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A Subclass of Univalent Functions Defined by Salagean Differential Operator

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Abstract

The main aim of this research is to introduce a new class $BT_n(m, \lambda, \alpha)$ defined by Salagean differential operator involving function $\Omega(z) \in A_n$ and important results are indicated.

Keywords: Univalent function, starlike function, convex function, Bazilevic function, Salagean Operator.

1 INTRODUCTION AND DEFINITIONS

We indicate by T_n the subclass of the class of function A_n which is of the form

$$\Omega(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \qquad (\mathbf{1})$$

consisting of function which are holomorphic in the open unit disk $U = \{z \in C : |z| < 1\}$ and $H(U)$ the space of analytic function in U, $n \in N = \{1,2,3, \dots\}$.

By $S_n^*(\alpha)$ denote a subclass of A_n consisting of starlike functions of order α, $0 \leq \alpha < 1$ which satisfies

$$R\left(\frac{z\Omega'(z)}{\Omega(z)}\right) > \alpha \qquad (z \in U).$$

Furthermore, a function $\Omega(z)$ associated with T_n is said to be convex of order α in U, if and only if

$$R\left(\frac{z\Omega''(z)}{\Omega'(z)}+1\right)>\alpha \qquad (z\in U),$$

for some $0\leq\alpha<1$. We denote by $K_n(\alpha)$ the class of functions in T_n which are convex of order α in U and denote by $R_n(\alpha)$ the class of function in A_n which satisfy

$$R(\Omega'(z))>\alpha \qquad (z\in U).$$

We also known that $K_n(\alpha)\subset S_n^*(\alpha)\subset T_n$.

Let D^m be Salagean differential operator [2] ,

$$D^m\colon A_n\longrightarrow A_n,\quad n\in N,$$

$$D^0\Omega(z)=\Omega(z)$$

$$D^1\Omega(z)=D\Omega(z)=z\Omega'(z),\ldots,$$

$$D^m\Omega(z)=D(D^{m-1}\Omega(z))=z(D^{m-1}\Omega(z))'.$$

For $\Omega(z)$ given by (1)**,** then

$$D^m\Omega(z)=z+\sum_{j=n+1}^{\infty}j^m a_j\,z^j \qquad (\mathbf{2})$$

Where $m\in N_0=N\cup\{0\}=\{0,1,2,3\ldots\}$.

Lemma 1.1[**1**]

Let p be holomorphic in U with $p(0)=1$ and if

$$R\left(1+\frac{zp'(z)}{p(z)}\right)>\frac{3\alpha-1}{2\alpha} \qquad (z\in U).$$

Then $R(p(z))>\alpha$ in U and $\frac{1}{2}\leq\alpha<1$.

2 MAIN RESULTS

Definition 2.1

A function $\Omega(z) \in A_n$ is said to be a member of the class $BT_n(m, \lambda, \alpha)$ if

$$\left|\frac{D^{m+1}\Omega(z)}{z}\left(\frac{D^m\Omega(z)}{z}\right)^{\lambda} - 1\right| < 1 - \alpha,\; z \in U,\; \lambda \geq -1 \text{ and } \frac{1}{2} \leq \alpha < 1 \qquad (\mathbf{3})$$

where D^m is the Salagean differential operator. Note that inequality (2.1) *implies that*

$$R\left(\frac{D^{m+1}\Omega(z)}{z}\left(\frac{D^m\Omega(z)}{z}\right)^{\lambda}\right) > \alpha \qquad 0 \leq \alpha < 1.$$

Remark 2.2

The family $BT_n(m, \lambda, \alpha)$ is a new general class of holomorphic functions which includes several new classes of holomorphic univalent functions along with some important ones. Such as, $BT_n(0, -1, \alpha) \equiv S_n^*(\alpha)$, $BT_n(1, -1, \alpha) \equiv K_n(\alpha)$ and $BT_n(0,0, \alpha) \equiv R_n(\alpha)$. Another impressive subclass is the special case $BT_n(0, \lambda, \alpha)$ which reduces to the Bazilevic function of order α which was studied by Singh [4].

Theorem 2.3

If $\Omega(z) \in A_n$ satisfies the condition

$$R\left(\frac{D^{m+2}\Omega(z)}{D^{m+1}\Omega(z)} + \lambda\frac{D^{m+1}\Omega(z)}{D^m\Omega(z)}\right) > \lambda + \frac{3\alpha - 1}{2\alpha} \qquad (z \in U). \qquad (\mathbf{4})$$

Then $\Omega(z) \in BT_n(m, \lambda, \alpha)$.

