Q-Cubic bi-quasi Ideals of Semigroups

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Abstract

In this paper, we introduce the notion of a Q-cubic bi-quasi ideal of semigroup and we characterize the regular semigroup in terms of a Q-cubic bi-quasi ideal of a semigroup.

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1. INTRODUCTION

In 1965, the fundamental concept of fuzzy sets was introduced by Zadeh [17]. At present, it is an important tool in science, engineering, computer science, control engineering, etc. In 1979, Kuroki [8, 9, 10] was given the idea of fuzzy ideal, fuzzy bi-ideals, and fuzzy interior ideals in semigroups. Later, concepts were expanded about interval-valued fuzzy sets that have many applications such as approximate reasoning, image processing, decision making, medicine, and mobile networks, etc. In 2006 [15], Narayanan and Manikanran initiated the notion of interval valued fuzzy ideal in semigroup. In 2012, Jun [6], introduced a new notion, called a cubic set, and investigated several properties and introduced cubic subsemigroups and cubic left (right) ideals of semigroups. Later, in 2015 Sadaf et al. [16], discussed cubic bi-ideal of a semigroup. In later years V. Chinnadurai and K. Bharathivelan[3], studied cubic ideal in $\Gamma$-semigroup and PO-$\Gamma$-semigroup.

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The idea of an intuitionistic Q-fuzzy set was first discussed by Atanassov [1, 2], as a generalization of the notion of a fuzzy set. Kyung Ho Kim [7] introduced on intuitionistic Q-Fuzzy semiprime ideals in Semigroups. Thillaigovindan et al. [12] discussed on interval-valued fuzzy quasi-ideals of semigroups. In 2020, the concept of fuzzy semigroups has been discussed in research on the prime fuzzy m-bi ideals in semigroups [14], Manahon et al. [13] studied on BF-semigroups and fuzzy BF-semigroups, and T. Gaketem [5] introduced cubic interior ideals in semigroups, etc.

The aim of this paper we define definition of Q-cubic bi-quasi ideal in semigroup and properties of Q-cubic bi-quasi ideals are investigated. Then we characterized regular semigroup in terms of Q-cubic bi-quasi ideal.

2. PRELIMINARIES

In this section, we give definitions that are used in this paper. By a subsemigroup of a semigroup S we mean a non-empty subset A of S such that A^2 ⊆ A, and by a left (right) ideal of S we mean a non-empty subset A of S such that SA ⊆ A(AS ⊆ A). By a two-sided ideal or simply an ideal, we mean a non-empty subset of a semigroup S that is both a left and a right ideal of S. A non-empty subset A of S is called an interior ideal of S if SAS ⊆ A. A subsemigroup A of a semigroup S is called a bi-ideal of S if ASA ⊆ A. A non-empty subset A of a semigroup S is called a quasi-ideal of S if AS ∩ SA ⊆ A. A subsemigroup A of a semigroup S is said to be left (right) bi-quasi ideal of S if SA ∩ ASA ⊆ A(AS ∩ ASA ⊆ A). A subsemigroup A of a semigroup S is said to be bi-quasi ideal of S if it is both a left bi-quasi and right bi-quasi ideal of S.

Definition 1. Let X and Q be non-empty sets. A mapping \( f : X \times Q \rightarrow [0, 1] \) is called a Q-fuzzy set of X over Q.

Definition 2. Let X and Q be a non-empty set. A mapping \( \bar{f} : X \times Q \rightarrow D[0, 1] \) is called interval Q-fuzzy set over Q, where \( D[0, 1] \) denote the family of all closed subinterval of \([0, 1]\) and \( \bar{f} = [f^-, f^+] \), where \( f^- \) and \( f^+ \) are Q-fuzzy sets of X such that \( f^-(x, q) \leq f^+(x, q) \) for all \( x \in X, q \in Q \).

Definition 3. Let X and Q be a non-empty sets. A Q-cubic set A is an object having the form \( A = \{(x, q, f(x, q), \omega(x, q)) : x \in X, q \in Q\} \) which is briefly denoted by \( A = (f, \omega) \) with respect to Q, where \( f : X \times Q \rightarrow D[0, 1] \) is an interval Q-fuzzy set over Q and \( \omega : X \times Q \rightarrow [0, 1] \) is a Q-fuzzy set over Q.

Definition 4. Let \( A = (f_A, \omega_A) \) be a Q-cubic set in X. Define \( U(A; \ell, n) = \{x \in X | \ell \leq f(x, q), \omega(x, q) \leq n\} \), where \( \ell \in D[0, 1] \) and \( n \in [0, 1] \) is called the Q-cubic level set of A.

For any non-empty subset I of a set X, the characteristic function of I is defined to be a structure \( \chi_I = \{(x, f_I(x, q), \omega_I(x, q)) : x \in X, q \in Q\} \) which is briefly denoted by \( \chi_I = (f_I, \omega_I) \).
for all $x, y$

Definition 5. Let $A$ be a Q-cubic set in a semigroup $S$ with respect to $Q$.

It satisfies the following conditions:

- $\omega$ denoted by $A$ (with respect to $Q$) satisfies $\omega(x, q) \geq \omega(y, q)$ for all $x, y, q \in Q$.

