Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 4 (2020), pp. 613-624

http://www.ripublication.com/gjpam.htm

Blow-up Phenomena for a Quasilinear Parabolic Equation with Nonlocal Boundary Conditions *

Xumin Li[1 †] **Zejian Cui**[2]

[1, 2] *College of Mathematic and Information,*
China West Normal University, NanChong, 637009 ,China.

Abstract

In this paper, we devote to investigating the blow-up of solutions of a quasilinear parabolic equation with non-local boundary conditions. By constructing appropriate auxiliary function, combined with the improved differential inequality technique, we establish conditions to guarantee the solution blows up in a finite time.Moreover, upper and lower bounds estimation of the blow-up time are derived.

Keywords: quasilinear parabolic equation, blow up,nonlocal boundary conditions,upper and lower bounds

1. INTRODUCTION

The blow-up phenomena of the solutions for nonlinear parabolic problems have aroused widespread concern of researchers.Recently,a vast literature on the study of blow-up phenomena and the bounds for the blow-up time of parabolic equations and system(see [1-10]).As we all know,the solution of the parabolic problems may remain bounded or blow up at finite time.When blow up occurs,the blow-up time cannot be determined explicitly.So,it's essential for us to do estimation about the upper and lower bounds of the blow-up time.In this paper,we consider the following quasilinear parabolic equation with nonlocal boundary conditions.

*Project Name:The Natural Science Fund of Department of Education of Sichuan Province(09ZA119)
†Corresponding author's e-mall:2295810379@qq.com;cuizejian@126.com

$$\begin{cases}(g(u))_t = \nabla\cdot(\rho(u)\nabla u) + b(t)f(u), & in\ D\times(0,t^*),\\ \dfrac{\partial u}{\partial\nu} = k(t)\displaystyle\int_D h(u)dx, & on\ \partial D\times(0,t^*),\\ u(x,0) = u_0(x)\geq 0, & in\ \overline{D},\end{cases} \tag{1.1}$$

where D represents a bounded convex region in $R^n(n \geq 2)$ with smooth boundary ∂D,$\frac{\partial u}{\partial\nu}$ is the outward normal derivatives on ∂D,and $(0,t^*)$ is the maximum existence interval of time.We assume that $g'(s) > 0$ for all $s > 0$,suppose that f and h are nonnegative $C^1(\overline{R_+})$ functions,b,k are the positive bounded$C^1(\overline{R_+})$ functions,ρ is positive $C^2(\overline{R_+})$ functions,and $u_0(x)$ is nonnegative $C^1(\overline{D})$ functions which satisfy the compatibility condition and $u_0(x)\not\equiv 0$,$x\in\overline{D}$. It follows from the parabolic maximum principle[16]that the solution of(1.1)is nonnegative in $x\in\overline{D}$ and $t\in[0,t^*)$.

In order to finish our research,we focus on the following works.Marras and Vernier Piro [11]studied the following equations

$$\begin{cases}u_t = \Delta u + k_1(t)f(u), & in\ \Omega\times(0,t^*),\\ \dfrac{\partial u}{\partial\nu} = k_2(t)\displaystyle\int_\Omega g(u)dx, & on\ \partial\Omega\times(0,t^*),\\ u(x,0) = u_0(x)\geq 0, & in\ \overline{\Omega},\end{cases} \tag{1.2}$$

where Ω is bounded convex domain in $R^N(N \geq 2)$ with smooth boundary.The authors proved that under proper conditions on data,the blow-up occurs at time t^*,and when it does ,they obtained an upper bound for blow-up time.Moreover,lower bounds for blow-up time to the cases of $\Omega \subset R^3$ were also obtained,respectively.In[12],Juntang Ding dealt with

$$\begin{cases}(b(u))_t = \nabla\cdot(\rho(u)\nabla u) + f(u), & in\ D\times(0,t^*),\\ \dfrac{\partial u}{\partial\nu} + \gamma u = 0, & on\ \partial D\times(0,t^*),\\ u(x,0) = u_0(x)\geq 0, & in\ \overline{D},\end{cases} \tag{1.3}$$

