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New Exact Solutions of the Time-space Fractional order $(1+3)-$Dimensional *Burger* Equation [*]

Dan Zhang [†1] **and Zejian Cui**[2]
[1,2] *College of Mathematic and information, China West Normal University, NanChong,637009 ,China.*

Abstract

In this paper, by using the modified auxiliary equation method and using *Matlab* software to solve the corresponding algebraic equations, two new accurate solutions of the time-space fractional order $(1+3)-$dimensional Burger equation:

$$\begin{aligned} &D_t^{\alpha}u - D_{xx}^{2\alpha}u - D_{yy}^{2\alpha}u - D_{zz}^{2\alpha}u - uD_x^{\alpha}u = 0 \\ &u = u(x,y,z,t), \alpha \in (0,1] \end{aligned} \tag{1}$$

are obtained, which greatly enriches the understanding of the system.
Keywords: time-space fractional order $(1+3)-$dimensional *Burger* equation,modified auxiliary equation method,exact solution

1. INTRODUCTION

Fractional calculus, also known as non-integer-order calculus, is an extension of integer-order calculus. We can use fractional calculus to model materials and processes that have memory and hereditary properties. At the same time, it can also be widely applied to problems in different fields such as abnormal diffusion, wave propagation, and turbulence.The 17th century German philosopher and mathematician *Leibnitz* invented the *Leibnitz* symbol.Under the *Leibnitz* symbol,the n order derivative of y with respect to x can be denoted as the quotient of dx and dy,that is,$\frac{d^n y}{dx^n}$.

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†Corresponding author‘s e-mall:2514650351@qq.com;cuizejian@126.com

With the doubt that n is a fraction,*Lacroix* proposed arbitrary derivative for the first time in 1819.At present, the most widely used definitions of fractional differential are the $Riemann-Liouville$ definition and *Caputo* definition[1, 2].But these two definitions cannot satisfy the derivative of the product of two functions.Until 2014,a new definition of fractional derivative has been proposed by Khalil et al[3]:

Definition Let $\alpha \in (n-1, n], n \in Z, t > 0$, and f be an $n-$differentiable at t, where $t > 0$.Then the conformable fractional derivative of f of order $---$ is defined as

$$D_t^\alpha f(t) = \lim_{\epsilon \to 0} \frac{f^{(n-1)}(t + \epsilon t^{n-\alpha}) - f^{(n-1)}(t)}{\epsilon} \tag{1.2}$$

Theorem Let $\alpha \in (n-1, n], n \in Z, t > 0$, and f be an $n-$differentiable at t, where $t > 0$.Then

(1)$D_t^\alpha(af + bg) = a(D_t^\alpha f) + b(D_t^\alpha g)$,$a, b \in R$.

(2)$D_t^\alpha(fg) = f(D_t^\alpha g) + g(D_t^\alpha f)$.

(3)$D_t^\alpha(\frac{f}{g}) = \frac{g(D_t^\alpha f) - f(D_t^\alpha g)}{g^2}$.

(4)$D_t^\alpha f = t^{(n-\alpha)} f^{(n)}(t)$.

(5)$D_t^\alpha(t^p) = pt^{p-\alpha}, p \in R$.

(6)$D_t^\alpha(\frac{t^\alpha}{\alpha}) = 1$.

(7)$D_t^\alpha f = 0$,for all constant functions $f(t) = \lambda$.

The evolution equation is an important branch of mathematical theory.In a broad sense,it is the general term of many important mathematical and physical equations containing time variable t. Fractionl evolution equations have been widely used in the modeling of many phenomena in engineering, physics and economics.In recent years,the fractional evolutional equation has been significantly developed.Many methods for solving integer order evolution equations have been extended to fractional evolution equations,such as Laplace transform method[4],Local fractional *Riccati* differential equation method[5], auxiliary equation method[6],extended direct algebra method[7],modified auxiliary equation method[8], singular manifold method[9], extended simple equation method[10],$\frac{G'}{G+G'}-$ expansion method[11],etc.In 1931, Richards proposed a nonlinear diffusion-convection equation, which can be used as a model to describe water flow in soil[12]. In this model,when the concentration-dependent diffusivity and the concentration-dependent hydraulic conductivity of the unsaturated medium reach a certain value,we get the $(1+3)-$dimensional integer-order Burger equation[13].In 1991, due to the discussion of the initial value of the *Burger* equation, the fractional *Burger* equation became active in the public eye[14].The spatial fractional *Burger* equation describes the physical process of the one-way propagation of weakly nonlinear sound waves through the inflatable tube. The fractional derivative is caused by the memory effect of wall friction passing through the boundary layer. At the same time, the model is also suitable for diving waves and bubbles. Undulations in liquids and other systems.In 2014, *MuhammadYounis* et al. used the $\frac{G'}{G}$ expansion method to solve the time-space fractional $(1+1)-$dimensional *Burger* equation, and obtained the hyperbolic function type solution, trigonometric function type Solution and rational function solution[15].In 2019, *A.A.Gaber* et al. used the generalized Kudryashov method to solve the time-space fractional $(1+3)-$dimensional Burger equation, obtained fractional trigonometric and exponential solutions, and obtained kink and singular kink solutions[16].In 2020, *S.O.Ajibola* et al. used the Laplace homotopy applied analysis method to solve the nonlinear approximate analytical solution of the time-space fractional $(1+1)-$dimensional *Burger* equation[17].Although the time-space fractional *Burger* equation has been widely studied, the exact solution of the time-space fractional $(1+3)-$dimensional Burger equation is rarely studied. This article will use the modified auxiliary equation method to solve the exact solution of the time-space fractional $(1+3)-$dimensional Burger equation under the background of the conformal fractional derivative, making the solution of the equation more diverse.