The whole cubic set $S$ in a semigroup $S$ is defined to be a structure

$$\chi_S = \{(x, \bar{f}_{xs}(x, q), \omega_{xs}(x, q)) : x \in S, q \in Q\},$$

with $\bar{f}_{xs}(x, q) = [1, 1]$ and $\omega_{xs}(x, q) = 0$. It will be briefly denoted by $\chi_S = (\bar{f}_{xs}, \omega_{xs})$.

For two Q-cubic sets $A = (f, \omega)$ and $B = (g, \upsilon)$ in a semigroup $S$, we define $A \subseteq B$ if and only if $f \subset g$ and $\omega \gtrless \upsilon$, where $f \subset g$ means that $f(x, q) \subseteq g(x, q)$ and $\omega \gtrless \upsilon$ means that $\omega(x, q) \geq \upsilon(x, q)$ for all $x \in S, q \in Q$.

The Q-cubic product of $A = (f, \omega)$ and $B = (g, \upsilon)$ is defined to be a Q-cubic set

$$A \sim B = \{(x, q), (f \circ g)(x, q), (\omega \circ \upsilon)(x, q)) : x \in S, q \in Q\}$$

$$\begin{align*}
(f \circ g)(x, q) &= \left\{ \begin{array}{ll}
\bigcup_{x = yz} \{f(y, q) \cap g(z, q)\} & \text{for some } x, y, z \in S, q \in Q; \\
0 & \text{otherwise.}
\end{array} \right.
\\
(\omega \circ \upsilon)(x, q) &= \left\{ \begin{array}{ll}
\bigwedge_{y = yz} \{\omega(y, q) \lor \upsilon(z, q)\} & \text{for some } x, y, z \in S, q \in Q; \\
1 & \text{otherwise.}
\end{array} \right.
\end{align*}$$

Let $A = (f, \omega)$ and $B = (g, \upsilon)$ be two Q-cubic sets in $S$. The intersection of $A$ and $B$ denoted by $A \cap B$ is the Q-cubic set $A \cap B = (\bar{f} \cap \bar{g}, \omega \cap \upsilon)$ with respect to $Q$, where $(\bar{f} \cap \bar{g})(x, q) = f(x, q) \cap g(x, q)$ and $(\omega \cap \upsilon)(x, q) = \omega(x, q) \lor \upsilon(x, q)$.

The union of $A$ and $B$ denoted by $A \cup B$ is the Q-cubic set $A \cup B = (\bar{f} \cup \bar{g}, \omega \lor \upsilon)$ with respect to $Q$, where $(\bar{f} \cup \bar{g})(x, q) = f(x, q) \cup g(x, q)$ and $(\omega \lor \upsilon)(x, q) = \omega(x, q) \land \upsilon(x, q)$.

Definition 5. A Q-cubic set $A = (f, \omega)$ of $S$ is called a Q-cubic subsemigroup of $S$ if it satisfies the following conditions:

1. $f(x, q) \cap f(y, q) \subseteq f(xy, q)$,
2. $\omega(xy, q) \leq \omega(x, q) \lor \omega(y, q)$

for all $x, y \in S$ and $q \in Q$. 

Definition 6. A Q-cubic set $A = (\bar{f}, \omega)$ of $S$ is called a Q-cubic left(resp.right) ideal of $S$ if it satisfies the following conditions:

1. $\bar{f}(y, q) \subseteq \bar{f}(xy, q)(\bar{f}(x, q) \subseteq \bar{f}(xy, q))$,
2. $\omega(xy, q) \leq \omega(y, q)(\omega(xy, q) \leq \omega(x, q))$

for all $x, y \in S$ and $q \in Q$.

A Q-cubic set $A = (\bar{f}, \omega)$ of $S$ is called a Q-cubic ideal of $S$ if it is both Q-cubic left ideal and Q-cubic right ideal of $S$.

3. Q-CUBIC BI-QUASI IDEALS OF SEMIGROUPS

In this section we define Q-cubic bi-quasi ideals in semigroup and investigation properties of Q-cubic bi-quasi ideals.

Definition 7. A Q-cubic subsemigroup $A = (\bar{f}, \omega)$ of $S$ is called a Q-cubic left(right) bi-quasi ideal of $S$ if it satisfies the following conditions:

1. $\bar{f}_x \circ \bar{f} \circ \bar{f} \subseteq \bar{f} (\bar{f} \circ \bar{f}_x \cap \bar{f} \circ \bar{f} \subseteq \bar{f})$,
2. $\omega \preceq \omega_x \circ \omega \circ \omega_x \circ \omega (\omega \preceq \omega \circ \omega_x \circ \omega \circ \omega_x \circ \omega)$,

A Q-fuzzy set $A = (\bar{f}, \omega)$ of semigroup $S$ is called a Q-cubic bi-quasi ideal if it is both Q-cubic left bi-quasi ideal and Q-cubic right bi-quasi ideal of $S$.

Theorem 8. Every Q-cubic left ideal of a semigroup $S$ is a Q-cubic left bi-quasi ideal of $S$.

Proof. Let $A = (\bar{f}, \omega)$ be a Q-cubic left ideal of a semigroup $S$. Let $x \in S$ and $q \in Q$. Then

$$ (\bar{f}_x \circ \bar{f})(x, q) = \bigcup_{x=yz} \{\bar{f}_x(y, q) \cap \bar{f}(z, q)\} $$

$$ = \bigcup_{x=yz} \{\bar{f}(z, q)\} $$

$$ \subseteq \bigcup_{x=yz} \{\bar{f}(yz, q)\} $$

$$ = \bigcup_{x=yz} \{\bar{f}(x, q)\} $$

$$ = \bar{f}(x, q). $$

Thus $\bar{f}_x \circ \bar{f} \circ \bar{f} \subseteq \bar{f}$. 