where $D\subset R^N(N \geq 2)$ is bounded convex domain with smooth boundary conditions which guarantee that blow up does not occur are obtained.the author proved that under certain conditions on data,the solution blows up in finite time.they derived an upper and lower bound for blow-up time.In[13],Xuhui Shen and Juntang Ding studied

$$\begin{cases} u_t = \nabla \cdot (\rho(u)\nabla u) + k_1(t)f(u), & in\ D \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = k_2(t) \displaystyle\int_D g(u)dx, & on\ \partial D \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0, & in\ \overline{D}, \end{cases} \tag{1.4}$$

where D is a bounded convex region in $R^n(n \geq 2)$,and the boundary is smooth.They impose conditions on data to get an upper bound of blow-up time when blow-up occurs.Furthermore,when $D \subset R^n(n \geq 3)$,the lower bounds of t^* are obtained.

Inspired by the aforementioned three papers,we study the blow-up problem of(1.1).The main difficulty in studying(1.1)is to construct appropriate auxiliary functions.while auxiliary functions defined in [11-13]are not suitable for our study,we need to build new auxiliary functions and use Sobolev inequalities to complete our research.The paper is organized as follows.In section 2,we give conditions on the data of (1.1)to guarantee the blow-up of the solution and derive an upper bound for the blow-up time in $D \subset R^n(n \geq 2)$.In section 3,we devoted to get a lower bound for the blow-up time in $D \subset R^n(n \geq 3)$ when blow-up does occur.

2. AN UPPER BOUND FOR THE BLOW-UP TIME T^*

In order to obtain an upper bound of t^* in $D \subset R^n(n \geq 2)$, We assume that functions ρ, f, g, h and b, k satisfy

$$b(t)f(s) \geq b_1 s^p, h(s) \geq s^q, \rho(s) \geq c_1, k(t) \geq b_2, g'(s) \leq d_1, s \geq 0 \tag{2.1}$$

where b_1, b_2 and p, q, c_1, d_1 are positive constants with

$$p > 1, q > 1 \tag{2.2}$$

then the auxiliary function $\Phi(t)$ is defined as follow

$$\Phi(t) = \int_D \phi(u(x, t))dx, t \geq 0, \qquad \phi(s) = \int_0^s g'(y)dy, s > 0 \tag{2.3}$$

Theorem 2.1. Let $u(x, t)$ be a classical solution of problem (1.1).we assume that (2.1)-(2.2)hold.Then the solution $u(x, t)$ blows up at time t^* in the measure $\Phi(t)$.

Proof. Making use of the divergence theorem and (2.1)-(2.3),we obtain

$$\begin{aligned}\Phi'(t) &= \int_D g'(u)u_t dx = \int_D \nabla\cdot(\rho(u)\nabla u)dx + b(t)\int_D f(u)dx \\ &= \int_{\partial D}\rho(u)\frac{\partial u}{\partial \nu}ds + b(t)\int_D f(u)dx \\ &= k(t)\int_{\partial D}\rho(u)ds\int_D h(u)dx + b(t)\int_D f(u)dx \\ &\geq c_1 b_2|\partial D|\int_D u^q dx + b_1\int_D u^p dx\end{aligned} \tag{2.4}$$

We apply the *Hölder* inequality to get

$$\int_D udx \leq |D|^{\frac{q-1}{q}}\left(\int_D u^q dx\right)^{\frac{1}{q}}, \int_D udx \leq |D|^{\frac{p-1}{p}}\left(\int_D u^p dx\right)^{\frac{1}{p}} \tag{2.5}$$

and then we have

$$\int_D u^q dx \geq |D|^{1-q}(\int_D udx)^q, \int_D u^p dx \geq |D|^{1-p}(\int_D udx)^p \tag{2.6}$$

inserting(2.6)into(2.4),then we obtain

$$\Phi'(t) \geq c_1 b_2|\partial D||D|^{1-q}(\int_D udx)^q + b_1|D|^{1-p}(\int_D udx)^p \tag{2.7}$$

