambda\big)^m(-d)^{n-1}a_n z^n}{4^{n-1}(n-1)!(k)_{n-1}} \tag{1.11}$$

where $m\in N_0 = N\cup\{0\}$ and $\lambda\ge 0$.

It can be easily verified from definition of (1.11) that

$$\lambda z\left(D_\lambda^m(k,d)f(z)\right)' = D_\lambda^{m+1}(k,d)f(z)-(1-\lambda)D_\lambda^m(k,d)f(z) \tag{1.12}$$

In the special cases of the $D_\lambda^m(k,d)f(z)$, we obtain the following operators related to the Bessel function:

i) Choosing $m=0$ in (1.11) we get the Deniz operator $B_k^c == z+\sum_{n=2}^{\infty}\frac{(-d)^{n-1}a_n z^n}{4^{n-1}(n-1)!(k)_{n-1}}$.

ii) Choosing $m=0$ and $b=d=1$ in (1.11) we obtain the operator $\mathrm{J}_v:\mathrm{A}\to\mathrm{A}$ related with Bessel function, defined by

$$\mathrm{J}_v f(z)=z+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}a_n z^n}{4^{n-1}(n-1)!(v+1)_{n-1}}.$$

iii) Choosing $m=0$, $b=1$ and $d=-1$ in (1.11) we obtain the operator $\mathrm{J}_v:\mathrm{A}\to\mathrm{A}$ related with modified Bessel function, defined by

$$\mathrm{J}_v f(z)=z+\sum_{n=2}^{\infty}\frac{(1)^{n-1}a_n z^n}{4^{n-1}(n-1)!(v+1)_{n-1}}.$$

iv) Choosing $m=0$, $b=2$ and $d=1$ in (1.11) we obtain the operator $\mathrm{S}_v:\mathrm{A}\to\mathrm{A}$ related with spherical Bessel function, defined by

$$\mathrm{S}_v f(z)=z+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}a_n z^n}{4^{n-1}(n-1)!\left(v+\frac{3}{2}\right)_{n-1}}.$$

Definition 1.1 *A function* $f(z)\in\mathrm{A}$ *is in the class* $\mathrm{S}^*(\alpha,\beta,\gamma)$, $\gamma\in\mathrm{C}\backslash\{0\}$ *if and only if*

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right)\right\}\right|^2+\beta>\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right|^2. \qquad (1.13)$$

Remark 1 *The above differential inequality can be equivalently written as*

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec e^{-i\alpha}\big(h(z)\cos\alpha+i\sin\alpha\big)=K_\alpha(z),$$

where $h(z)=1-\frac{\beta}{\cos^2\alpha}+\frac{2}{\pi}\left(log\frac{1+\sqrt{\frac{(z+\theta)}{(1+\theta z)}}}{1-\sqrt{\frac{(z+\theta)}{(1+\theta z)}}}\right)^2$, $\theta=\left(\frac{e^\mu-1}{e^\mu+1}\right)^2$ and $\mu=\frac{\sqrt{\beta}\,\pi}{2\cos\alpha}$.

Definition 1.2 *A function* $f(z)\in \mathrm{A}$ *is in the class* $Q_\lambda^m\big(k,d,\alpha,\beta,\gamma\big)$,*if and only if*

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\big(\mathrm{J}(\lambda,m,k,d,z)-1\big)\right)\right\}\right|^2+\beta>\left|\frac{1}{\gamma}\big(\mathrm{J}(\lambda,m,k,d,z)-1\big)\right|^2,$$

where

$$\mathrm{J}(\lambda,m,k,d,z)=\frac{z\left(D_\lambda^m(k,d)f(z)\right)'}{D_\lambda^m(k,d)f(z)}. \tag{1.14}$$

Since $K_\alpha(z)=e^{i\alpha}\big(h(z)\cos\alpha+i\sin\alpha\big)$ is a convex univalent function, we can write Definition 1.1 in subordination form

$$f\in Q_\lambda^m\big(k,d,\alpha,\beta,\gamma\big)\Leftrightarrow f\in \mathrm{A} \ \textit{and}\ \mathrm{J}\big(\lambda,m,k,d,z\big)\prec K_\alpha(z)\quad (z\in U).$$

Special Cases

For $\lambda=0$ and $f\in Q_\lambda^m\big(k,d,\alpha,\beta,\gamma\big)$ we have $D_\lambda^m(k,d)f(z)\in \mathrm{S}^*\big(\alpha,\beta,\gamma\big)$.