And

\[
(\omega_{xS} \circ \omega)(x, q) = \bigwedge_{x = yz} \{\omega_{xS}(y, q) \lor \omega(z, q)\}
\]

\[
= \bigwedge_{x = yz} \{\omega(z, q)\}
\]

\[
\geq \bigwedge_{x = yz} \{\omega(yz, q)\}
\]

\[
= \bigwedge_{x = yz} \{\omega(x, q)\}
\]

\[
= \omega(x, q).
\]

Then \(\omega_{xS} \circ \omega \geq \omega \circ \omega_{xS} \circ \omega \geq \omega\).

Hence \(A = (\bar{f}, \omega)\) be a Q-cubic left bi-quasi ideal of the semigroup \(S\). \(\square\)

**Theorem 9.** Every Q-cubic left ideal of a semigroup \(S\) is a Q-cubic right bi-quasi ideal of \(S\).

**Proof.** Let \(A = (\bar{f}, \omega)\) be a Q-cubic left ideal of a semigroup \(S\). Let \(x \in S\) and \(q \in Q\). We have \((\bar{f}_{xS} \circ \bar{f})(x, q) \subseteq \bar{f}(x, q)\) and \((\omega_{xS} \circ \omega)(x, q) \geq \omega(x, q)\). Then

\[
(\bar{f} \circ \bar{f}_{xS} \circ \bar{f})(x, q) = \bigcup_{x = abc} \{\bar{f}(a, q) \cap (\bar{f}_{xS} \circ \bar{f})(bc, q)\}
\]

\[
\subseteq \bigcup_{x = abc} \{\bar{f}(a, q) \cap \bar{f}(bc, q)\}
\]

\[
\subseteq \bar{f}(x, q).
\]

Thus \(\bar{f} \circ \bar{f}_{xS} \cap \bar{f} \circ \bar{f}_{xS} \circ \bar{f} \subseteq \bar{f}\).

And

\[
(\omega \circ \omega_{xS} \circ \omega)(x, q) = \bigwedge_{x = abc} \{\omega(a, q) \lor (\omega_{xS} \circ \omega)(bc, q)\}
\]

\[
\geq \bigwedge_{x = abc} \{\omega(a, q) \lor \omega(bc, q)\}
\]

\[
\geq \omega(x, q).
\]

Now \(\omega \circ \omega_{xS} \geq \omega \circ \omega_{xS} \circ \omega \geq \omega\).

Hence \(A = (\bar{f}, \omega)\) be a Q-cubic right bi-quasi ideal of the semigroup \(S\). \(\square\)
**Theorem 10.** Every Q-cubic right ideal of a semigroup $S$ is a Q-cubic right bi-quasi ideal of $S$.

**Proof.** Let $A = (\bar{f}, \omega)$ be a Q-cubic right ideal of a semigroup $S$. Let $x \in S$ and $q \in Q$. Then

$$
(f \circ \bar{f}_{\chi_S})(x, q) = \bigcup_{x = yz} \{f(y, q) \cap \bar{f}_{\chi_S}(z, q)\}
$$

$$
= \bigcup_{x = yz} \{f(y, q)\}
$$

$$
\subseteq \bigcup_{x = yz} \{\bar{f}(yz, q)\}
$$

$$
= \bigcup_{x = yz} \{\bar{f}(x, q)\}
$$

$$
= \bar{f}(x, q).
$$

Thus $f \circ \bar{f}_{\chi_S} \cap f \circ \bar{f}_{\chi_S} \subseteq \bar{f}$.

And

$$
(\omega \circ \omega_{\chi_S})(x, q) = \bigwedge_{x = yz} \{\omega(y, q) \lor \omega_{\chi_S}(z, q)\}
$$

$$
= \bigwedge_{x = yz} \{\omega(z, q)\}
$$

$$
\geq \bigwedge_{x = yz} \{\omega(yz, q)\}
$$

$$
= \bigwedge_{x = yz} \{\omega(x, q)\}
$$

$$
= \omega(x, q).
$$

Then $\omega \circ \omega_{\chi_S} \lor \omega \circ \omega_{\chi_S} \circ \omega \supseteq \omega$.

Hence $A = (\bar{f}, \omega)$ be a Q-cubic right bi-quasi ideal of the semigroup $S$. 

**Corollary 11.** Every Q-cubic right ideal of a semigroup $S$ is a Q-cubic left bi-quasi ideal of $S$.

**Corollary 12.** Every Q-cubic right(left) ideal of a semigroup $S$ is a Q-cubic bi-quasi ideal of $S$.

**Theorem 13.** Let $S$ be a semigroup and $A = (\bar{f}, \omega)$ be a non-empty Q-fuzzy set of $S$. A Q-fuzzy set $A = (\bar{f}, \omega)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$ if and only if the Q-cubic level set $U(A; \bar{t}, n)$ of $A$ is a left bi-quasi ideal of a semigroup $S$ for every $\bar{t} \in D[0, 1], n \in [0, 1], \text{ where } U(A; \bar{t}, n) \neq \emptyset$. 