from (2.1)we have $\phi(u) \leq \int_0^u d_1 dy = d_1 u$,and then we obtain

$$\int_D udx \geq \frac{1}{d_1}\int_D \phi(u)dx = \frac{1}{d_1}\Phi(t) \tag{2.8}$$

so from (2.7)we have

$$\Phi'(t) \geq j_1(\Phi(t))^q + j_2(\Phi(t))^p, t \geq 0 \tag{2.9}$$

where $j_1 = c_1 b_2|\partial D||D|^{1-q}\frac{1}{d_1^q}, j_2 = b_1|D|^{1-p}\frac{1}{d_1^p}$
we integrate(2.9)over $[0,t]$ to obtain

$$t \leq \int_{\Phi(0)}^{\Phi(t)} \frac{d\eta}{j_1\eta^q + j_2\eta^p} \tag{2.10}$$

it is easy to find the solution must blow up at t^* in the measure $\Phi(t)$,passing to the limit as $t \to t^*$,inequality (2.10)becomes

$$t^* \leq \int_{\Phi(0)}^{+\infty} \frac{d\eta}{j_1\eta^q + j_2\eta^p} \tag{2.11}$$

3. A LOWER BOUND FOR THE BLOW-UP TIME T^*

In order to obtain a lower bound of t^* in this section, We restrict $D \subset R^n$ $(n \geq 3)$ and suppose that functions ρ, f, g, h, k, b and positive constants n satisfy

$$b(t)f(s) \leq b_1 s^p, h(s) \leq s^q, c_2 \leq \rho(s) \leq c_3, k(t) \leq b_2, g'(s) \geq d_2, s \geq 0, t \geq 0 \tag{3.1}$$

where b_1, b_2 and p, q, c_2, c_3, d_2 are positive constants with

$$p > 1, q > 1, \tag{3.2}$$

It follows from the reference[14] we obtain that

$$\left(\int_D \left(u^{\frac{n(p-1)}{2}}\right)^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2n}} \leq C\left(\int_D u^{n(p-1)}dx + \int_D |\nabla u^{\frac{n(p-1)}{2}}|^2 dx\right)^{\frac{1}{2}} \tag{3.3}$$

where $C = C(n, D)$ is a Sobolev embedding constant depending on n and D.In this section,we need to use the Sobolev inequality (3.3).We construct the following auxiliary function

$$\Psi(t) = \int_D \psi(u(x,t))dx, \quad \psi(s) = (m+1)\int_0^s g'(y)y^m dy, s \geq 0 \tag{3.4}$$

where

$$m > \max\{1, \frac{n(p-1)}{2} - 1, q-1\} \tag{3.5}$$

Theorem 3.1. Let $u(x,t)$ be a classical solution of problem (1.1).we assume that (3.1)-(3.2)hold and $u(x,t)$ become unbounded in the measure $\Psi(t)$ at some finite time t^*. Then t^* is bounded below by

$$t^* \geq \int_{\Psi(0)}^{+\infty} \frac{d\tau}{\kappa_1\tau^{\frac{m+q}{m+1}} + \kappa_2\tau^{\frac{m+2q-1}{m+1}} + \kappa_3\tau^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}} + \kappa_4\tau^{\frac{m+p}{m+1}}} \tag{3.6}$$

where

$$\kappa_1 = A_1 \frac{1}{d_2}^{\frac{m+q}{m+1}}, \kappa_2 = A_2 \frac{1}{d_2}^{\frac{m+2q-1}{m+1}}, \kappa_3 = A_3 \frac{1}{d_2}^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}, \kappa_4 = A_4 \frac{1}{d_2}^{\frac{m+p}{m+1}} \tag{3.7}$$

$$\begin{aligned} A_1 &= \frac{b_2 c_3 (m+1) N}{\rho_0} |D|^{\frac{m-q+2}{m+1}}, A_2 = \frac{m d_0 b_2 c_3}{\rho_0 \varepsilon_1} |D|^{\frac{m-q+2}{m+1}} \\ A_3 &= b_1 (m+1) (\sqrt{2} C_s)^{\frac{n(p-1)}{m+1}} \left(\frac{1}{\varepsilon_2}\right)^{\frac{n(p-1)}{2(m+1)-n(p-1)}} \left(1 - \frac{n(p-1)}{2(m+1)}\right) \\ A_4 &= b_1 (m+1) (\sqrt{2} C_s)^{\frac{n(p-1)}{m+1}} \end{aligned} \tag{3.8}$$

