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Interior Bias Odd Domination of a Graph

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Abstract

This paper aims at the study of a new concept Interior Bias Odd Domination of graphs. An interior dominating set D of a non complete connected graph $G = (V, E)$ is said to be an interior bias odd dominating set if every vertex of V/D has odd number of neighbours in D and is denoted by $ibod\ set$. The interior bias odd domination number, $\gamma_{ibod}(G)$ is the minimum cardinality taken over all $ibod\ sets$ of G. Some bounds for $\gamma_{ibod}(G)$, exact values for some particular classes of graphs and the relation between interior bias odd domination number and some other domination parameters are presented in this paper.

Keywords: bias odd domination, interior bias odd domination, subdivision graphs.

AMS subject classification: 05C69

1. INTRODUCTION

Let $G = \big(V(G), E(G)\big)$ be a simple graph and let $v \in V(G)$. The open neighbourhood and the closed neighbourhood of v are denoted by $N(v) = \{u \in V(G): uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ respectively. The degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges incident with v in G. A vertex $v \in V(G)$ is called pendant vertex if $deg(v) = 1$. The vertex adjacent to pendant vertex is called support vertex. The distance between two vertices u and v, denoted $d(u, v)$ is the length of the shortest $u - v$ path. The eccentricity $e(v)$ of a vertex v of a connected graph G is

$\max_{u \in V(G)} d(u,v)$. The radius of a graph G is denoted by $rad\,(G)$ and is defined as $rad\,(G) = \min_{v \in V(G)} e(v)$. A vertex v is a central vertex if $e(v) = rad\,(G)$. The set of all central vertices of the graph G is the center of G. The floor function of a real number x is the greatest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$.The ceiling function of a real number x is the lowest integer greater than or equal to x and it is denoted by $\lceil x \rceil$. A set $D \subseteq V(G)$ is a dominating set of G, if every vertex in $V(G)\backslash D$ is adjacent to some vertex in D. The dominating set D is a minimal dominating set if no proper subset D' of D is a dominating set. The minimum cardinality of the dominating set is known as the domination number and is denoted by $\gamma(G)$. Let x and z be two distinct vertices of G. A vertex y distinct from x and z is said to lie between x and z if $d(x,z) = d(x,y) + d(y,z)$. A vertex v is an interior vertex if for every $u \in N(v)$ there exists a vertex $w \in N(v)$ such that v lies between u and w. The set of all interior vertices of the graph G is denoted by $Int\,(G)$. A dominating set $D \subseteq V(G)$ is an interior dominating set if every $v \in D$ is a vertex of $Int\,(G)$, in other words for every $v \in D, |N(v)| \geq 2$ and for all $x \in N(v)$ there exists a vertex $y \in N(v)$ such that $d(x,y) = d(x,v) + d(v,y)$. The interior domination number of a graph G, denoted by $\gamma_{Id}(G)$ is defined as the minimum cardinality of the interior dominating set and the corresponding set is denoted as γ_{Id} set. A dominating set $D \subseteq V(G)$ is said to be a bias odd dominating set if every vertex in $V(G)\backslash D$ has odd number of neighbours in D. The minimum cardinality of the bias odd dominating set is denoted by $\gamma_{bo}(G)$. A bias odd dominating set with the cardinality $\gamma_{bo}(G)$ is called as γ_{bo} set. An interior dominating set $C \subseteq V(G)$ is called a connected interior dominating set if the induced subgraph $\langle C \rangle$ is connected. The minimum cardinality of a connected interior dominating set of G, denoted by $\gamma_{cId}(G)$, is called the connected interior domination number. A connected interior dominating set of cardinality $\gamma_{cId}(G)$ is called a $\gamma_{cId}\ set$ of G. An interior dominating set $D \subseteq V(G)$ in which for every vertex $v \in V(G)\backslash D, |N(v) \cap D| \equiv 1(mod\ 2)$ is said to be an interior bias odd dominating set and is denoted by $ibod\ set$. The minimum cardinality of the $ibod\ set$ of graph G is called interior bias odd domination number and is denoted by $\gamma_{ibod}(G)$. Bistar $B_{\boldsymbol{m,n}}$ is the graph obtained by connecting the central vertices of star graphs $K_{1,m}$ and $K_{1,n}$ by an edge. The wheel graph W_n is obtained by joining the cycle C_n to K_1. In symbols, $W_n = C_n + K_1$. In W_n, the vertices of the cycle are called the rim vertices and the vertex of K_1 is called as apex vertex. A helm graph denoted by H_n is a graph obtained by attaching a single edge and a vertex to each vertex of the outer circuit of a wheel graph W_n. The prism graph Y_n is obtained from the graph Cartesian product of the cycle C_n and the path P_2 i.e.$Y_n = C_n \square P_2$. The ladder graph L_n is obtained from the graph Cartesian product of the path P_n and path P_2. In symbols, $L_n = P_n \square P_2$. The graph $K_m(P_n)$ is obtained by connecting a complete graph K_m to the path P_n with a bridge. $C_m(P_n)$ is the graph obtained by connecting a cycle C_m