Lemma 16. It is straightforward.

Proof. Assume that $A = (\bar{f}, \omega)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$. Let $U(A; \bar{t}, n) = \emptyset$, $\bar{t} \in D[0, 1]$, $n \in [0, 1]$. Let $x \in SU(A; \bar{t}, n) \cap U(A; \bar{t}, n)$. Then $x = ba = cde$ where $b, d \in S$ and $a, c, e \in U(A; \bar{t}, n)$. Then $\bar{t} \subseteq (\bar{f}_x \circ \bar{f})(x, q)$ and $\bar{t} \subseteq (\bar{f} \circ \bar{f}_x \circ \bar{f})(x, q)$ implies that $\bar{t} \subseteq \bar{f}(x, q)$ and $(\omega_x \circ \omega)(x, q) \leq n$ and $(\omega \circ \omega_x \circ \omega)(x, q) \leq n$ implies that $\omega(x, q) \leq n$. Then $x \in U(A; \bar{t}, n)$. Therefore $U(A; \bar{t}, n)$ is a left bi-quasi ideal of the semigroup $S$.

Conversely suppose that $U(A; \bar{t}, n)$ is a left bi-quasi ideal of the semigroup $S$, for all $\bar{t} \in Im(\bar{f})$ and $n \in Im(\omega)$. Let $x, y \in S, q \in Q$. Then $\bar{f}(x, q) = \bar{t}_1, \bar{f}(y, q) = \bar{t}_2, \omega(x, q) = n_1, \omega(y, q) = n_2, \bar{t}_1 \geq \bar{t}_2$ and $n_1 \leq n_2$. Then $x, y \in U(A; \bar{t}, n)$. We have $SU(A; \bar{t}, m) \cap U(A; \bar{t}, n) SU(A; \bar{t}, m) \subseteq U(A; \bar{t}, m)$, for all $\bar{t} \in Im(\bar{f})$ and $m \in Im(\omega)$. Suppose $\bar{t} = \min\{Im(\bar{f})\}$ and $n = \max\{Im(\omega)\}$. Then $SU(A; \bar{t}, n) \cap U(A; \bar{t}, n) SU(A; \bar{t}, n) \subseteq U(A; \bar{t}, n)$. Therefore $\bar{f}_x \circ \bar{f} \cap \bar{f} \circ \bar{f}_x \circ \bar{f} \subseteq \bar{f}$ and $\omega \leq \omega_x \circ \omega \gamma \omega \circ \omega_x \circ \omega$. Hence $A = (\bar{f}, \omega)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$. □

Corollary 14. Let $S$ be a semigroup and $A = (\bar{f}, \omega)$ be a non-empty Q-fuzzy set of $S$. A Q-fuzzy set $A = (\bar{f}, \omega)$ is a Q-cubic right bi-quasi ideal of a semigroup $S$ if and only if the Q-cubic level set $U(A; \bar{t}, n)$ of $A$ is a right bi-quasi ideal of a semigroup $S$ for every $\bar{t} \in D[0, 1], n \in [0, 1]$, where $U(A; \bar{t}, n) \neq \emptyset$.

Corollary 15. Let $S$ be a semigroup and $A = (\bar{f}, \omega)$ be a non-empty Q-fuzzy set of $S$. A Q-fuzzy set $A = (\bar{f}, \omega)$ is a Q-cubic bi-quasi ideal of a semigroup $S$ if and only if the Q-cubic level set $U(A; \bar{t}, n)$ of $A$ is a bi-quasi ideal of a semigroup $S$ for every $\bar{t} \in D[0, 1], n \in [0, 1]$, where $U(A; \bar{t}, n) \neq \emptyset$.

Lemma 16. For non-empty subsets $G$ and $H$ of a semigroup $S$, we have

1. $\bar{f}_x \circ \bar{f}_y = \bar{f} \chi_{GH}$,
2. $\bar{f}_x \cap \bar{f}_y = \bar{f} \chi_{GH}$,
3. $\omega_x \circ \omega_y = \omega \chi_{GH}$,
4. $\omega \chi_x \gamma \omega_y = \omega \chi_{GH}$.

Proof. It is straightforward. □

Theorem 17. Let $I$ be a non-empty subset of a semigroup $S$ and $\chi_I = (\bar{f}_I, \omega_I)$ be the characteristic function of $I$. Then $I$ is a left bi-quasi ideal of a semigroup $S$ if and only if $\chi_I = (\bar{f}_I, \omega_I)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$. 

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Proof. Suppose $I$ is a left bi-quasi ideal of $S$. Then $I$ is a subsemigroup of $S$ and $SI \cap ISI \subseteq I$. Obviously $\chi_I = (f_{xI}, \omega_{xI})$ is a Q-cubic subsemigroup of $S$. And

$$(f_{xs} \circ f_{xi} \cap f_{xs} \circ f_{xi})(x, q) = (f_{xs} \circ f_{xi})(x, q) \cap (f_{xs} \circ f_{xi})(x, q)$$

$$= f_{xsI}(x, q) \cap f_{xsisI}(x, q)$$

$$= f_{xsisI}(x, q).$$

Thus, $f_{xs} \circ f_{xi} \cap f_{xs} \circ f_{xi} \subseteq f_{xi}$. Similarly, we can show that $\omega_{xs} \circ \omega_{sx} = \omega \circ \omega_{xs} \circ \omega \geq \omega$. Hence $\chi_I = (f_{xI}, \omega_{xI})$ is a Q-cubic left bi-quasi ideal of $S$.