2 PRELIMINARY RESULTS

Lemma 1 *[13] Let* h *be a convex univalent function in* U *with* $Re\big(\lambda h(z)+\mu\big)>0$, *where* $\mu\in\mathrm{C},\lambda\in\mathrm{C}\setminus\{0\},z\in U$. *If* p *is analytic in* U *with* $p(0)=h(0)$, *then*

$$p(z)+\frac{zp'(z)}{\lambda p(z)+\mu}\prec h(z).$$

Lemma 2 *[12] Let* d *be a complex number with* $Re\, d > 0$*. Suppose that* $\Psi : \mathbb{C}^2 \to \mathbb{C}$ *is continuous and satisfies the conditions* $Re\, \Psi(ix, y) \le 0$*, when* x *is real and*

$$y \le (-|d - ix|^2)/(2 Re\, d).$$

If p is analytic in U with $p(0) = d$ and $Re\left[\Psi\left(p(z), zp'(z)\right)\right] > 0$ for $z \in U$, then $Re\ p(z) > 0$ in U.

Lemma 3 *[18] Let* f *and* g *be in the class* $k\mathcal{V}$ *and* $\mathcal{S}^*$ *respectively. Then, for every function* F *analytic in* U*, we have*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{co}[F(U)],\ z \in U,$$

where $\overline{co}[F(U)]$, denotes the closed convex hull of the set $F(U)$.

Lemma 4 *[16] Let the function* $\phi(z)$ *given by*

$$\phi(z) = \sum_{n=1}^{\infty} B_n z^n$$

be convex in U. Suppose also that the function $h(z)$ given by

$$h(z) = \sum_{n=1}^{\infty} h_n z^n$$

is holomorphic in U. If $h(z) \prec \phi(z)$, $z \in U$, then $|h_n| \le |B_1|$, $n \in N = \{1, 2, 3, \cdots\}$.

3 COEFFICIENT INEQUALITIES

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let $\sum$ denote the class of bi-univalent functions defined in the unit disk U.

Definition 3.1 *Let* $h : U \to \mathbb{C}$ *be a convex univalent function such that* $h(0) = 1$ *and* $h(\bar{z}) = \overline{h(z)}$*, for* $z \in U$ *and* $Re(h(z)) > 0$*. A function* $f \in \sum$ *is said to be in the class*

$S^*_{\Sigma}(\alpha,\gamma), \gamma \in \mathbb{C}\setminus\{0\}$ *if the following conditions are satisfied:*

$$f \in \Sigma, \quad e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right) \prec h(z)\cos\alpha + i\sin\alpha, \quad z \in U \tag{3.1}$$

and

$$e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{zg'(\omega)}{g(\omega)}-1\right)\right) \prec h(\omega)\cos\alpha + i\sin\alpha, \quad \omega \in U, \tag{3.2}$$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots, \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Definition 3.2 *A function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $S^{\lambda,m}_{\Sigma}(\alpha,\gamma,k,d,h)$ *if the following conditions are satisfied:*

$$e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{z\left(D_\lambda^m(k,d)f(z)\right)'}{D_\lambda^m(k,d)f(z)}-1\right)\right) \prec h(z)\cos\alpha + i\sin\alpha, \quad z \in U \tag{3.3}$$

and

$$e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{\omega\left(D_\lambda^m(k,d)g(\omega)\right)'}{D_\lambda^m(k,d)g(\omega)}-1\right)\right) \prec h(\omega)\cos\alpha + i\sin\alpha, \quad \omega \in U, \tag{3.4}$$

where $g(\omega) = f^{-1}(\omega), \gamma \in \mathbb{C}\setminus\{0\}, 0 \le \lambda \le 1, z, \omega \in U$ on specializing the parameter λ
.