to the path P_n with a bridge. For a graph G, the split graph is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The split graph of a graph G is denoted as $spl(G)$. A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge by a path uwv. The graph obtained from G by subdividing every edge e of G exactly once is called the subdivision graph of G and is denoted by $S(G)$. The concept of interior domination was introduced by Anto Kinsley and Caroline [5]. Connected interior domination was introduced by Casinillo, Lagumbay and Abad [7].

2. BIAS ODD DOMINATION OF GRAPHS

Definition 2.1:

A set $D \subseteq V(G)$ is said to be a **bias odd dominating set** if for every $v \in V(G)\backslash D, |N(v) \cap D| \equiv 1 \pmod 2$. The minimum cardinality of the bias odd dominating set of G is the bias odd domination number and is denoted as $\gamma_{bo}(G)$. The bias odd dominating set with cardinality $\gamma_{bo}(G)$ is called as $\gamma_{bo}\ set$. A **Weak Odd Dominated set** (***WOD set***) is a set $B \subseteq V(G)$ such that there exists $D \subseteq V(G)\backslash B$ with $B \subseteq Odd(D) = \{v \in V(G)\backslash D: |N(v) \cap D| \equiv 1 (mod\ 2)\}$. This concept was introduced by Sylvain Gravier & Co in [3]. The bias odd domination number can be obtained from $\zeta(G)$ which is the cardinality of the largest $WOD\ set$. For any graph $G = (V(G), E(G))$ of order n, $\zeta(G) = \max_{B\ WOD} |B| = \max_{D \subseteq V(G)} |Odd\ (D)|$. By correlating both definitions, it is obvious that $\gamma_{bo}(G) = n - \zeta(G)$.

Example 2.2: The bias odd domination of the graph in figure 1 is given.

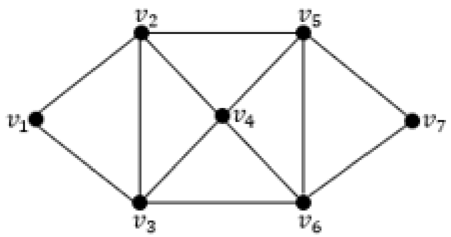

Figure 1: Graph G

For the graph G, $D = \{v_2, v_7\}$ is the minimum dominating set and so $\gamma(G) = 2$.

But $D_1 = \{v_2, v_4, v_6\}$ is the minimum bias odd dominating set. Hence $\gamma_{bo}(G) = 3$.

Theorem 2.3: For any graph G of order n, $\frac{n}{\Delta+1} \leq \gamma_{bo}(G) \leq n - \Delta$.

Proof: In [3] it is proved that the largest $WOD\ set$ of a graph G of order n and maximum degree Δ satisfies $\Delta \leq \zeta(G) \leq \frac{n.\Delta}{\Delta+1}$. Therefore, $n-\Delta \geq n-\zeta(G) \geq n-\frac{n.\Delta}{\Delta+1}$. Since $\gamma_{bo}(G) = n-\zeta(G)$, we have $n-\Delta \geq \gamma_{bo}(G) \geq \frac{n}{\Delta+1}$. Hence $\frac{n}{\Delta+1} \leq \gamma_{bo}(G) \leq n-\Delta$.

Note 2.4: For any graph $G, \gamma(G) \leq \gamma_{bo}(G)$, since every bias odd dominating set is a dominating set.

Theorem 2.5: For any connected graph $G, \gamma_{bo}(G) = 1$ if and only if $\gamma(G) = 1$.

Proof: Assume $\gamma_{bo}(G) = 1$ then there exists a vertex which dominates every vertex of G. Hence $\gamma(G) = 1$. Conversely assume $\gamma(G) = 1$. Let $D = \{x\}$ where $x \in V(G)$ be a minimum dominating set. Therefore every vertex in $V(G)\backslash\{x\}$ has exactly one neighbour in D. Thus $\gamma_{bo}(G) = 1$.