Proof. Making use of the divergence theorem and (3.1),we obtain

$$\begin{aligned} \Psi'(t) &= (m+1) \int_D u^m g'(u) u_t dx = (m+1) \int_D u^m (g(u))_t dx \\ &= (m+1) \int_D u^m \nabla \cdot (\rho(u) \nabla u) dx + b(t)(m+1) \int_D u^m f(u) dx \\ &= (m+1) \int_D \nabla \cdot (u^m \rho(u) \nabla u) dx - m(m+1) \int_D u^{m-1} \rho(u) |\nabla u|^2 dx + b(t)(m+1) \int_D u^m f(u) dx \\ &= (m+1) \int_{\partial D} u^m \rho(u) \frac{\partial u}{\partial \nu} ds - m(m+1) \int_D u^{m-1} \rho(u) |\nabla u|^2 dx + b(t)(m+1) \int_D u^m f(u) dx \\ &\le b_2 c_3 (m+1) \int_{\partial D} u^m ds \int_D u^q dx - c_2 m(m+1) \int_D u^{m-1} |\nabla u|^2 dx + b_1 (m+1) \int_D u^{m+p} dx \\ &= b_2 c_3 (m+1) \int_{\partial D} u^m ds \int_D u^q dx - \frac{4 m c_2}{m+1} \int_D |\nabla u^{\frac{m+1}{2}}|^2 dx + b_1 (m+1) \int_D u^{m+p} dx \end{aligned} \tag{3.9}$$

Firstly, for the first term of the right side of (3.9),we use the divergence theorem,and then obtain following inequality(see [15])

$$\int_{\partial D} u^m ds \le \frac{N}{\rho_0} \int_D u^m dx + \frac{m d_0}{\rho_0} \int_D u^{m-1} |\nabla u| dx \tag{3.10}$$

where $\rho_0 = \min_{x \in \partial D} (x \cdot \nu)$, $d_0 = \max_{x \in \overline{D}} |x|$. According to *Hölder* inequality we have

$$\int_D u^m dx \le |D|^{\frac{1}{m+1}} \left(\int_D u^{m+1} dx \right)^{\frac{m}{m+1}} \tag{3.11}$$

$$\begin{aligned}\int_D u^{m-1}|\nabla u|dx &\le \left(\int_D u^{m-1}dx\right)^{\frac{1}{2}}\left(\int_D u^{m-1}|\nabla u|^2dx\right)^{\frac{1}{2}}\\ &= \frac{2}{m+1}\left(\int_D u^{m-1}dx\right)^{\frac{1}{2}}\left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{1}{2}}\end{aligned} \tag{3.12}$$

By *Hölder* inequality we have

$$\int_D u^{m-1}dx \le |D|^{\frac{2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m-1}{m+1}} \tag{3.13}$$

and

$$\int_D u^{m-1}|\nabla u|dx \le \frac{2}{m+1}|D|^{\frac{1}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m-1}{2(m+1)}}\left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{1}{2}} \tag{3.14}$$

We substitute(3.11)-(3.14)into(3.10)to derive

$$\int_{\partial D} u^m ds \le \frac{N}{\rho_0}|D|^{\frac{1}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m}{m+1}} + \frac{2md_0}{\rho_0(m+1)}|D|^{\frac{1}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m-1}{2(m+1)}}\left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{1}{2}} \tag{3.15}$$

Apply the *Hölder* inequality again,by (3.5)we have

$$\int_D u^q dx \le |D|^{\frac{m+1-q}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{q}{m+1}} \tag{3.16}$$

