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Majorization Theorem for Concavifiable Functions

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Abstract

In the present article, we would extend the majorization theorem from concave function to concavifiable function and our article gives some results of different authors of different articles.

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1. INTRODUCTION

The concepts of generalised concavity have been introduced and investigated by different authors in different articles, e.g., $Hanson$, $Mangasarian$, $Ponstein$, $Karamardian$, $Greenberg$ and $Pierskalla$ (see articles in [23]).

A word of motivation is in order. Generalised concavity is important for several reasons. First, generalised concave functions arise naturally in applications. For example, unimodal probability density functions (like the gamma). Second, generalised concavity proves that this is useful in extreme value problems. Particularly in the theory of nonlinear programming, authors: $Mangasarian\, and\, Zangwill$, in different articles have presented the case for the utility of generalised concavity (see [23]).

Some other applications, in microeconomic theory and production functions that are usually assumed to be concave over some or all of their domains, resulting in diminishing returns to input factors (see [21]). Further that concavity of a function replaces the second derivative test to separate local max, min or saddle, moreover, for a concave function a critical point which is local max (min) is global (see [22]).

Therefore, we say that concave function's theory has become an especial domain of inequality's theory it means they have closed relationship.

While convex theory plays an important role in several fields of physical sciences. This theory attracts many engineers and economists including mathematicians due to number of applications and important results in the following respective fields [18] such as; differential equations, operations research, functional analysis, geometry, control theory, optimization, probability theory, operator theory, information theory, integral operator theory, numerical integration etc. The theory of convex functions also acts an important part in other fields of sciences as: mechanics, statistics, finance, engineering, physics, management sciences and economics.

Here, we state useful definition which is extracted from [19, 20] for concave function. Throughout the article L is an interval in $\mathbb{R}$.

Definition 1.1. *A function $\Psi : L \to \mathbb{R}$, known as* concave *if the given inequality holds*

$$\Psi\left(\sigma u_1 + (1-\sigma)\, u_2\right) \;\geq\; \sigma\, \Psi(u_1) \;+\; (1-\sigma)\, \Psi(u_2) \tag{1}$$

$\forall\, u_1, u_2 \in L$ *and* $\sigma \in [0,1]$.

Remark 1.2. *The following are the remarks about strictly concave, convex and strictly convex functions and recalled from [9, 15].*

(1) If inequality (1) *is strict for each* $u_1 \neq u_2$ *and* $\sigma \in (0,1)$*, then* Ψ *is called* strictly concave.

(2) If inequality (1) *is reversed, then* Ψ *is called* convex *and if it is strict for each* $u_1 \neq u_2$ *and* $\sigma \in (0,1)$*, then* Ψ *is called* strictly convex.

For more study for higher order convex and concave functions (see [1, 10, 11]).

1.1. Majorization

The basic idea of majorization has come from measure of variety of m-tuple (m-dimensional) components of vector and it is nearly linked to convexity and concavity. The main contributors are Hardy, Littlewood & Polya, who discussed interesting basic ideas about the majorization in their book "Inequality". Questions related to majorization were worked on by the comparatively few research scholars who were inspired by the book " Theory of Majorization and Its Application ", they put effort in order to rearrange ideas and to separate the literature valiantly. They have also given proofs on fundamental consequences and references to multiple point of view with respect to the wide range of applied discipline.

The application of theory of majorization is present in many fields such as pure and applied mathematics and engineering as well.

Here we state some definitions and results that would be used in sequel manner.

For fixed $m \geq 2$, $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$ denote 2 $m-$tuples and $u_{[1]} \geq u_{[2]} \geq \cdots \geq u_{[m]}, \quad v_{[1]} \geq v_{[2]} \geq \cdots \geq v_{[m]}$ be the ordered components.

Definition 1.3. *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$*,*

$$\mathbf{u} \prec \mathbf{v} \quad \textit{if} \quad \begin{cases} \sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]} \quad , & k \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^{m} u_{[i]} = \sum_{i=1}^{m} v_{[i]} \quad , & \end{cases}$$

where $\mathbf{u} \prec \mathbf{v}$*,* $\mathbf{u}$ is majorized by $\mathbf{v}$ *or* $\mathbf{v}$ majorizes $\mathbf{u}$.

In [4] Hardy $et.al.$ has introduced this above notation for majorization.

We provide the following theorem of majorization involving concave function from [16].

Theorem 1.4. *Let continuous function* $\Psi : L \to \mathbb{R}$ *be concave and* $\mathbf{u} = (u_1, \ldots, u_m)$*, and* $\mathbf{v} = (v_1, \ldots, v_m)$ *be two* m*-tuples, such that* $u_i, v_i \in L$ $(i = 1, \ldots, m)$*. If* $\mathbf{u}$ *majorizes* $\mathbf{v}$*, then*

$$\sum_{i=1}^{m} \Psi(u_i) \geq \sum_{i=1}^{m} \Psi(v_i), \tag{2}$$

holds.