Bias Odd Domination number for some graphs 2.6:

(i) $\gamma_{bo}(K_n) = 1$

(ii) $\gamma_{bo}(K_{m,n}) = 2$

(iii) $\gamma_{bo}(P_n) = \left\lceil \frac{n}{3} \right\rceil$ for n$\geq$ 2

(iv) $\gamma_{bo}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil & otherwise \end{cases}$

(v) $\gamma_{bo}(W_n) = 1$

(vi) $\gamma_{bo}(H_n) = \begin{cases} n & if\ n\ is\ odd \\ n+1 & if\ n\ is\ even \end{cases}$

(vii) $\gamma_{bo}(spl(P_n)) = \begin{cases} \gamma + 1 & if\ n = 2m+1, m\ even \\ \gamma & otherwise \end{cases}$

(viii) $\gamma_{bo}(C_m(P_n)) = \begin{cases} \left\lfloor \frac{m+n+1}{3} \right\rfloor & if\ m \equiv 0(mod\ 3) \\ \left\lfloor \frac{m+n+2}{3} \right\rfloor & if\ m \equiv 1(mod\ 3) \\ \left\lfloor \frac{m+n+1}{3} \right\rfloor + 1 & if\ m \equiv 2(mod\ 3) \end{cases}$

3. INTERIOR BIAS ODD DOMINATION OF GRAPHS

Definition 3.1:

An interior dominating set $D \subseteq V(G)$ in which for every $v \in V(G)\backslash D, |N(v) \cap D| \equiv$

$1(mod\ 2)$ is said to be an interior bias odd dominating set and is denoted as $ibod\ set$. The minimum cardinality of the $ibod\ set$ of graph G is called **interior bias odd domination number** and is denoted by $\gamma_{ibod}\,(G)$.

Example 3.2: The interior bias odd domination of the graph in figure 2 is given.

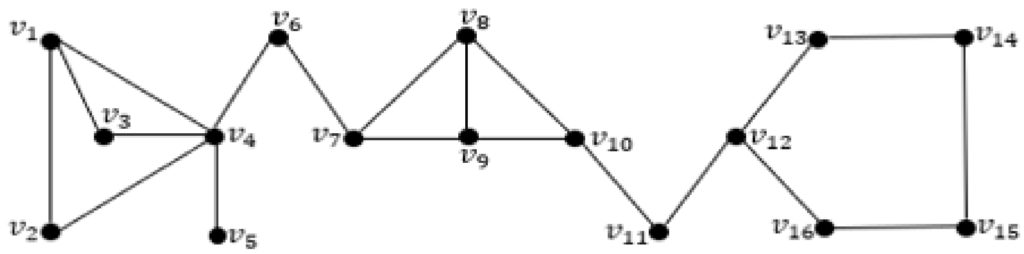

Figure 2: Graph H

$Int\ H = \{v_4, v_6, v_7, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$. $D = \{v_4, v_9, v_{12}, v_{15}\}$ is the minimum dominating set. Hence $\gamma(H) = 4$. But D is not an $ibod\ set$. The set $D_1 = \{v_4, v_6, v_{10}, v_{11}, v_{12}, v_{13}, v_{16}\}$ is an $ibod\ set$ of H but not minimum. $D_2 = \{v_4, v_6, v_7, v_{11}, v_{13}, v_{16}\}$ is the minimum $ibod\ set$. Hence $\gamma_{ibod}\,(H) = 6$.

Theorem 3.3: Let G be a connected graph of order $n \geq 3$, then $\gamma_{ibod}\,(G) = 1 \Leftrightarrow G = K_1 + H$ for every H with $\gamma(H) \neq 1$.

Proof: Let $v_1, v_2, \dots\dots, v_n$ be the vertices of graph H. Join $K_1 = \{u\}$ to H. Let $G = K_1 + H$. Then $uv_k \in E(G)\ \forall\ k = 1\ to\ n$. Assume $\gamma_{ibod}\,(G) = 1$. To prove $\gamma(H) \neq 1$. Suppose $\gamma(H) = 1$ then there exists a vertex $v_i \in V(H) \ni: v_iv_j \in E(G)\ \forall\ j \neq i$. Then the vertex u is non interior since $d(v_i, v_j) \neq 2, \forall\ j \neq i$ and similarly we can prove that the vertex v_i is non interior. Since $d(v_i, v_j) = 1, \forall\ j \neq i$ and $d(u, v_k) = 1\ \forall\ k = 1\ to\ n$ every vertex of $V(G)\backslash\{v_i, u\}$ is non interior. Thus $Int(G) = \phi$ and $ibod\ set$ doesnot exist for G which is a contradiction to the assumption. Hence $\gamma(H) \neq 1$.

Conversely assume $\gamma(H) \neq 1$. Then for every vertex $v_k \in V(H)$ there exist some vertex v_j such that $v_kv_j \notin E(H)$. Clearly $\{u\}$ is an interior vertex and dominates every vertex of G. Hence $\gamma(K_1 + H) = 1$. Since $\gamma(G) = 1$ implies $\gamma_{bo}\,(G) = 1, D = \{u\}$ forms an $ibod\ set$. Thus $\gamma_{ibod}\,(G) = 1$. Hence the proof.