By (3.15)and (3.16),we obtain

$$\begin{aligned}&\int_{\partial D} u^m ds\int_D u^q dx\\ &\le \frac{N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}} + \frac{2md_0}{\rho_0(m+1)}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{2(m+1)}}\left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{1}{2}}\\ &= \frac{N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}} + \frac{2md_0}{\rho_0(m+1)}|D|^{\frac{m-q+2}{m+1}}\left(\frac{1}{\varepsilon_1}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}}\right)^{\frac{1}{2}}\left(\varepsilon_1\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{1}{2}}\\ &\le \frac{N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}} + \frac{2md_0}{\rho_0(m+1)}|D|^{\frac{m-q+2}{m+1}}\left(\frac{1}{2\varepsilon_1}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}} + \frac{\varepsilon_1}{2}\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)\\ &= \frac{N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}} + \frac{md_0}{\varepsilon_1\rho_0(m+1)}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}}\\ &\qquad + \frac{\varepsilon_1 md_0}{\rho_0(m+1)}|D|^{\frac{m-q+2}{m+1}}\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\end{aligned} \tag{3.17}$$

Substitute (3.17)into(3.9),we derive

$$\begin{aligned}\Psi'(t) \le & \frac{b_2c_3(m+1)N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}} \\ & + \frac{md_0b_2c_3}{\rho_0\varepsilon_1}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}} \\ & + \frac{\varepsilon_1 md_0b_2c_3}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\int_D |\nabla u^{\frac{m+1}{2}}|^2dx - \frac{4mc_2}{m+1}\int_D |\nabla u^{\frac{m+1}{2}}|^2dx + b_1(m+1)\int_D u^{m+p}dx\end{aligned} \tag{3.18}$$

Secondly, we estimate the last term of the right side of(3.18).By (3.2),(3.5)and apply the *Hölder* inequality and sobolev inequality(3.3),we obtain

$$\begin{aligned}\int_D u^{m+p} & \le \left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left(\int_D \left(u^{\frac{m+1}{2}}\right)^{\frac{2n}{n-2}}dx\right)^{\frac{(n-2)(p-1)}{2(m+1)}} \\ & \le \left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left(C_s^{\frac{2n}{n-2}}\left(\int_D u^{m+1}dx + \int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n}{n-2}}\right)^{\frac{(n-2)(p-1)}{2(m+1)}} \\ & = C_s^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left(\int_D u^{m+1}dx + \int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n(p-1)}{2(m+1)}}\end{aligned} \tag{3.19}$$

using the inequality

$$(\alpha+\beta)^\mu \le 2^\mu(\alpha^\mu+\beta^\mu), \alpha > 0, \beta > 0, \mu > 0 \tag{3.20}$$

then we derive

$$\begin{aligned}\int_D u^{m+p} & \le (\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left[\left(\int_D u^{m+1}dx\right)^{\frac{n(p-1)}{2(m+1)}} + \left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n(p-1)}{2(m+1)}}\right] \\ & = (\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+p}{m+1}} \\ & \quad + (\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left(\int_D |\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n(p-1)}{2(m+1)}}\end{aligned} \tag{3.21}$$

we know $1-\frac{n(p-1)}{2(m+1)}=\frac{2(m+1)-n(p-1)}{2(m+1)}$,apply the Young inequality with ε,we have

$$\begin{aligned}
&(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)}}\left(\int_D|\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n(p-1)}{2(m+1)}}\\
&=(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left[\left(\frac{1}{\varepsilon_2}\right)^{\frac{n(p-1)}{2(m+1)-n(p-1)}}\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}\right]^{1-\frac{n(p-1)}{2(m+1)}}\left(\varepsilon_2\int_D|\nabla u^{\frac{m+1}{2}}|^2dx\right)^{\frac{n(p-1)}{2(m+1)}}\\
&\le(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\frac{1}{\varepsilon_2}\right)^{\frac{n(p-1)}{2(m+1)-n(p-1)}}\left(1-\frac{n(p-1)}{2(m+1)}\right)\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}\\
&\qquad+(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\frac{\varepsilon_2 n(p-1)}{2(m+1)}\int_D|\nabla u^{\frac{m+1}{2}}|^2dx
\end{aligned}\tag{3.22}$$