2. MODIFIED AUXILIARY EQUATION METHOD

For nonlinear equations with independent variables $x_1, x_2, x_3, ..., x_n, t$

$$L(u, D_{x_1}^{\alpha}u, D_{x_2}^{\beta}u, D_{x_3}^{\gamma}u, ..., D_{x_n}^{\rho}u, D_t^{\eta}u, ..., D_{x_1}^{\alpha}D_{x_1}^{\alpha}u, D_{x_2}^{\beta}D_{x_2}^{\beta}u, D_{x_3}^{\gamma}D_{x_3}^{\gamma}u, ...) = 0, \alpha, \beta, \gamma, \rho, \eta, \in (0,1] \tag{2.1}$$

Where u is an unknown function,L is a polynomial in u and its partial fractional derivatives which involve the highest order derivatives and nonlinear terms.The main steps of solving the $Eq.(2.1)$ by using the modified auxiliary equation method are as follows.

Step 1:Using the traveling wave

$$u(x_1, x_2, x_3, ..., x_n, t) = U(\theta), \theta = \frac{k_1 x_1^{\alpha}}{\alpha} + \frac{k_2 x_2^{\beta}}{\beta} + \frac{k_3 x_3^{\gamma}}{\gamma} + ... + \frac{k_n x_n^{\rho}}{\rho} + \frac{ct^{\eta}}{\eta} \tag{2.2}$$

Where $k_1, k_2, k_3, ..., k_n, c$ are nonzero constants.Using the properties of the conformable fractional derivative,the traveling wave $Eq.(2.2)$ is used to reduce $Eq.(2.1)$ to ordinary differential equation(ODE):

$$N(U, U', U'', U''', ...) = 0 \tag{2.3}$$

Where N is a polynomial in $U(\theta)$ and its derivatives with respect to θ.
Step 2:Suppose the solution of $Eq.(2.3)$ can be expressed as

$$U(\theta) = \sum_{r=1}^{n} a_r \kappa^{rf(\theta)} + a_0 + \sum_{r=1}^{n} b_r \kappa^{-rf(\theta)} \tag{2.4}$$

Where $a_r(r = 1, ..., n), b_r(r = 1, ..., n), a_0$ are arbitrary constants to be determined and the positive integer n is determined by the homogeneous balance method,meanwhile,$f'(\theta)$ satisfies the following equation

$$f'(\theta) = \frac{p\kappa^{-f(\theta)} + m + q\kappa^{f(\theta)}}{ln\kappa} \tag{2.5}$$

Where p, m, q, are arbitrary constants,and κ is arbitrary constant to be determined.

For $\kappa^f(\theta)$, we have the following instructions[18]:
(1)We can see that our new method is exactly same to the $\frac{G'}{G}-$expansion method when $\kappa^{f(\theta)} = \frac{G'(\theta)}{G(\theta)}, p = -\mu, m = -\lambda, q = 1, \mu, \lambda$ are arbitrary constants or $\kappa^{f(\theta)} = d + \psi(\theta), p = \mu, m = \lambda, q = v - 1, \mu, \lambda, v, d$ are arbitrary constants or $\kappa^{f(\theta)} = \frac{\frac{G'(\theta)}{G(\theta)}}{\delta + \frac{G'(\theta)}{G(\theta)}}, p = -\mu, m = 0, q = -1, \mu, \delta$ are arbitrary constants,then the equation (2.5) becomes $(\frac{G'}{G})' = (\frac{G'}{G})^2 - \lambda(\frac{G'}{G}) - \mu$ or $(d + \psi)' = (v - 1)(d + \psi)^2 + \lambda(d + \psi) + \mu$ or $(\frac{\frac{G'}{G}}{\delta + \frac{G'}{G}})' = -(\frac{\frac{G'}{G}}{\delta + \frac{G'}{G}})^2 - \mu$ accordingly.
(2)We can see that our new method is exactly same to the $e^{-i\phi(\theta)}-$expansion method when $\kappa^{f(\theta)} = e^{-i\phi(\theta)}, p = -\mu, m = -\lambda, q = -1, \mu, \lambda$ are arbitrary constants,then the equation (2.5) becomes $(e^{-i\phi})' = -(e^{-i\phi})^2 - \lambda e^{-i\phi} - \mu$.
(3)We can see that our new method is exactly same to the extended tanh-function method when $\kappa^{f(\theta)} = \phi(\theta), p = b, m = 0, q = 1, b$ is arbitrary constant,then the equation (2.5) becomes $\phi' = \phi^2 + b$.
(4)We can see that our new method is exactly same to the Kudryashov and modified Kudryashov methods when $\kappa^{f(\theta)} = \phi(\theta), p = 0, m = -q = -ln(\kappa)$,then the equation (2.5) becomes $\phi' = ln(\kappa)\phi^2 - ln(\kappa)\phi$.
(5)We can see that our new method is exactly same to the improved $tan\frac{\phi(\theta)}{2}-$expansion method when $\kappa^{f(\theta)} = tan\frac{\phi(\theta)}{2}, p = b + c, m = \kappa, q = c - b, b, c$ are arbitrary constants,then the equation (2.5) becomes $(tan\frac{\phi}{2})' = (c - b)(tan\frac{\phi}{2})^2 + \kappa tan\frac{\phi}{2} + (b + c)$.