Conversely suppose that $\chi_I = (f_{xI}, \omega_{xI})$ is a Q-cubic left bi-quasi ideal of $S$. Then $I$ is a subsemigroup of $S$. We have

$$(f_{xs} \circ f_{xi})(x, q) \cap (f_{xs} \circ f_{xi})(x, q) \subseteq f_{xi}(x, q)$$

$$\Rightarrow f_{xsI}(x, q) \cap f_{xsisI}(x, q) \subseteq f_{xi}(x, q)$$

$$\Rightarrow f_{xsisI}(x, q) \subseteq f_{xi}(x, q).$$

Thus $SI \cap ISI \subseteq I$. Hence $I$ is a left bi-quasi ideal of a semigroup $S$. \hfill \Box

Corollary 18. Let $I$ be a non-empty subset of a semigroup $S$ and $\chi_I = (f_{xI}, \omega_{xI})$ be the characteristic function of $I$. Then $I$ is a right bi-quasi ideal of a semigroup $S$ if and only if $\chi_I = (f_{xI}, \omega_{xI})$ is a Q-cubic right bi-quasi ideal of a semigroup $S$.

Corollary 19. Let $I$ be a non-empty subset of a semigroup $S$ and $\chi_I = (f_{xI}, \omega_{xI})$ be the characteristic function of $I$. Then $I$ is a bi-quasi ideal of a semigroup $S$ if and only if $\chi_I = (f_{xI}, \omega_{xI})$ is a Q-cubic bi-quasi ideal of a semigroup $S$.

Theorem 20. if $A = (f, \omega)$ and $B = (\bar{g}, \nu)$ are Q-cubic bi-quasi ideals of a semigroup $S$, then $A \cap B = (f \cap \bar{g}, \omega \cap \nu)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$.

Proof. Let $A = (f, \omega)$ and $B = (\bar{g}, \nu)$ be Q-cubic bi-quasi ideals of a semigroup $S$. Then

$$(f_{xs} \circ f \cap \bar{g})(x, q) = \bigcup_{x=ab} \{ f_{xs}(a, q) \cap (f \cap \bar{g})(b, q) \}$$

$$= \bigcup_{x=ab} \{ f_{xs}(a, q) \cap f(b, q) \cap \bar{g}(b, q) \}$$

$$= \bigcup_{x=ab} \{ \{ f_{xs}(a, q) \cap f(b, q) \} \cap \{ f_{xs}(a, q) \cap \bar{g}(b, q) \} \}$$

$$= \bigcup_{x=ab} \{ f_{xs}(a, q) \cap f(b, q) \} \cap \bigcup_{x=ab} \{ f_{xs}(a, q) \cap \bar{g}(b, q) \}$$

$$= (f_{xs} \circ f)(x, q) \cap (f_{xs} \circ \bar{g})(x, q)$$

$$= (f_{xs} \circ f \cap f_{xs} \circ \bar{g})(x, q).$$
Therefore $\tilde{f}_x \circ \tilde{f} \cap \tilde{g} = \tilde{f}_x \circ \tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{g}$.

$$(\tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{f} \cap \tilde{g})(x, q) = \bigcup_{x=abc} \{(\tilde{f} \cap \tilde{g})(a, q) \cap (\tilde{f}_x \circ \tilde{f} \cap \tilde{g})(bc, q)\}
= \bigcup_{x=abc} \{(\tilde{f} \cap \tilde{g})(a, q) \cap ((\tilde{f}_x \circ \tilde{f} \cap \tilde{g})(bc, q))\}
= \bigcup_{x=abc} \{(\tilde{f}(a, q) \cap (\tilde{f}_x \circ \tilde{f})(bc, q)) \cap \{\tilde{g}(a, q) \cap (\tilde{f}_x \circ \tilde{g})(bc, q)\}\}
= (\tilde{f} \circ \tilde{f}_x \circ \tilde{f}_{\cap} \tilde{g})(x, q) \cap (\tilde{g} \circ \tilde{f}_x \circ \tilde{g})(x, q) = (\tilde{f} \circ \tilde{f}_x \circ \tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{g})(x, q).$$

Therefore $\tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{f} \cap \tilde{g} = \tilde{f} \circ \tilde{f}_x \circ \tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{g}$. 

Then $\tilde{f}_x \circ (\tilde{f} \cap \tilde{g}) \cap (\tilde{f} \cap \tilde{g}) \circ \tilde{f}_x \circ (\tilde{f} \cap \tilde{g}) = \tilde{f}_x \circ \tilde{f} \cap \tilde{f} \circ \tilde{f}_x \circ \tilde{f} \cap \tilde{g} \circ \tilde{f}_x \circ \tilde{g} \circ \tilde{f}_x \circ \tilde{g} \circ \tilde{f}_x \circ \tilde{g}$.

Similarly, we can show that $\omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \cap \omega_{\gamma} \cap \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \circ \omega_{\times} \circ \omega_{\gamma} \circ \omega_{\gamma} \cap \omega_{\gamma} \cap \omega_{\gamma}$. Therefore $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \gamma, \gamma)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$.