Note 3.4: The existence of an $ibod\ set$ for every connected nontrivial graph is uncertain.

Example 3.5: A graph having both γ_{Id} set and γ_{bo} set but no $ibod$ set is shown in figure 3.

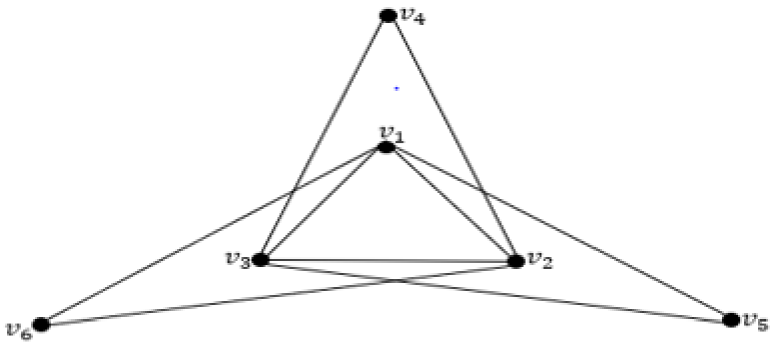

Figure 3: Split graph of C_3 $(spl(C_3))$

$Int\ (spl(C_3)) = \{v_1, v_2, v_3\}$. $D_1 = \{v_1, v_2\}$ is a minimum interior dominating set. Hence $\gamma_{Id}\left(spl(C_3)\right) = 2$. The set $D_2 = \{v_1, v_4, v_5, v_6\}$ is a minimum bias odd dominating set. Thus $\gamma_{bo}\left(spl(C_3)\right) = 4$. But there is no set $D \subseteq\ Int\ (spl(C_3)) \ni: D$ is an $ibod\ set$.

The following theorem characterize some graphs for which an $ibod\ set$ doesn't exist.

Theorem 3.6: An $ibod\ set$ does not exists for any connected graph G of order $n \geq 3$ with at least two vertices of degree $n - 1$.

Proof: Let G be a connected graph of order $n \geq 3$ with at least two vertices of degree $n - 1$. Let A be the set of all vertices of graph G with degree $n - 1$. Any vertex of $V(G) \backslash A$ is non interior since every vertex of A is adjacent to every other vertices of G. Let $x \in A$. If x is interior, for every $y \in N(x)$ where $y \in A$ there exists a vertex $z \in N(x)$ such that $d(y, z) = d(y, x) + d(x, z)$. Then $d(y, z) = 2$ which is a contradiction to $y \in A$. Therefore $x \in A$ is non interior. Since $x \in A$ is arbitrary, any vertex of A is non interior. Hence $Int(G) = \phi$. Thus interior bias odd domination doesn't exist.

Note 3.7: $ibod\ set$ does not exist for Complete graph because K_n does not contain interior points.

It is assumed that all connected nontrivial graphs considered in the upcoming theorems have nonempty interior sets which admits interior bias odd domination.

Theorem 3.8: For a non-complete connected graph G, $\gamma(G) \leq \gamma_{Id}\,(G) \leq \gamma_{ibod}\,(G)$.

Proof: Let D_{Id} be the minimum interior dominating set. Then $\gamma_{Id}\,(G) = |D_{Id}|$. Let D be the minimum $ibod\ set$ of graph G. If every member of $V \backslash D_{Id}$ has odd number of neighbours in D_{Id} then $|D_{Id}| = |D|$ otherwise $|D_{Id}| < |D|$. Hence $|D_{Id}| \le |D|$ and so $\gamma_{Id}(G) \le \gamma_{ibod}(G)$. Also we have $\gamma(G) \le \gamma_{Id}(G)$ and hence $\gamma(G) \le \gamma_{ibod}(G)$. Therefore $\gamma(G) \le \gamma_{Id}(G) \le \gamma_{ibod}(G)$.

Note 3.9: For any connected graph of order $n \ge 3, 1 \le \gamma_{ibod}(G) \le |Int\ G|$.

The theorem 3.3 characterizes all connected graphs G of order $n \ge 3$, that attain the lower bound of $\gamma_{ibod}(G)$. Every vertex of an $ibod\ set$ of G belongs to $Int(G)$. Then for any graph G, $ibod\ set \subseteq Int(G)$ and so $\gamma_{ibod}(G) \le |Int\ G|$. Hence $1 \le \gamma_{ibod}(G) \le |Int\ G|$.

Note 3.10: If D is an $ibod\ set$ of a connected graph, $n \ge 3$, then every support vertex is in D.

Theorem 3.11: In a tree T of order $n \ge 3$, $Int\ T$ forms an $ibod\ set$.