Remark 1.5. (i) *It is clear that if* Ψ *is concave then -*Ψ *is convex and vice versa.*

(ii) *The reversed inequality (2) of above is known in the literature as Karamata's inequality [5, 6, 8].*

In the following we provide the definition of concavifiable function as $S.\ Zlobec$ discussed $Convexifiable\ function$ in his article "Characterization of convexifiable function" [24](see also [8]).

Definition 1.6. *[16] Let a continuous function $\Psi : L \to \mathbb{R}$ defined on compact interval $L \subset \mathbb{R}$, consider a function $F : L \times \mathbb{R} \to \mathbb{R}$ stated as*

$$F(u, \sigma) = \Psi(u) - \frac{\sigma}{2}u^2.$$

If $F(u, \sigma)$ is concave function in the interval L for some $\sigma = \sigma^*$, then $F(u, \sigma)$ is said to be concavification of Ψ and σ^* is its concavifier on L. Function Ψ is concavifiable if it has a concavification.

A remark about concavifiable function are given by authors in their article [16] that is in the following as $Muhammad\ Adil\ Khan$ had given in his article "Majorization theorem for convexifiable functions".

Remark 1.7. *If σ^* is a concavifier of Ψ, then for each $\sigma \geq \sigma^*$.*

Concavifiable functions have been studied on $\mathbb{R}$. The class of concavifiable functions is large: beside concave and twice continuous differentiable function.

In this article, we would extend inequality (2) and its weighted version from concave to concavifiable functions.

2. MAIN RESULTS

The following are the results for (2) and its weighted version in the form of concavifiable functions.

Theorem 2.1. *Let continuous function $\Psi : L \to \mathbb{R}$ be concavifiable on the compact interval L and σ its concavifier. Let $\mathbf{u} = (u_1, \ldots, u_m)$, and $\mathbf{v} = (v_1, \ldots, v_m)$ be two m-tuples, such that $u_i, v_i \in L$ $(i = 1, \ldots, m)$. If $\mathbf{u}$ majorizes $\mathbf{v}$, then*

$$\sum_{i=1}^{m} \Psi(u_i) \geq \sum_{i=1}^{m} \Psi(v_i) - \frac{\sigma}{2} \sum_{i=1}^{m} (v_i^2 - u_i^2), \tag{3}$$

holds.

Proof. Since Ψ is concavifiable with concavifier σ, so $F(u, \sigma) = \Psi(u) - \frac{\sigma}{2}u^2$ is a concave function and $\mathbf{u}$ majorizes $\mathbf{v}$. Therefore, by applying $F(u, \sigma)$ instead of $\Psi(u)$ in inequality (2) we obtain our required inequality (3). $\square$

Remark 2.2. *By putting $\sigma = 0$ in above theorem we recapture Theorem 2.1 of [16].*

Remark 2.3. *If we put $\Psi = -\Psi$ in Theorem 2.1, then we get Theorem 2 of [8].*

In the following theorem we give result for weighted concavifiable function.

Theorem 2.4. *Let continuous function $\Psi : L \to \mathbb{R}$ be concavifiable on the compact interval L and σ its concavifier. Let $\mathbf{u} = (u_1, \ldots, u_m)$, and $\mathbf{v} = (v_1, \ldots, v_m)$ be two decreasing m-tuples such that $u_i, v_i \in L$ $(i = 1, \ldots, m)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ be a real m-tuples such that*

$$\sum_{i=1}^{k} r_i u_i \geq \sum_{i=1}^{k} r_i v_i \qquad \text{for} \qquad k = 1, \ldots, m-1, \tag{4}$$

and

$$\sum_{i=1}^{m} r_i u_i = \sum_{i=1}^{m} r_i v_i \tag{5}$$

Then

$$\sum_{i=1}^{m} r_i \Psi(u_i) \geq \sum_{i=1}^{m} r_i \Psi(v_i) - \frac{\sigma}{2} \sum_{i=1}^{m} r_i (v_i^2 - u_i^2). \tag{6}$$

Proof. By putting $\Psi = -\Psi$ in Theorem 3 of [8] then we get desired result. □

Remark 2.5. *By substituting several conditions on the m-tuples $\mathbf{u}$,$\mathbf{v}$ and $\mathbf{p}$, weighted versions of inequality (2) and their integral versions have been proved (see [2, 3, 17]) and some of the reference therein. In the similar way, by using Theorem 2.4 we can get all such results for concavifiable functions.*

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