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A Companion of Weighted Ostrowski's type Inequality for Functions whose 1^{st} Derivatives are Bounded with Applications

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Abstract

In this article, we would get generalisation of companion of Ostrowski's type integral inequality involving weights for differentiable functions whose 1^{st} derivatives are bounded. The present article recaptures the results of M. W. Alomari's article. Application is also deduced for numerical integration.

Keywords and phrases: Ostrowski's inequality, Differentiable mapping, Bounded mapping, Numerical Integration

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1. INTRODUCTION

In 1938, A. M. Ostrowski gave an inequality in his article [10]. Now-a-days this inequality is called Ostrowski inequality and this result had obtained by applying the Montgomery identity.

Here, we present an inequality from article [4] that is given below. Throughout the article $K \subset \mathbb{R}$ and K^o is the interior of the interval K.

Proposition 1.1. *Suppose $\rho : K \to \mathbb{R}$ is a differentiable function in the interval K^o such that $\rho' \in L[j,k]$, where $j,k \in K$ and $j < k$. If $|\rho'(\theta)| \leq \mathfrak{M} \;\forall\; \theta \in (j,k)$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left|\rho(\theta) - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it can not be replaced by the smaller one.

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Gruŝs inequality [7, 9].

Proposition 1.2. *Let $\rho, g : [j,k] \to \mathbb{R}$ be both integrable functions such that $m_1 \leq \rho(\dagger) \leq M_1$ and $m_2 \leq g(\dagger) \leq M_2 \;\forall\; \dagger \in [j,k]$, where m_1, M_1, m_2, M_2 are real constants. Then*

$$\left|\frac{1}{k-j}\int_j^k \rho(\dagger)g(\dagger)d\dagger - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger \cdot \frac{1}{k-j}\int_j^k g(\dagger)d\dagger\right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2). \tag{1.2}$$

In [6], S. S. Dragomir has derived the following companion of the Ostrowski inequality.

Proposition 1.3. *Let $\rho : K \to \mathbb{R}$ be an absolutely continuous function on $[j,k]$. Then we have the inequalities*

$$\left|\frac{\rho(\theta) + \rho(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \begin{cases} \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right](k-j)\|\rho'\|_\infty, \quad \rho' \in L_\infty[j,k], \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\theta - j}{k-j}\right)^{q+1} + \left(\frac{\frac{j+k}{2} - \theta}{k-j}\right)^{q+1}\right]^{\frac{1}{q}}(k-j)^{\frac{1}{q}}\|\rho'\|_{[j,k],p}\,, \\ \qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ and\ \rho' \in L_p[j,k], \\ \left[\frac{1}{4} + \left|\frac{\theta - \frac{3j+k}{4}}{k-j}\right|\right]\|\rho'\|_{[j,k],1}\,, \end{cases} \tag{1.3}$$

$\forall\; \theta \in [j, \frac{j+k}{2}]$.

In 2011, M. W. Alomari has proved the following result about a companion inequality for differentiable functions whose darivatives are bounded (see [1]).

Proposition 1.4. *Let $\rho : K \to \mathbb{R}$ be a differentiable function in the interval K^o and let $j, k \in K$ with $j < k$. If $\rho' \in L^1[j,k]$ and $m_2 \leq \rho'(\theta) \leq M_2$, for all $\theta \in [j,k]$, then the following inequality holds*

$$\left|\frac{\rho(\theta)+\rho(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq (k-j)\left[\frac{1}{16} + \left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right](M_2 - m_2), \quad (1.4)$$

$\forall\, \theta \in [j, \frac{j+k}{2}]$.

In 2002, S. S. Dragomir [5] established some inequalities for this companion for functions of bounded variation. In 2009, Z. Liu [8] introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. In 2009, Barnett *et. al* [3] have derived some companions for Ostrowski inequality and the generalised trapezoid inequality. In 2012, M. W. Alomari [2] obtained a companion inequality of Ostrowski's type using $Grüss$ result with applications.

In the present article we would prove a companion of weighted Ostrowsk's type inequality by applying $Grüss$ result and then we would give its applications.

2. GENERALISATION OF COMPANION OF OSTROWSKI'S TYPE INEQUALITY

Under present section we would give our results about companion of Ostrowski's type inequality which are as follow:

Theorem 2.1. *Let $\rho : [j,k] \to \mathbb{R}$ be a differentiable function in the interval (j,k) and $j < k$ and $w : [j,k] \to \mathbb{R}$ is a integrable function. If $\rho' \in L^1[j,k]$ and $m_2 \leq \rho'(\dagger) \leq M_2$, for all $\dagger \in [j,k]$, then*

$$\left|\rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k \rho(\dagger)w(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2 - m_2) \quad (2.1)$$

holds $\forall\, \theta \in [j, \frac{j+k}{2}]$.