**Corollary 21.** If $A = (\tilde{f}, \omega)$ and $B = (\tilde{g}, \nu)$ are Q-cubic bi-quasi ideals of a semigroup $S$, then $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \gamma, \nu)$ is a Q-cubic right bi-quasi ideal of a semigroup $S$.

**Corollary 22.** If $A = (\tilde{f}, \omega)$ and $B = (\tilde{g}, \nu)$ are Q-cubic bi-quasi ideals of a semigroup $S$, then $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \gamma, \nu)$ is a Q-cubic bi-quasi ideal of a semigroup $S$.

**Theorem 23.** If $A = (\tilde{f}, \omega)$ and $B = (\tilde{g}, \nu)$ are Q-cubic right ideals and a Q-cubic left ideal of a semigroup $S$ respectively. Then $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \nu, \nu)$ is a Q-cubic left bi-quasi ideal of a semigroup $S$.

**Proof.** It following Theorem 20. □

**Corollary 24.** If $A = (\tilde{f}, \omega)$ and $B = (\tilde{g}, \nu)$ are Q-cubic right ideals and a Q-cubic left ideal of a semigroup $S$ respectively. Then $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \gamma, \nu)$ is a Q-cubic right bi-quasi ideal of a semigroup $S$.

**Corollary 25.** If $A = (\tilde{f}, \omega)$ and $B = (\tilde{g}, \nu)$ are Q-cubic right ideals and a Q-cubic left ideal of a semigroup $S$ respectively. Then $A \cap B = (\tilde{f} \cap \tilde{g}, \omega \cap \gamma, \nu)$ is a Q-cubic bi-quasi ideal of a semigroup $S$.

**Definition 26.** A semigroup $S$ is called regular if for all $a \in S$ there exists $x \in S$ such that $a = axa$.

**Definition 27.** A Q-cubic subsemigroup $A = (\tilde{f}, \omega)$ of $S$ is called a Q-cubic quasi ideal of $S$ if it satisfies the following conditions:
Proof. Let $f_{xS} \circ \bar{f} \cap f_{xS} \subseteq \bar{f}$.

2. $\omega \leq \omega_{xS} \circ \omega \gamma \omega \circ \omega_{xS}$.

**Theorem 28.** If $A = (\bar{f}, \omega)$ be a Q-cubic quasi ideal of a regular semigroup $S$. Then $A = (\bar{f}, \omega)$ is a Q-cubic ideal of a semigroup $S$.

**Proof.** Assume that $A = (\bar{f}, \omega)$ is a Q-cubic quasi-ideal of $S$ and let $x, y \in S, q \in Q$. Then

$$f(xy, q) \supseteq (f \circ f_{xS})(xy, q) \cap (f_{xS} \circ \bar{f})(xy, q)$$

$$= \bigcup_{xy=ab} \{f(a, q) \cap f_{xS}(b, q)\} \cap \bigcup_{xy=ij} \{f_{xS}(i, q) \cap \bar{f}(j, q)\}$$

$$\supseteq f(x, q) \cap f_{xS}(y, q) \cap f_{xS}(x, q) \cap \bar{f}(y, q)$$

$$= \bar{f}(x, q) \cap \bar{f}(y, q).$$

Thus $f(xy, q) \supseteq \bar{f}(x, q) \cap \bar{f}(y, q)$. And similarly we can show that $\omega(xy, q) \leq \omega(x, q) \lor \omega(y, q)$.

**Hence $A = (\bar{f}, \omega)$ is a Q-cubic subsemigroup of $S$.** Let $x, y, z \in S, q \in Q$. Then

$$f(xyz, q) \supseteq (f \circ f_{xS})(xyz, q) \cap (f_{xS} \circ \bar{f})(xyz, q)$$

$$= \bigcup_{xyz=ab} \{f(a, q) \cap f_{xS}(b, q)\} \cap \bigcup_{xyz=ij} \{f_{xS}(i, q) \cap \bar{f}(j, q)\}$$

$$\supseteq f(x, q) \cap f_{xS}(y, q) \cap f_{xS}(z, q) \cap \bar{f}(z, q)$$

$$= \bar{f}(x, q) \cap \bar{f}(z, q).$$

Thus $f(xyz, q) \supseteq \bar{f}(x, q) \cap \bar{f}(z, q)$. And similarly we can show that $\omega(xyz, q) \leq \omega(x, q) \lor \omega(z, q)$. Hence $A = (\bar{f}, \omega)$ is a Q-cubic bi-ideal of $S$. Since $S$ is regular, $A = (\bar{f}, \omega)$ is a Q-cubic bi-ideal of $S$ and $x, y \in S$ we have $xy \in (xSx)S \subseteq xSx$. Thus there exists $k \in S$ such that $xy = xkx$. So

$$\bar{f}(xy, q) = f(xkx, q) \supseteq \bar{f}(x, q) \cap \bar{f}(z, q) = \bar{f}(x, q).$$

And similarly $\omega(xy, q) \leq \omega(x, q)$. Thus, $A = (\bar{f}, \omega)$ is a Q-cubic right ideal of $S$. Similarly, we can show that $\bar{f}(xy, q) \supseteq \bar{f}(y, q)$ and $\omega(xy, q) \leq \omega(y, q)$. Thus $A = (\bar{f}, \omega)$ is a Q-cubic left ideal of $S$. Hence $A = (\bar{f}, \omega)$ is a Q-cubic ideal of $S$. \qed

**Theorem 29.** Let $S$ be a regular semigroup. Then $A = (\bar{f}, \omega)$ is a Q-cubic left bi-quasi ideal of $S$ if and only if $A = (\bar{f}, \omega)$ is a Q-cubic quasi ideal of $S$.