Theorem 1 *Let* f *given by (1.1) be in the class* $S^*_{\Sigma}(\alpha,\gamma)$, *then*

$$|a_2| \le \sqrt{|\gamma||B_1|\cos\alpha}$$

and

$$|a_3| \le 4|\gamma||B_1|\cos\alpha.$$

Theorem 2 *Let f given by (1.1) be in the class* $\mathcal{S}_{\Sigma}^{\lambda,m}(\alpha,\gamma,k,d,h)$, *then*

$$|a_2| \leq \sqrt{\frac{2|\gamma||B_1|\cos\alpha}{\frac{(1+2\lambda)^m d^2}{8(k)_2} - \frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}}}$$

and

$$|a_3| \leq \frac{2|\gamma|^2|B_1|\cos\alpha}{\frac{(1+2\lambda)^m d^2}{8(k)_2} - \frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}}.$$

Proof. It follows from (3.3) and (3.4) that

$$e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{z\left(D_{\lambda}^{m}(k,d)f(z)\right)'}{D_{\lambda}^{m}(k,d)f(z)}-1\right)\right)=p(z)\cos\alpha+i\sin\alpha, \quad z\in U \tag{3.5}$$

and

$$e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{\omega\left(D_{\lambda}^{m}(k,d)g(\omega)\right)'}{D_{\lambda}^{m}(k,d)g(\omega)}-1\right)\right)=q(\omega)\cos\alpha+i\sin\alpha, \quad \omega\in U, \tag{3.6}$$

where $p(z) \prec h(z)$ and $q(\omega) \prec h(\omega)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + \cdots$$

respectively. It follows from (3.5) and (3.6) that

$$\frac{e^{i\alpha}}{\gamma}\frac{(1+\lambda)^m(-d)}{4(1)!(k)_1}a_2 = p_1\cos\alpha \tag{3.7}$$

$$\frac{e^{i\alpha}}{\gamma}\left(\frac{(2)(1+2\lambda)^m d^2}{4^2(2)!(k)_2}a_3 - \left(\frac{(1+\lambda)^m(-d)}{4(1)!(k)_1}\right)^2 a_2^2\right) = p_2\cos\alpha \tag{3.8}$$

$$\frac{-e^{i\alpha}}{\gamma}\frac{(1+\lambda)^m(-d)}{4(1)!(k)_1}a_2 = q_1\cos\alpha \tag{3.9}$$

and

$$\frac{e^{i\alpha}}{\gamma}\left(\frac{(4)(1+2\lambda)^m d^2}{4^2(2)!(k)_2}a_2^2-\frac{(2)(1+2\lambda)^m d^2}{4^2(2)!(k)_2}a_3-\left(\frac{(1+\lambda)^m(-d)}{4(1)!(k)_1}\right)^2 a_2^2\right)=q_2\cos\alpha. \quad (3.10)$$

From (3.7) and (3.9), we obtain

$$p_1=-q_1.$$

Adding (3.8) and (3.10), we get

$$\frac{e^{i\alpha}}{\gamma}\left(\frac{(1+2\lambda)^m d^2}{8(k)_2}-\frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}\right)a_2^2=(p_2+q_2)\cos\alpha. \quad (3.11)$$

Since $p,q\in h(U)$, applying Lemma 4, we have

$$|p_m|=\left|\frac{p^{(m)}(0)}{m!}\right|\le|B_1|,\quad m\in N \quad (3.12)$$

$$|q_m|=\left|\frac{q^{(m)}(0)}{m!}\right|\le|B_1|,\quad m\in N. \quad (3.13)$$

Applying (3.12), (3.13) and Lemma 4 for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_2|\le\sqrt{\frac{2|\gamma|\,|B_1|\cos\alpha}{\frac{(1+2\lambda)^m d^2}{8(k)_2}-\frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}}}. \quad (3.14)$$