Proof: Let $B = \{ v \in V(T) : deg(v) = 1\}$. Consider $D = V(T) \setminus B$. The support vertices belonging to D dominates the corresponding pendant vertices in B and at most itself. Hence D is a dominating set. For any tree, since every non pendant vertex is interior $D = Int\ T$. Also every $v \in B$ has exactly one neighbour in D. Hence D forms an $ibod\ set$ of T but need not be a minimum $ibod\ set$. Thus $\gamma_{ibod}(T)$ is either equal to $|D|$ or less than $|D|$. Therefore $\gamma_{ibod}(T) \le |V(T) \setminus B|$.

Theorem 3.12: If T is a tree with order $n \ge 3$, then $\frac{n-l+2}{3} \le \gamma_{ibod}(T) \le n - l$, where l denotes the number of pendant vertices of T.

Proof: For any tree T with order $n \ge 3$, Delavina [2] proved $\gamma(T) \ge \frac{n-l+2}{3}$, l be the pendant vertices. Since any tree of order $n \ge 3$ is a non complete connected graph by theorem 3.8,

$\gamma(T) \le \gamma_{ibod}(T)$. Hence $\frac{n-l+2}{3} \le \gamma_{ibod}(T)$. By the previous theorem, $\gamma_{ibod}(T) \le n - l$. Thus $\frac{n-l+2}{3} \le \gamma_{ibod}(T) \le n - l$. Moreover, the upper bound is sharp. If every vertex of $Int(T)$ is a support vertex then $\gamma_{ibod}(T) = n - l$. And the lower bound is attained for $P_n, n \equiv 0\ mod\ 3$.

Result 3.13: For any tree T with x possible cut vertices, $\frac{x+2}{3} \le \gamma_{ibod}(T) \le x$.

Proof: Since for any tree of order $n \ge 3$ and with l pendant vertices, $n - l(T) = x$ the result follows from the previous theorem.

Theorem 3.14: For any tree T of order $n \ge 3$, $\gamma_{ibod}(T) \le \gamma_{cId}(T)$.

Proof: Let the number of pendant vertices be l. If C is a $\gamma_{cId}\ set$ of T, then every cut vertex is in C. Hence $\gamma_{cId}(T) = n - l$. Since $\gamma_{ibod}(T) \le n - l$ it is quite obvious that

$\gamma_{ibod}(T) \leq \gamma_{cId}\,(T)$.

Corollary 3.15: If T is a tree of order $n \geq 3$, then $\gamma_{ibod}(T) \leq n-2$.

Proof: For any tree T, the number of pendant vertices $l \geq 2$ and so $n-l \leq n-2$. Hence by the previous theorem, $\gamma_{ibod}(T) \leq n-2$.

Theorem 3.16: Let G be a nontrivial connected graph and H be any graph. Then for the Corona product of G and H, $\gamma_{ibod}(G \circ H) = |V(G)|$.

Proof: In [8] it is proved that for any nontrivial connected graph G and for any graph H, $Int\ (G \circ H) = V(G)$. For any $S \subset V(G), S$ is not a dominating set of $G \circ H$. Hence $\gamma_{Id}\ set$ of $G \circ H$ is $V(G)$. And for every $v \in V(G \circ H)\backslash V(G), |N(v) \cap V(G)| = 1$. Thus $\gamma_{Id}\ set$ itself forms the minimum $ibod\ set$. Hence $\gamma_{ibod}(G \circ H) = |V(G)|$.

Result 3.17: For the Cartesian product of any nontrivial connected graphs G and H, $\gamma_{ibod}(G\square H) = \gamma_{bo}(G\square H)$.

Proof: By the lemma proved in [8] for any nontrivial connected graphs G and H,

$Int\ (G\square H) = V(G\square H)$. Then $D \subseteq V(G\square H)$ is an $ibod\ set$ if and only if D is a bias odd dominating set of $G\square H$. Hence $\gamma_{ibod}(G\square H) = \gamma_{bo}(G\square H)$.

Theorem 3.18: For any connected graph G of order n, $\left\lceil\frac{n}{\Delta+1}\right\rceil \leq \gamma_{ibod}(G) \leq n-l$, l denotes the pendant vertices of G.

Proof: Suppose $\gamma_{ibod}(G) = m$. Let $D = \{u_1, u_2, \dots \dots .., u_m\}$ be a minimum $ibod\ set$ of G. Since u_i dominates $1 + deg\ u_i$ vertices of G for $1 \leq i \leq m$ and the vertices of D dominate all n vertices of $G, \sum_{i=1}^{m}(1 + deg\ u_i) \geq n$. However, $1 + deg\ u_i \leq 1 + \Delta(G)\ \forall\ i = 1\ to\ m$.