Thirdly, substitute (3.21)and(3.22)into(3.18),we obtain

$$\begin{aligned}
\Psi'(t)\le{}&\frac{b_2c_3(m+1)N}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}}+\frac{md_0b_2c_3}{\rho_0\varepsilon_1}|D|^{\frac{m-q+2}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}}\\
&+b_1(m+1)(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\frac{1}{\varepsilon_2}\right)^{\frac{n(p-1)}{2(m+1)-n(p-1)}}\left(1-\frac{n(p-1)}{2(m+1)}\right)\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}\\
&+b_1(m+1)(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\left(\int_D u^{m+1}dx\right)^{\frac{m+p}{m+1}}\\
&+b_1(m+1)(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}\frac{\varepsilon_2 n(p-1)}{2(m+1)}\int_D|\nabla u^{\frac{m+1}{2}}|^2dx\\
&+\frac{\varepsilon_1 md_0b_2c_3}{\rho_0}|D|^{\frac{m-q+2}{m+1}}\int_D|\nabla u^{\frac{m+1}{2}}|^2dx-\frac{4mc_2}{m+1}\int_D|\nabla u^{\frac{m+1}{2}}|^2dx
\end{aligned}\tag{3.23}$$

let $\varepsilon_1=\frac{2\rho_0c_2}{b_2d_0c_3(m+1)|D|^{\frac{m-q+2}{m+1}}}$, $\varepsilon_2=\frac{4mc_2}{b_1n(p-1)(m+1)(\sqrt{2}C_s)^{\frac{n(p-1)}{m+1}}}$,we have

$$\begin{aligned}
\Psi'(t)\le{}&A_1\left(\int_D u^{m+1}dx\right)^{\frac{m+q}{m+1}}+A_2\left(\int_D u^{m+1}dx\right)^{\frac{m+2q-1}{m+1}}\\
&+A_3\left(\int_D u^{m+1}dx\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}+A_4\left(\int_D u^{m+1}dx\right)^{\frac{m+p}{m+1}}
\end{aligned}\tag{3.24}$$

where A_1,A_2,A_3,A_4 are given in(3.7).by (3.1)and (3.4),we have

$$\psi(u)=(m+1)\int_0^u g'(y)y^m dy\ge(m+1)\int_0^u d_2y^m dy=d_2u^{m+1}\tag{3.25}$$

and

$$\int_D u^{m+1}dx \le \frac{1}{d_2}\int_D \psi(u)dx = \frac{1}{d_2}\Psi(t) \tag{3.26}$$

Now,substitute (3.26)into (3.24),we derive

$$\begin{aligned}
\Psi'(t) &\le A_1\left(\frac{1}{d_2}\Psi(t)\right)^{\frac{m+q}{m+1}} + A_2\left(\frac{1}{d_2}\Psi(t)\right)^{\frac{m+2q-1}{m+1}} + A_3\left(\frac{1}{d_2}\Psi(t)\right)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}} + A_4\left(\frac{1}{d_2}\Psi(t)\right)^{\frac{m+p}{m+1}}\\
&= A_1\frac{1}{d_2}^{\frac{m+q}{m+1}}\Psi(t)^{\frac{m+q}{m+1}} + A_2\frac{1}{d_2}^{\frac{m+2q-1}{m+1}}\Psi(t)^{\frac{m+2q-1}{m+1}} + A_3\frac{1}{d_2}^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}\Psi(t)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}}\\
&\quad + A_4\frac{1}{d_2}^{\frac{m+p}{m+1}}\Psi(t)^{\frac{m+p}{m+1}}
\end{aligned} \tag{3.27}$$

then we have

$$\Psi'(t) \le \kappa_1\Psi(t)^{\frac{m+q}{m+1}} + \kappa_2\Psi(t)^{\frac{m+2q-1}{m+1}} + \kappa_3\Psi(t)^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}} + \kappa_4\Psi(t)^{\frac{m+p}{m+1}} \tag{3.28}$$

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, are given in(3.6).