**Proof.** Let $A = (\bar{f}, \omega)$ is a Q-cubic left bi-quasi ideal of $S$ and $x \in S, q \in Q$. Thus,

$$(\bar{f}_{xS} \circ \bar{f})(x, q) \cap (f \circ f_{xS} \circ \bar{f})(x, q) \subseteq \bar{f}(x, q)$$

and $\omega(x, q) \leq (\omega_{xS} \circ \omega)(x, q) \gamma (\omega \circ \omega_{xS})$. Therefore,
\( \omega_{xs} \circ \omega)(x, q). \) Suppose \((\bar{f}_{xs} \circ \bar{f})(x, q) \supseteq \bar{f}(x, q) \). Since \( S \) is regular, there exists \( y \in S \) such that \( x = xyx. \) Then

\[
(\bar{f} \circ \bar{f}_{xs} \circ \bar{f})(x, q) = \bigcup_{x = xyx} \{ \bar{f}(xy, q) \cap (\bar{f}_{xs} \circ \bar{f})(x, q) \}
\]

\[
\supseteq \bigcup_{x = xyx} \{ \bar{f}(x, q) \cap \bar{f}(x, q) \}
\]

\[
= \bar{f}(x, q).
\]

Which is a contradiction. Therefore \( A = (\bar{f}, \omega) \) is a \( Q \)-cubic quasi ideal of \( S \). By Theorem 28, converse is true.

**Corollary 30.** Let \( S \) be a regular semigroup. Then \( A = (\bar{f}, \omega) \) is a \( Q \)-cubic right bi-quasi ideal of \( S \) if and only if \( A = (\bar{f}, \omega) \) is a \( Q \)-cubic quasi ideal of \( S \).

**Corollary 31.** Let \( S \) be a regular semigroup. Then \( A = (\bar{f}, \omega) \) is a \( Q \)-cubic bi-quasi ideal of \( S \) if and only if \( A = (\bar{f}, \omega) \) is a \( Q \)-cubic quasi ideal of \( S \).

**Theorem 32.** Let \( S \) be a semigroup. \( S \) is a regular semigroup if and only if \( B = SB \cap BSB, \) for every bi-quasi ideal of \( S \).

**Theorem 33.** Let \( S \) be a semigroup. Then \( S \) is a regular if and only if \( \bar{f} = \bar{f}_{xs} \circ \bar{f} \cap \bar{f} \subseteq \bar{f} \) and \( \omega = \omega_{xs} \circ \omega \cap \gamma \circ \omega_{xs} \circ \omega \), for any \( Q \)-cubic left bi-quasi ideal of a semigroup \( S \).

**Proof.** Let \( A = (\bar{f}, \omega) \) be a \( Q \)-cubic left bi-quasi ideal of the regular semigroup \( S \). Then \( \bar{f}_{xs} \circ \bar{f} \cap \bar{f} \subseteq \bar{f} \) and \( \omega \leq \omega_{xs} \circ \omega \cap \gamma \circ \omega_{xs} \circ \omega \). Let \( x \in S, q \in Q \). Since \( S \) is regular, there exists \( a \in S \) such that \( x = xax. \) Thus

\[
(\bar{f} \circ \bar{f}_{xs} \circ \bar{f})(x, q) = \bigcup_{x = xax} \{ \bar{f}(x, q) \cap (\bar{f}_{xs} \circ \bar{f})(ax, q) \}
\]

\[
= \bigcup_{x = xax} \{ \bar{f}(x, q) \cap \bigcup_{ax = yz} \{ \bar{f}_{xs}(y, q) \cap \bar{f}(z, q) \} \}
\]

\[
\supseteq \bigcup_{x = xax} \{ \bar{f}(x, q) \cap \bar{f}(x, q) \}
\]

\[
= \bar{f}(x, q).
\]

Similarly, \((\bar{f}_{xs} \circ \bar{f})(x, q) \supseteq \bar{f}(x, q), \omega(x, q) \supseteq (\omega_{xs} \circ \omega)(x, q) \) and \( \omega(x, q) \supseteq (\omega \circ \omega_{xs} \circ \omega)(x, q) \). Therefore \( \bar{f} = \bar{f}_{xs} \circ \bar{f} \cap \bar{f}_{xs} \circ \bar{f} \) and \( \omega = \omega_{xs} \circ \omega \cap \gamma \circ \omega \circ \omega_{xs} \circ \omega \).

Conversely suppose that let \( B \) be a left bi-quasi ideal of a semigroup \( S \). Then by Theorem 17, \( \chi_B = (\bar{f}_{xs}, \omega_B) \) be a \( Q \)-cubic bi-interior ideal of the semigroup \( S \). Thus

\[
\bar{f}_{xb}(x, q) = (\bar{f}_{xs} \circ \bar{f}_{xb})(x, q) \cap (\bar{f}_{xs} \circ \bar{f}_{xb})(x, q)
\]

\[
= \bar{f}_{xs}(x, q) \cap \bar{f}_{xb}(x, q)
\]

\[
= \bar{f}_{xb}(x, q).
\]

Therefore \( B = SB \cap BSB. \) By Theorem 32, \( S \) is regular semigroup. 