Subtracting (3.10) from (3.8), we get

$$\frac{e^{i\alpha}}{\gamma}\left(\frac{(1+2\lambda)^m d^2}{8(k)_2}a_3-\frac{(1+2\lambda)^m d^2}{8(k)_2}a_2^2\right)=(p_2-q_2)\cos\alpha \quad (3.15)$$

or, equivalently

$$a_3=\left(\frac{\gamma}{e^{i\alpha}}\right)\left(\frac{8(k)_2(p_2-q_2)\cos\alpha}{(1+2\lambda)^m d^2}\right)+\left(\frac{\gamma}{e^{i\alpha}}\right)^2\left(\frac{(p_2+q_2)\cos\alpha}{\frac{(1+2\lambda)^m d^2}{8(k)_2}-\frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}}\right).$$

Applying (3.12), (3.13) and Lemma 4 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{2|\gamma|^2|B_1|\cos\alpha}{\frac{(1+2\lambda)^m d^2}{8(k)_2} - \frac{(1+\lambda)^{2m} d^2}{8(k)_1^2}}.$$

This complete the proof of Theorem 2.

Corollary 1 *Let* $f \in \mathcal{A}$ *be bi convex function of order* β *then* $|a_2| \leq \sqrt{1-\beta}$ *and* $|a_3| \leq 1-\beta$.

Corollary 2 *Let* $f \in \mathcal{A}$ *satisfy the condition* $1+\frac{zf''(z)}{f'(z)} \prec h(z)$ *and* $1+\frac{zg''(w)}{g'(w)} \prec h(w)$ *then* $|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2+2B_1-2B_2|}}$ *and* $|a_3| \leq \frac{1}{2}(B_1+|B2-B1|)$.

Corollary 3 *Let* f *be given by (1.1) and* $g = f^{-1}$. *If* f *and* g *satisfies the condition* $(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right) \prec h(z)$

and $(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right) \prec g(w)$,

then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+\lambda)|B_1^2+(1+\lambda)(B_1-B_2)|}} \tag{3.16}$$

and

$$|a_3| \leq \frac{B_1+|B2-B1|}{1+\lambda}. \tag{3.17}$$

Remark 2 *Put* $\lambda = 1$ *in corollary 3, we get the result* $|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}}$ *and* $|a_3| \le \frac{1}{2}(B_1 + |B2 - B1|)$ *of corollary 2.*

4 CLOSURE PROPERTY

Theorem 3 *Let*

$$F(\lambda, m, k, d, f(z)) = D_\lambda^m(k,d) f(z). \tag{4.1}$$

Then $f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$ if and only if $F(\lambda,m,k,d,f(z)) \in S^*(\alpha,\beta,\gamma)$.

Proof. Let $F(\lambda,m,k,d,f(z)) \in S^*(\alpha,\beta,\gamma)$, then

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{zF'(\lambda,m,k,d,f(z))(z)}{F(\lambda,m,k,d,f(z))}-1\right)\right)\right\}\right|^2 + \beta > \left|\frac{1}{\gamma}\left(\frac{zF'(\lambda,m,k,d,f(z))(z)}{F(\lambda,m,k,d,f(z))}-1\right)\right|^2. \tag{4.2}$$

Thus (4.1) together with (4.2) implies

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\left(J(\lambda,m,k,d,z)-1\right)\right)\right\}\right|^2 + \beta > \left|\frac{1}{\gamma}\left(J(\lambda,m,k,d,z)-1\right)\right|^2,$$

where $J(\lambda,m,k,d,z)$ is given by (1.14). Therefore $f(z) \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$. Converse is immediate.

Theorem 4 *For* $m \ge 1$, $Q_\lambda^{m+1}(k,d,\alpha,\beta,\gamma) \subset Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

Proof. Let $f \in Q_\lambda^{m+1}(k,d,\alpha,\beta,\gamma)$ and

$$\frac{z\left(D_\lambda^m(k,d) f(z)\right)'}{D_\lambda^m(k,d) f(z)} = p(z), \tag{4.3}$$

where p is analytic in U and $p(0) = 1$.