Therefore, $n \leq \sum_{i=1}^{m}(1 + deg\ u_i) \leq \sum_{i=1}^{m}(1 + \Delta(G))$

$$= m\big(1 + \Delta(G)\big)$$

$$\therefore n \leq\ m\big(1 + \Delta(G)\big)$$

Since $\gamma_{ibod}(G) = m,\ n \leq \gamma_{ibod}(G)\,\big(1 + \Delta(G)\big)$. Hence $\left\lceil\frac{n}{\Delta+1}\right\rceil \leq \gamma_{ibod}(G)$.

Let $B = \{\, v \in V(G): deg(v) = 1\}$. Since every $v \in B$ is non interior, $v \notin ibod\ set$. Thus if a graph G has l pendant vertices, $\gamma_{ibod}(G) \leq\ n-l$. Hence $\left\lceil\frac{n}{\Delta+1}\right\rceil \leq \gamma_{ibod}(G) \leq n-l$.

Remark 3.19: In the Corona product of graphs G and $K_1, G \circ K_1$ where G be any connected graph the upper bound is sharp. Also, the lower bound is sharp for the path $P_n, n \equiv 0,1(mod\ 3)$.

Theorem 3.20:

(a)For $n \geq 2,\ \gamma_{ibod}(K_{1,n}) = 1$.

(b)For $m, n \geq 2,\ \gamma_{ibod}(K_{m,n}) = 2$.

(c)For $n \geq 4,\ \gamma_{ibod}(W_n) = 1$.

(d)For $n \geq 3,\ \gamma_{ibod}(H_n) = \begin{cases} \gamma & if\ n\ is\ odd \\ \gamma + 1 & if\ n\ is\ even \end{cases}$

Proof:

(a)Let v be the central vertex in $K_{1,n}$. $Int\ (K_{1,n}) = \{v\}$ and v dominates every vertex of $K_{1,n}$. The set $D = \{v\}$ is the minimum $ibod\ set$ also D is both $\gamma_{Id}\ set$ and $\gamma_{bo}\ set$. Hence $\gamma_{ibod}(K_{1,n}) = 1$.

(b)Since every vertex of $K_{m,n}$ is interior, $Int\ (K_{m,n}) = V(K_{m,n})$ and so $\gamma_{ibod}(K_{m,n})$ is same as $\gamma_{bo}(K_{m,n})$. Hence $\gamma_{ibod}(K_{m,n}) = 2$.

(c)For the wheel $W_n = C_n + K_1, n \geq 4$ the apex vertex is the only interior vertex and it dominates $\Delta(W_n) + 1$ vertices. The apex vertex forms the minimum $ibod\ set$. Thus, $\gamma_{ibod}(W_n) = 1$.

(d)Helm H_n has $2n + 1$ vertices. The set of all rim vertices of the wheel W_3 from which H_3 is obtained forms $Int\ H_3$. In $H_n, n \geq 4$ except pendant vertices all vertices are interior. Denote the apex vertex as u and D be the set of all support vertices of H_n. D is the minimum dominating set of H_n. Hence $|D| = \gamma(H_n)$. If n is odd, then D is the required minimum $ibod\ set$. If n is even, then D is a $\gamma_{Id}\ set$ but fails to be a bias odd dominating set since u has even number of neighbours in D. Hence $D \cup \{u\}$ forms the minimum $ibod\ set$.

Thus, $\gamma_{ibod}(H_n) = \begin{cases} \gamma & if\ n\ is\ odd \\ \gamma + 1 & if\ n\ is\ even \end{cases}$

Theorem 3.21: For $n \geq 3, \gamma_{ibod}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil & otherwise \end{cases}$

Proof: Let $V(P_n) = \{v_i : i = 1\ to\ n\}$ be the vertex set of the path P_n taken in order.

$Int\ P_n = V(P_n) \backslash \{v_1, v_n\}$.

Case (i): $n \equiv 0(mod\ 3)$

If $n \equiv 0(mod\ 3)$, then $\{v_i : i \equiv 2(mod\ 3)\}$ forms the minimum $ibod\ set$. Hence in this case, $\gamma_{ibod}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Case (ii): $n \equiv 1(mod\ 3)$

The minimum $ibod\ set$ is $\{v_i : i \equiv 2(mod\ 3)\} \cup \{v_{n-1}\}$, hence $\gamma_{ibod}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Case (iii): $n \equiv 2(mod\ 3)$

The set $\{v_i : i \equiv 2(mod\ 3) \backslash \{v_n\}\} \cup \{v_{n-2}, v_{n-1}\}$ is the minimum $ibod\ set$.