Corollary 34. Let $S$ be a semigroup. Then $S$ is a regular if and only if $\tilde{f} = \tilde{f}_x \circ \bigcap \tilde{f} \circ \tilde{f}_x \circ \tilde{f}$ and $\omega = \omega_x \circ \omega \circ \omega_x \circ \omega$, for any $Q$-cubic right bi-quasi ideal of a semigroup $S$.

Corollary 35. Let $S$ be a semigroup. Then $S$ is a regular if and only if $\tilde{f} = \tilde{f}_x \circ \bigcap \tilde{f} \circ \tilde{f}_x \circ \tilde{f}$ and $\omega = \omega_x \circ \omega \gamma \circ \omega \circ \omega_x \circ \omega$ or $\tilde{f} = \tilde{f} \circ \tilde{f}_x \cap \tilde{f} \circ \tilde{f}_x \circ \tilde{f}$ and $\omega = \omega \circ \omega_x \gamma \circ \omega_x \circ \omega$, for any $Q$-cubic bi-quasi ideal of a semigroup $S$.

Theorem 36. Let $S$ be a semigroup. Then $S$ is a regular if and only if $\tilde{f} \cap \tilde{g} \circ \tilde{f} \cap \tilde{f}$ and $\omega \gamma \nu \subseteq \nu \circ \omega \gamma \circ \nu \circ \omega$, for every $Q$-cubic left bi-quasi ideal $A = (\tilde{f}, \omega)$ and every $Q$-cubic ideal $B = (\tilde{g}, \nu)$ of a semigroup $S$.

Proof. Let $S$ be a regular semigroup and $x \in S$. Then there exists $y \in S$ such that $x = xyx$.

$$(\tilde{f} \circ \tilde{g} \circ \tilde{f})(x, q) = \bigcup_{x = xyz} \{(\tilde{f} \circ \tilde{g})(xy, q) \cap \tilde{f}(x, q)\}$$

$$= \bigcup_{x = xyz} \bigcup_{xy = xyz} \{\tilde{f}(x, q) \cap \tilde{g}(yx, yx) \cap \tilde{f}(x, q)\}$$

$$\supseteq \{\tilde{f}(x, q) \cap \tilde{g}(x, q)\} \cap \tilde{f}(x, q)$$

$$= \tilde{f}(x, q) \cap \tilde{g}(x, q)$$

$$= (\tilde{f} \cap \tilde{g})(x, q).$$

And

$$(\tilde{g} \circ \tilde{f})(x, q) = \bigcup_{x = xyz} \{\tilde{g}(xy, q) \cap \tilde{f}(x, q)\}$$

$$\supseteq \tilde{g}(x, q) \cap \tilde{f}(x, q)$$

$$= (\tilde{g} \cap \tilde{f})(x, q).$$

Similarly we can prove $(\omega \circ \nu \circ \omega)(x, q) \subseteq (\nu \gamma \omega)(x, q)$ and $(\nu \circ \omega)(x, q) \subseteq (\nu \gamma \omega)(x, q)$. Hence $\tilde{f} \cap \tilde{g} \subseteq \tilde{g} \circ \tilde{f} \circ \tilde{g} \circ \tilde{f}$ and $\omega \gamma \nu \subseteq \nu \circ \nu \gamma \circ \nu \circ \omega$. Conversely suppose that the condition holds. Let $A = (\tilde{f}, \omega)$ be a Q-cubic left bi-quasi ideal. We have $\tilde{f} \cap \tilde{f}_x \subseteq \tilde{f}_x \circ \bigcap \tilde{f} \circ \tilde{f} \circ \tilde{f}_x \circ \tilde{f}$ and $\omega \gamma \nu \subseteq \nu \circ \omega \gamma \circ \nu \circ \omega$, for every $Q$-cubic right bi-quasi ideal $A = (\tilde{f}, \omega)$ and every $Q$-cubic ideal $B = (\tilde{g}, \nu)$ of a semigroup $S$.

Corollary 37. Let $S$ be a semigroup. Then $S$ is a regular if and only if $\tilde{f} \cap \tilde{g} \subseteq \tilde{g} \circ \tilde{f} \circ \tilde{g} \circ \tilde{f}$ and $\omega \gamma \nu \subseteq \nu \circ \omega \gamma \circ \nu \circ \omega$, for every $Q$-cubic right bi-quasi ideal $A = (\tilde{f}, \omega)$ and every $Q$-cubic ideal $B = (\tilde{g}, \nu)$ of a semigroup $S$.

Corollary 38. Let $S$ be a semigroup. Then $S$ is a regular if and only if $\tilde{f} \cap \tilde{g} \subseteq \tilde{g} \circ \tilde{f} \circ \tilde{g} \circ \tilde{f}$ and $\omega \gamma \nu \subseteq \nu \circ \omega \gamma \circ \nu \circ \omega$, for every $Q$-cubic bi-quasi ideal $A = (\tilde{f}, \omega)$ and every $Q$-cubic ideal $B = (\tilde{g}, \nu)$ of a semigroup $S$. 

□
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