Proof: For $z \in U$, define an analytic function $p(z)$ with $p(0) = 1$ by

$$p(z) = \frac{D^{m+1}\Omega(z)}{z}\left(\frac{D^m\Omega(z)}{z}\right)^{\lambda}$$

By simple differentiation it implies that

$$\ln p(z) = \ln\left(D^{m+1}\Omega(z)\right) - \ln(z) + \lambda\ln\left(D^m\Omega(z)\right) - \lambda\ln(z)$$

$$\frac{p'(z)}{p(z)} = \frac{\left(D^{m+1}\Omega(z)\right)'}{\left(D^{m+1}\Omega(z)\right)} - \frac{1}{z} + \lambda\frac{\left(D^m\Omega(z)\right)'}{\left(D^m\Omega(z)\right)} - \lambda\frac{1}{z}$$

Multiply through by z,

$$\frac{zp'(z)}{p(z)} = \frac{z\left(D^{m+1}\Omega(z)\right)'}{\left(D^{m+1}\Omega(z)\right)} - 1 + \lambda\frac{z\left(D^{m}\Omega(z)\right)'}{\left(D^{m}\Omega(z)\right)} - \lambda$$

$$\frac{zp'(z)}{p(z)} = \frac{z\left(D^{m+1}\Omega(z)\right)'}{\left(D^{m+1}\Omega(z)\right)} + \lambda\frac{z\left(D^{m}\Omega(z)\right)'}{\left(D^{m}\Omega(z)\right)} - 1 - \lambda$$

$$\frac{zp'(z)}{p(z)} = \frac{D^{m+2}\Omega(z)}{D^{m+1}\Omega(z)} + \lambda\frac{D^{m+1}\Omega(z)}{D^{m}\Omega(z)} - 1 - \lambda$$

So that

$$R\left(1+\frac{zp'(z)}{p(z)}\right) = R\left(1+\frac{D^{m+2}\Omega(z)}{D^{m+1}\Omega(z)} + \lambda\frac{D^{m+1}\Omega(z)}{D^{m}\Omega(z)} - 1 - \lambda\right) > \frac{3\alpha-1}{2\alpha}$$

$$R\left(1+\frac{zp'(z)}{p(z)}\right) = R\left(\frac{D^{m+2}\Omega(z)}{D^{m+1}\Omega(z)} + \lambda\frac{D^{m+1}\Omega(z)}{D^{m}\Omega(z)} - \lambda\right) > \frac{3\alpha-1}{2\alpha}$$

$$R\left(1+\frac{zp'(z)}{p(z)}\right) = R\left(\frac{D^{m+2}\Omega(z)}{D^{m+1}\Omega(z)} + \lambda\frac{D^{m+1}\Omega(z)}{D^{m}\Omega(z)}\right) > \lambda + \frac{3\alpha-1}{2\alpha}$$

Which by lemma (1.1), implies

$$R\left(\frac{D^{m+1}\Omega(z)}{z}\left(\frac{D^{m}\Omega(z)}{z}\right)^{\lambda}\right) > \alpha \qquad\qquad \frac{1}{2} \leq \alpha < 1.$$

Remark 2.4

When $n = 1$ and $m = 0$ in Theorem 2.3, we have the theorem below;

Theorem 2.5 $[\mathbf{3}]$

If $\Omega(z) \in A$ satisfies

$$R\left(\left(1+\frac{z\Omega''(z)}{f(z)}\right) + \lambda\frac{z\Omega'(z)}{f(z)}\right) > \lambda + \frac{3\alpha-1}{2\alpha}, \qquad z \in U$$

Then $R(p(z)) > \alpha$ in U and $\frac{1}{2} \leq \alpha < 1$.

We have the following interesting corollaries as a result of Theorem 2.3.

Corollary 2.6

If $\Omega(z) \in A_n$ and

$$R\left(\frac{2z\Omega''(z) + z^2\Omega'''(z)}{z\Omega''(z) + \Omega'(z)} - \frac{z\Omega''(z)}{\Omega'(z)}\right) > -\frac{1}{2}, \qquad (z \in U).$$

Then

$$R\left(\frac{z\Omega''(z)}{\Omega'(z)} + 1\right) > \frac{1}{2}.$$

That is $\Omega(z)$ is convex of order $\frac{1}{2}$.

Proof: Putting $m = 1, \lambda = -1$ and $\alpha = \frac{1}{2}$ into $(\mathbf{4})$, it implies that

$$R\left(\frac{D^3\Omega(z)}{D^2\Omega(z)} + (-1)\frac{D^2\Omega(z)}{D^1\Omega(z)}\right) > -1 + \frac{1}{2}$$

Then

$$R\left(\frac{z\Omega'(z)}{\Omega(z)} + 1\right) > \frac{1}{2}.$$

Corollary 2.7

If $\Omega(z) \in A_n$ and

$$R\left(\frac{2z^2\Omega''(z) + z^3\Omega'''(z)}{z^2\Omega''(z) + z\Omega'(z)}\right) > -\frac{1}{2}, \qquad (z \in U).$$