Thus $\gamma_{ibod}(P_n) = \left\lceil \frac{n}{3} \right\rceil + 1$. Hence $\gamma_{ibod}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil & otherwise \end{cases}$

Note 3.22: In comparison with $\gamma_{bo}(P_n)$, the above theorem can be restated as

$$\gamma_{ibod}(P_n) = \begin{cases} \gamma_{bo}(P_n) + 1 & if\ n \equiv 2(mod\ 3) \\ \gamma_{bo}(P_n) & otherwise \end{cases}$$

Theorem 3.23: For the cycle of order $n \geq 4$, $\gamma_{ibod}(C_n) = \gamma_{bo}(C_n)$.

Proof: Let the vertex set of cycle C_n be $V(C_n) = \{v_i : i = 1\ to\ n\}$ where vertices are taken in order. Since $Int\ C_n = V(C_n)$, $\gamma_{ibod}(C_n) = \gamma_{bo}(C_n)$.

Hence a $\gamma_{bo}\ set$ of C_n,

$$D = \begin{cases} \{v_i : i \equiv 1(mod\ 3)\} \cup \{v_n\} & if\ n \equiv 2(mod\ 3) \\ \{v_i : i \equiv 1(mod\ 3)\} & otherwise \end{cases}$$

forms the minimum $ibod\ set$ of C_n.

Hence $\gamma_{ibod}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil & otherwise \end{cases}$

Observation 3.24:

(a)For the prism graph Y_n with $n \geq 4$, $\gamma_{ibod}(Y_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor$.

(b)For the ladder graph L_n with $n \geq 2$, $\gamma_{ibod}(L_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

In Y_n and L_n all vertices are interior. Hence $\gamma_{ibod}(Y_n) = \gamma_{bo}(Y_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor$ and $\gamma_{ibod}(L_n) = \gamma_{bo}(L_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Theorem 3.25:

For $m \geq 3$, $\gamma_{ibod}(K_m(P_n)) = \begin{cases} n & if\ n = 1,2 \\ \gamma_{ibod}(P_n) + 2 & if\ n \equiv 0(mod\ 3) \\ \gamma_{ibod}(P_n) & otherwise \end{cases}$

Proof: Let v be the vertex with degree $\Delta\big(K_m(P_n)\big)$ and let $V(P_n) = \{u_i: i = 1\ to\ n\}$ be the vertex set of P_n where $vu_1 \in E\big(K_m(P_n)\big)$. $Int\ \big(K_m(P_n)\big)$=$\{v, u_1, u_2, \dots \dots, u_{n-1}\}$. If $n = 1, D_1 = \{v\}$ is the minimum $ibod\ set$. Hence $\gamma_{ibod}(K_m(P_1)) = 1$. If $n = 2, D_2 = \{v, u_1\}$ is the minimum $ibod\ set$. Thus $\gamma_{ibod}(K_m(P_2)) = 2$.

Case (i): $n \equiv 0\ (mod\ 3)$

The minimum $ibod\ set$ is $\{u_i: i \equiv 1\ (mod\ 3)\} \cup \{v, u_{n-1}\}$.

Hence $\gamma_{ibod}\big(K_m(P_n)\big) = \left\lceil \frac{n}{3} \right\rceil + 2$.

Case (ii): $n \equiv 1\ (mod\ 3)$

The set $\{u_i: i \equiv 0\ (mod\ 3)\} \cup \{v\}$ is the minimum $ibod\ set$.

Hence $\gamma_{ibod}\big(K_m(P_n)\big) = \left\lceil \frac{n}{3} \right\rceil$.

Case (iii): $n \equiv 2\ (mod\ 3)$

The set $\{u_i: i \equiv 0\ (mod\ 3)\} \cup \{v, u_{n-1}\}$ forms the minimum $ibod\ set$.

Thus $\gamma_{ibod}\big(K_m(P_n)\big) = \left\lceil \frac{n}{3} \right\rceil + 1$.

We observe that if $n \neq 1\&2$, $\gamma_{ibod}(K_m(P_n)) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 2 & if\ n \equiv 0(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 1(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod\ 3) \end{cases}$

Comparing these values with $\gamma_{ibod}(P_n)$, we get

$$\gamma_{ibod}(K_m(P_n)) = \begin{cases} n & if\ n = 1,2 \\ \gamma_{ibod}(P_n) + 2 & if\ n \equiv 0(mod\ 3) \\ \gamma_{ibod}(P_n) & otherwise \end{cases}$$

Result 3.26: For any vertex of a graph G, if the induced subgraph of its closed neighbourhood forms a star then it is a interior vertex.

Proof: Let u be a vertex of graph G such that $\langle N_G[u] \rangle$ be a star. Then for every $x \in N_G(u)$ there exists $y \in N_G(u) \ni: d(x, u) + d(u, y) = 2$. Thus $u \in Int\ G$.