From (1.12) and (4.3) and after some simplification, we obtain

$$\frac{D_\lambda^{m+1}(k,d)f(z)}{D_\lambda^m(k,d)f(z)} = \lambda p(z) + (1-\lambda). \tag{4.4}$$

By logarithmic differentiation of (4.4), we have

$$\frac{z\left(D_\lambda^{m+1}(k,d)f(z)\right)'}{D_\lambda^{m+1}(k,d)f(z)} = p(z) + \frac{\lambda z p'(z)}{(1-\lambda)+\lambda p(z)}. \tag{4.5}$$

Since $f \in Q_\lambda^{m+1}(k,d,\alpha,\beta,\gamma)$, so

$$p(z) + \frac{\lambda z p'(z)}{(1-\lambda)+\lambda p(z)} \prec K_\alpha(z).$$

Thus by using Lemma 1, $p(z) \prec K_\alpha(z)$ and hence $f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

Let us consider the Bernardi integral operator F_μ given by

$$F_\mu f(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt. \tag{4.6}$$

For μ with $Re\,\mu > -1$ the operator has the property $F_\mu : \mathcal{A} \to \mathcal{A}$, (see, for instance, [10], p.11).

Theorem 5 *Let* $\mu > -1$ *and* $f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$, *then* $F_\mu f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

Proof. Let $f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$ and

$$\frac{z\left(D_\lambda^m(k,d)F_\mu(f)(z)\right)'}{D_\lambda^m(k,d)F_\mu(f)(z)} = p(z), \tag{4.7}$$

where the function $p(z)$ is analytic in U and $p(0) = 1$. From (4.6), we have

$$z\left(F_\mu(f)\right)'(z) + \mu F_\mu(f)(z) = (\mu+1)f(z)$$

and so

$$z\left(D_\lambda^m(k,d)F_\mu(f)(z)\right)' = (\mu+1)(D_\lambda^m(k,d)f(z) - \mu\left(D_\lambda^m(k,d)F_\mu(f)(z)\right). \tag{4.8}$$

Then, by using equations (4.7) and (4.8), we obtain

$$\frac{(\mu+1)(D_\lambda^m(k,d)f(z)}{D_\lambda^m(k,d)F_\mu(f)(z)} = p(z)+\mu \qquad (4.9)$$

By logarithmic differentiation of(4.9), we have

$$\frac{z\left(D_\lambda^m(k,d)f(z)\right)'}{D_\lambda^m(k,d)f(z)} = p(z)+\frac{zp'}{p(z)+\mu}, \ (z\in\mu).$$

Hence by Lemma1, we conclude that $p(z)\prec K_\alpha(z)$ in U which implies that $F_\mu f\in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

Theorem 6 *Let* $f\in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$ *and* $\psi\in k\mathrm{V}$. *If* $0<\gamma\le 1$, *then* $\psi * f\in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

Proof. Let $F=\psi * f$. If $f\in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$ then the condition (4.3) is satisfied with $p\prec K_\alpha(z)$. Using the usual convolution properties and (4.3),

$$p(z)=\frac{z\left(D_\lambda^m(k,d)F(z)\right)'}{D_\lambda^m(k,d)F(z)}$$

$$=\frac{z\left(D_\lambda^m(k,d)(\psi * f)\right)'}{D_\lambda^m(k,d)(\psi * f)}$$

$$=\frac{z\left(\psi * D_\lambda^m(k,d)f\right)'}{\psi * D_\lambda^m(k,d)f}$$

$$=\frac{\psi * z\left(D_\lambda^m(k,d)f\right)'}{\psi * D_\lambda^m(k,d)f}$$

$$=\frac{\psi *\left(P(z)F(z)\right)}{\psi * F(z)}. \qquad (4.10)$$

By Theorem(3), the function $F(z) = D_\lambda^m(k,d) f(z) \in S^*(\alpha,\beta,\gamma)$.

Hence, by Lemma (3) , we have

$$\frac{z\left(D_\lambda^m(k,d)F(z)\right)'}{D_\lambda^m(k,d)F(z)} \in \overline{co}[F(U)] \subset K_\alpha(z).$$

Since $K_\alpha(z)$ is a convex univalent and $p(z) \prec K_\alpha(z)$.

Hence $F = \psi * f \in Q_\lambda^m(k,d,\alpha,\beta,\gamma)$.

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