Then

$$R(z\Omega''(z) + \Omega'(z)) > \frac{1}{2}.$$

Proof: Putting $m = 1, \lambda = 0$ and $\alpha = \frac{1}{2}$ into $(\mathbf{4})$, it implies that

$$R\left(1 + \frac{D^3\Omega(z)}{D^2\Omega(z)} + (0)\frac{D^2\Omega(z)}{D^1\Omega(z)} - 1\right) > 0 + \frac{1}{2}$$

Then

$$R(z\Omega''(z)+\Omega'(z))>\frac{1}{2}.$$

Corollary 2.8

If $\Omega(z)\in A_n$ and

$$R\left(\frac{z\Omega''(z)}{\Omega'(z)}-\frac{z\Omega'(z)}{\Omega(z)}\right)>-\frac{3}{2},\qquad (z\in U).$$

Then

$$R\left(\frac{z\Omega'(z)}{\Omega(z)}\right)>\frac{1}{2}.$$

That is $\Omega(z)$ is starlike of order $\frac{1}{2}$.

Proof: Putting $m=0,\lambda=-1$ and $\alpha=\frac{1}{2}$ into $(\mathbf{4})$, we get

$$R\left(\frac{D^2\Omega(z)}{D^1\Omega(z)}+(-1)\frac{D^1\Omega(z)}{D^0\Omega(z)}\right)>(-1)+\frac{1}{2}$$

Then

$$R\left(\frac{z\Omega'(z)}{\Omega(z)}\right)>\frac{1}{2}.$$

Corollary 2.9

If $\Omega(z)\in A_n$ and

$$R\left(1+\frac{z\Omega'(z)}{\Omega(z)}\right)>\frac{1}{2},\qquad (z\in U).$$

Then

$$R(\Omega'(z))>\frac{1}{2}.$$

Hence, if the function $\Omega(z)$ is convex of order $\frac{1}{2}$ then $\Omega(z)\in BT_n(0,0,\alpha)\equiv R_n\left(\frac{1}{2}\right)$.

Proof: Putting $m = 0, \lambda = 0$ and $\alpha = \frac{1}{2}$ into $(\mathbf{4})$, it implies that

$$R\left(\frac{D^2\Omega(z)}{D^1\Omega(z)} + (0)\frac{D^1\Omega(z)}{D^0\Omega(z)}\right) > (0) + \frac{1}{2}$$

Then

$$R(\Omega'(z)) > \frac{1}{2}.$$

Corollary 2.10

If $\Omega(z) \in A_n$ and

$$R\left(2\left(\frac{z\Omega''(z)}{\Omega'(z)} + 1\right) - \frac{z\Omega'(z)}{\Omega(z)}\right) > 0, \qquad (z \in U).$$

Then

$$R\left(\frac{z^{\frac{1}{2}}\Omega'(z)}{\Omega^{\frac{1}{2}}(z)}\right) > \frac{1}{2}.$$

That is $\Omega(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{1}{2}$ in U.

Proof: Putting $m = 0, \lambda = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$ into $(\mathbf{4})$, we get

$$R\left(\frac{D^2\Omega(z)}{D^1\Omega(z)} + \left(-\frac{1}{2}\right)\frac{D^1\Omega(z)}{D^0\Omega(z)}\right) > -\frac{1}{2} + \frac{1}{2}$$

Then

$$R\left(\frac{z^{\frac{1}{2}}\Omega'(z)}{\Omega^{\frac{1}{2}}(z)}\right) > \frac{1}{2}.$$

Corollary 2.11

If $\Omega(z) \in A_n$ and

$$R\left(2\left(\frac{z\Omega''(z)}{\Omega'(z)}+1\right)+\frac{z\Omega'(z)}{\Omega(z)}\right) > 1, \qquad (z \in U).$$

Then

$$R\left(\frac{\Omega^{\frac{1}{2}}\Omega'(z)}{z^{\frac{1}{2}}(z)}\right) > \frac{1}{2}.$$

That is $\Omega(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{3}{2}$ in U.

Proof: Putting $m = 0, \lambda = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ into $(\mathbf{4})$, it implies that

$$R\left(\frac{D^2\Omega(z)}{D^1\Omega(z)}+\left(\frac{1}{2}\right)\frac{D^1\Omega(z)}{D^0\Omega(z)}\right) > \frac{1}{2}+\frac{1}{2}$$

Then

$$R\left(\frac{\Omega^{\frac{1}{2}}\Omega'(z)}{z^{\frac{1}{2}}(z)}\right) > \frac{1}{2}.$$

CONCLUSION

In this present paper, we describe new subclass of univalent functions applying Salagean differential operator, and some of its properties were created. The obtained results include the properties of certain subclasses of univalent functions.

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