Result 3.27: Let G be a connected graph. Every non pendant vertex of $S(G)$ is interior.

Proof: Let u be an arbitrary vertex of G such that $deg(u) \geq 2$. Then u is a non pendant vertex of $S(G)$. Any two vertices $v, w \in N_G(u)$ have either $d_G(v, w) = 1$ or $d_G(v, w) > 1$. But in $S(G)$, $N_{S(G)}(u)$ forms the set of pairwise nonadjacent vertices.

Hence $\langle N_{S(G)}[u]\rangle$ is a star. By the previous result, $u \in Int\, S(G)$. Since u is arbitrary, every non pendant vertex of $S(G)$ belongs to $Int\,(S(G))$.

Theorem 3.28: For $n \geq 2$, $\gamma_{ibod}(S(K_{1,n})) = \begin{cases} n & if\ n\ is\ odd \\ n+1 & if\ n\ is\ even \end{cases}$

Proof: Let u be the central vertex of $K_{1,n}$. The subdivision graph $S(K_{1,n})$ has $2n+1$ vertices and $2n$ edges. Consider $D = V(S(K_{1,n}))\backslash V(K_{1,n})$ which is the $\gamma_{Id}\ set$ of $S(K_{1,n})$. Hence $\gamma_{Id}(S(K_{1,n})) = |D| = n$. If n is odd then D itself forms the minimum $ibod\ set$, hence $\gamma_{ibod}(S(K_{1,n})) = n$. If n is even, D fails to hold the bias odd domination condition for u. Hence $D \cup \{u\}$ is the minimum $ibod\ set$ and thus $\gamma_{ibod}(S(K_{1,n})) = n+1$.

Hence $\gamma_{ibod}(S(K_{1,n})) = \begin{cases} n & if\ n\ is\ odd \\ n+1 & if\ n\ is\ even \end{cases}$

Theorem 3.29: For $B_{m,n}$ bistar of $m+n+2$ vertices, $\gamma_{ibod}(S(B_{m,n})) = m+n+1$.

Proof: Let x and y be the central vertices of $B_{m,n}$. Let $x_1, x_2, \dots\dots, x_m$ and $y_1, y_2, \dots\dots, y_n$ be the vertices adjacent with x and y respectively. Each edge xx_i is subdivided into two edges xx_i' and $x_i'x_i$ where $i = 1\ to\ m$. Similarly each edge yy_j is subdivided into two edges yy_j' and $y_j'y_j$ where $j = 1\ to\ n$. The bridge xy of $B_{m,n}$ is subdivided into two edges xz and zy. Thus $S(B_{m,n})$ has $2(m+n)+3$ vertices. In the graph $S(B_{m,n})$, except pendant vertices all the vertices are interior. Let $A' = \{\{x_i' : i = 1\ to\ m\} \cup \{y_j' : j = 1\ to\ n\}\}$. The vertices of A' dominates $2(m+n)+2$ vertices of $S(B_{m,n})$ and $|A'| = m+n$.

Case (i) m & n even

If both m & n even, $A' \cup \{z\}$ forms the minimum $ibod\ set$ since for every vertex in $V(S(B_{m,n}))\backslash A' \cup \{z\}$ has odd number of neighbours in $A' \cup \{z\}$.

Hence $\gamma_{ibod}(S(B_{m,n})) = m+n+1$.

Case (ii) m & n odd

If both m & n odd, $A' \cup \{x\}$ and $A' \cup \{y\}$ forms the minimum $ibod\ set$.

Hence $\gamma_{ibod}(S(B_{m,n})) = m+n+1$.

Case (iii) m odd (even) & n even (odd)

If m odd (even) & n even (odd), $A' \cup \{y\}$ and $A' \cup \{x\}$ forms the minimum $ibod\ sets$ respectively. Thus $\gamma_{ibod}(S(B_{m,n})) = m+n+1$.

Observation 3.30:

(a) For $S(P_n \circ K_1)$ with $n \geq 2$, $\gamma_{ibod}\big(S(P_n \circ K_1)\big) = \begin{cases} \frac{3n-1}{2} & if\ n\ is\ odd \\ \frac{3n}{2} & if\ n\ is\ even \end{cases}$

(b) For $S(C_n \circ K_1))$ with $n \geq 3$, $\gamma_{ibod}\big(S(C_n \circ K_1)\big) = \begin{cases} \frac{3(n+1)}{2} & if\ n\ is\ odd \\ \frac{3n}{2} & if\ n\ is\ even \end{cases}$

4. CONCLUSION

In this article, few bounds on interior bias odd domination number are presented. Some results are proved on subdivision graphs with respect to this new parameter. The future work can be carried on computing interior bias odd domination number for various graphs and can be derive like Nordhaus-Gaddum type results.

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