Proof. For the sake of proof we state the weighted kernel as;

$$P(\theta,\dagger)=\begin{cases}\displaystyle\int_j^{\dagger} w(u)du, & \text{if} \quad \dagger\in[j,\theta],\\[2ex] \displaystyle\int_{\frac{j+k}{2}}^{\dagger} w(u)du, & \text{if} \quad \dagger\in(\theta,j+k-\theta],\\[2ex] \displaystyle\int_k^{\dagger} w(u)du, & \text{if} \quad \dagger\in(j+k-\theta,k],\end{cases}$$

$\forall\, \theta\in[j,\frac{j+k}{2}]$.

Applying by parts formula of integration, obtain

$$\begin{aligned}\int_j^k P(\theta,\dagger)\rho'(\dagger)d\dagger &= \rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger+\rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger\\ &- \int_j^k \rho(\dagger)w(\dagger)d\dagger. \end{aligned}\tag{2.2}$$

It is clear that $\forall\, \dagger\in[j,k]$ and $\theta\in[j,\frac{j+k}{2}]$, we have

$$\theta-\frac{j+k}{2}\le P(\theta,\dagger)\le\theta-j\,.$$

Applying Proposition 1.2 to the mappings $P(\theta,\cdot)$ and $\rho'(\cdot)$, we obtain

$$\begin{aligned}&\left|\int_j^k P(\theta,\dagger)\rho'(\dagger)d\dagger-\int_j^k P(\theta,\dagger)d\dagger\cdot\frac{1}{k-j}\int_j^k\rho'(\dagger)d\dagger\right|\\ &\le\frac{1}{4}\left(\theta-j-\left(\theta-\frac{j+k}{2}\right)\right)(M_2-m_2)=\frac{1}{8}(k-j)(M_2-m_2),\end{aligned}\tag{2.3}$$

$\forall\, \theta\in[j,\frac{j+k}{2}]$. Since $\int_j^k P(\theta,\dagger)d\dagger=0$, then (2.3) implies

$$\left|\int_j^k P(\theta,\dagger)\rho'(\dagger)d\dagger\right|\le\frac{1}{8}(k-j)(M_2-m_2).\tag{2.4}$$

Finally, we obtain desired result (2.1) from (2.4). □

Remark 2.2. If put $w=\frac{1}{k-j}$ in Theorem 2.1, then we recapture the Theorem 5 of [2].

Corollary 2.3. *In the inequality* (2.1)*, select*
(i) $\theta=j$*, obtain*

$$\left|\rho(j)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger+\rho(k)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger-\int_j^k w(\dagger)\rho(\dagger)d\dagger\right|\le\frac{1}{8}(k-j)(M_2-m_2)\tag{2.5}$$

(ii) $\theta = \frac{j+k}{2}$, *obtain*

$$\left|\rho(\frac{j+k}{2})\int_j^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2), \qquad (2.6)$$

(iii) $\theta = \frac{3j+k}{4}$, *obtain*

$$\left|\rho(\frac{3j+k}{4})\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(\frac{j+3k}{4})\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2), \qquad (2.7)$$

(iv) $\theta = \frac{2j+k}{3}$, *obtain*

$$\left|\rho(\frac{2j+k}{3})\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(\frac{j+2k}{3})\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2). \qquad (2.8)$$

In the following we present special case of (iv) of Corollary 2.3.

Special Case: If put $w = \frac{1}{k-j}$ in (iv) of Corollary 2.3, then we get

$$\left|\frac{\rho(\frac{2j+k}{3}) + \rho(\frac{j+2k}{3})}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2).$$

Remark 2.4. (i) By putting $w = \frac{1}{k-j}$ in (i) of Corollary 2.3, we recapture the Corollary 1(1) of [2].

(ii) By putting $w = \frac{1}{k-j}$ in (ii) of Corollary 2.3, we recapture the Corollary 1(3) of [2].

(iii) By putting $w = \frac{1}{k-j}$ in (iii) of Corollary 2.3, we recapture the Corollary 1(2) of [2].

Ostrowski's type inequality can be defined in the form of following corollary.

Corollary 2.5. *Let the assumptions of Theorem* 2.1 *be valid. Further, if* ρ *is symmetric about the* θ*-axis, i.e.,* $\rho(j+k-\theta) = \rho(\theta)$, *then*

$$\left|\rho(\theta)\int_j^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2) \qquad (2.9)$$

holds $\forall\ \theta \in [j, \frac{j+k}{2}]$. *For instance, select* $\theta = j$, *we have*

$$\left|\rho(j)\int_j^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \leq \frac{1}{8}(k-j)(M_2-m_2). \qquad (2.10)$$

Remark 2.6. By putting $w = \frac{1}{k-j}$ in Corollary 2.5, we recapture the Corollary 2 of [2].