At last,integrating (3.28)from 0 to t,we obtain

$$t \ge \int_{\Psi(0)}^{\Psi(t)} \frac{d\tau}{\kappa_1\tau^{\frac{m+q}{m+1}} + \kappa_2\tau^{\frac{m+2q-1}{m+1}} + \kappa_3\tau^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}} + \kappa_4\tau^{\frac{m+p}{m+1}}} \tag{3.29}$$

Thus ,if u blow up in the measure of $\Psi(t)$ at some finite time t^*,passing the limit as $t \to t^*$,we have

$$t^* \ge \int_{\Psi(0)}^{+\infty} \frac{d\tau}{\kappa_1\tau^{\frac{m+q}{m+1}} + \kappa_2\tau^{\frac{m+2q-1}{m+1}} + \kappa_3\tau^{\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}} + \kappa_4\tau^{\frac{m+p}{m+1}}} \tag{3.30}$$

where $\frac{m+q}{m+1}$, $\frac{m+2q-1}{m+1}$, $\frac{2(m+p)-n(p-1)}{2(m+1)-n(p-1)}$, $\frac{m+p}{m+1} > 1$ inview of(3.2)and(3.5).

REFERENCES

[1] K.Baghaei, M.B.Ghaemi, M.Hesaaraki, Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source, Appl. Math. Lett. 27(2014)49-52.

[2] L.L.Zhang,N.Zhang,L.X.Li,Blow-up solutions and global existence for a kind of quasilinear reaction-diffusion equations,Z.Anal.Anwend.33(2014)247-258.

[3] Z.J.Cui, Z.D.Yang, Blow-up of solutions for nonlinear parabolic equation with nonlocal source and nonlocal boundary condition, Appl. Math. Comput. 224(2013)1-8.

[4] K.Baghaei,M.Hesaaraki, Blow-up phenomena for a system of semilinear parabolic equations with nonlinear boundary conditions, Math. Methods Appl. Sci. 38(2015)527-536.

[5] M.Winkler,Blow-up in a degenerate paraolic equation, Indiana Univ. Math .J. 53(2004)1415-1442.

[6] X.H.Shen,J.T.Ding, Blow-up analysis in quasilinear reaction-diffusion problems with weighted nonlocal source,Comput.Math,Anal,75(2018)1288- 1301.

[7] L.E.Payne, G.A.Philippin, S.Vernier Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, II, Nonlinear Anal. TMA73(2010)971-978.

[8] J.C.Song,Lower bounds for the blow-up time in a nonlocal reaction-diffusion problem,Appl.Math.Lett.24(2011),793-796.

[9] H.H.Lu, M.X.Wang, Global solutions and blow-up problems for a nonlinear degenerate parabolic system coupled via nonlocal sources. J. Math. Anal. Appl. 333(2007)984-1007.

[10] L.E.Payne,G.A.Philippin,P.W.Schaefer.Blow-up phenomena for some nonlinear parabolic problems,Nonlinear Anal.TMA69(2008)3495-3502.

[11] M.Marras,S.Vernier Piro, Reaction diffusion problems under non-local boundary conditions with blow-up solutions,J Inequal Appl,167(2014)1-11.

[12] J.T.Ding,Global and blow-up of solutions for nonlinear parabolic equations with Robin boundary conditions,Comput.Math,Appl, 65(2013)1808-1822.

[13] J.T.Ding,W.Kou, Blow-up solutions for reaction diffusion equations with nonlocal boundary conditions.J Math Anal Appl,470(2019)1-15.

[14] H.Brezis,Functional Analysis, Sobolev Spaces and Partial Differential Equations,New York,Springer-Verlag,2011.

[15] L.E.Payne, G.A.Philippin, S.Vernier Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition I, Z Angew Math Phys, 61(2010)999-1007.

[16] M.H.Protter, H.F.Weinberger, Maximum Principles in Differential Equations, Prentice Hall, Englewood Cliffs, New Jersey, 1967.