3. APPLICATION TO NUMERICAL INTEGRATION

Let $K_n : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ be a division of the interval $[j,k]$ and $h_i = \theta_{i+1} - \theta_i$, $(i = 0, 1, 2, \cdots, n-1)$.

Consider the quadrature formula

$$Q_n(K_n,\rho) := \sum_{i=0}^{n-1}\left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger + \rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right]. \tag{3.1}$$

We give following result.

Theorem 3.1. *Let $\rho : K \to \mathbb{R}$ be a differentiable function in the interval K^o and $w : [j,k] \to \mathbb{R}$ is a integrable function, where $j, k \in K$ with $j < k$. If $\rho' \in L^1[j,k]$ and $m_2 \leq \rho'(\theta) \leq M_2$, for all $\theta \in [j,k]$, then the following holds*

$$\int_j^k w(\dagger)\rho(\dagger)d\dagger = Q_n(K_n,\rho) + R_n(K_n,\rho), \tag{3.2}$$

where $Q_n(K_n,\rho)$ is stated as above and the following remainder $R_n(K_n,\rho)$ satisfies the estimates

$$|R_n(K_n,\rho)| \leq \frac{1}{8}(M_2 - m_2)h_i. \tag{3.3}$$

Proof. Applying inequality (2.7) on the intervals $[\theta_i, \theta_{i+1}]$, we get

$$\begin{aligned} R_i(K_i,\rho) &= \int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger \right. \\ &+ \left. \rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right]. \end{aligned} \tag{3.4}$$

Summing (3.4) over i from 0 to $n-1$, then

$$\begin{aligned} R_n(K_n,\rho) &= \sum_{i=0}^{n-1}\int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \sum_{i=0}^{n-1}\left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger \right. \\ &+ \left. \rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right], \end{aligned}$$

which follows the form of (2.7), i.e.

$$\begin{aligned}|R_n(K_n,\rho)| &= \left|\sum_{i=0}^{n-1}\int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \sum_{i=0}^{n-1}\left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger\right.\right. \\ &+ \left.\left.\rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right]\right| \\ &\leq \frac{1}{8}(M_2-m_2)\sum_{i=0}^{n-1} h_i.\end{aligned}$$

This completes the required proof. □

Remark 3.2. By putting $w=\frac{1}{k-j}$ in Theorem 3.1, we recapture the result of Theorem 6 of [2].

4. CONCLUSION

In this article our target was to generalise the results of [2]. We have obtained generalisation of companion of Ostrowski's type integral inequality involving weights for differentiable functions whose 1^{st} derivatives are bounded. By applying suitable substitutions we have recaptured the results of M. W. Alomari's article. Moreover, we have given applications to numerical integration.

REFERENCES

[1] M. W. Alomari, A companion of Ostrowski's inequality with applications, *Trans. J. Math. Mech.*, **3** (1) (2011), 9—14.

[2] M. W. Alomari, A companion of Ostrowski's inequality for mappings whose first derivatives are bounded and applications in numerical integration, *Kragujevac Journal of Mathematics*, **36** 1 (2012), 77—82.

[3] N.S. Barnett, S.S. Dragomir and I. Gomma, A companion for the Ostrowski and the generalised trapezoid inequalities, *J. Mathematical and Computer Modelling*, 50 (2009), 179-–187.

[4] S. S. Dragomir and T. M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academics Publishers, Dordrecht, 2002.

[5] S.S. DRAGOMIR, A companion of Ostrowski's inequality for functions of bounded variation and applications, *RGMIA Preprint*, Vol. 5 Supp. (2002) article No. 28. [http://ajmaa.org/RGMIA/papers/v5e/COIFBVApp.pdf]

[6] S. S. DRAGOMIR, Some companions of Ostrowski's inequality for absolutely continuous functions and applications, *Bull. Korean Math. Soc.*, **42** (2005), No. 2, pp. 213—230.

[7] G. GRÜSS, Über das Maximum des absoluten Betrages von, *Math. Z.*, **39** (1) (1935), 215–226.

[8] Z. LIU, Some companions of an Ostrowski type inequality and applications, *J. Ineq. Pure & Appl. Math.*, Volume 10 (2009), Issue 2, Article 52, 12 pp.

[9] D. S. MITRINOVIĆ , J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[10] A. M. OSTROWSKI, *Ü*ber die Absolutabweichung einer Differentiebaren Funktion von Ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.