Although the method in this article is similar to the above method when the parameters are set to some of the above-mentioned specific values, the above method is also limited for solving the exact solution of the equation. The types of solutions and the number of solutions we can get are relatively small. of. The method in this article generalizes the above methods more, so that the solutions we can get are richer and more comprehensive.

Step 3:solving the $Eq.(2.5)$ and getting the value of $f(\theta)$.
When $q \neq 0, \Delta = m^2 - 4pq < 0$:

$$\kappa^{f(\theta)} = \frac{\sqrt{4pq - m^2}tan(\frac{\sqrt{4pq-m^2}\theta}{2}) - m}{2q} \tag{2.6}$$

When $q \neq 0, p = -q, \Delta = m^2 + 4q^2 < 0$:

$$\kappa^{f(\theta)} = \frac{\sqrt{-4q^2 - m^2}tan(\frac{\sqrt{-4q^2-m^2}\theta}{2}) - m}{2q} \tag{2.7}$$

When $q \neq 0, p = q, \Delta = m^2 - 4q^2 < 0$:

$$\kappa^{f(\theta)} = \frac{\sqrt{4q^2 - m^2}tan(\frac{\sqrt{4q^2-m^2}\theta}{2}) - m}{2q} \tag{2.8}$$

When $q \neq 0, p \neq 0, m = 0, \Delta = -4pq < 0$:

$$\kappa^{f(\theta)} = \frac{\sqrt{pq}tan(\sqrt{pq}\theta)}{q} \tag{2.9}$$

When $q \neq 0, \Delta = m^2 - 4pq > 0$:

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 - 4pq}coth(\frac{\sqrt{m^2-4pq}\theta}{2})}{2q} \tag{2.10}$$

or

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 - 4pq}th(\frac{\sqrt{m^2-4pq}\theta}{2})}{2q} \tag{2.11}$$

When $q \neq 0, p = -q, \Delta = m^2 + 4q^2 > 0$:

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 + 4q^2}coth(\frac{\sqrt{m^2+4q^2}\theta}{2})}{2q} \tag{2.12}$$

or

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 + 4q^2}th(\frac{\sqrt{m^2+4q^2}\theta}{2})}{2q} \tag{2.13}$$

When $q \neq 0, p = q, \Delta = m^2 - 4q^2 > 0$:

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 - 4q^2}coth(\frac{\sqrt{m^2-4q^2}\theta}{2})}{2q} \tag{2.14}$$

or

$$\kappa^{f(\theta)} = \frac{-m - \sqrt{m^2 - 4q^2} th(\frac{\sqrt{m^2-4q^2}\theta}{2})}{2q} \tag{2.15}$$

When $q \neq 0, p \neq 0, m = 0, \Delta = -4pq > 0$:

$$\kappa^{f(\theta)} = \frac{-\sqrt{-pq} coth(\sqrt{-pq}\theta)}{q} \tag{2.16}$$

or

$$\kappa^{f(\theta)} = \frac{-\sqrt{-pq} th(\sqrt{-pq}\theta)}{q} \tag{2.17}$$

When $q \neq 0, m = 0, p = -q$:

$$\kappa^{f(\theta)} = coth(-q\theta) \tag{2.18}$$

When $q \neq 0, m = 0, p = q$:

$$\kappa^{f(\theta)} = tan(q\theta) \tag{2.19}$$

When $q = 0, m \neq 0$:

$$\kappa^{f(\theta)} = \frac{e^{m\theta} - p}{m} \tag{2.20}$$

When $q = 0, m = \kappa, p = 2\kappa$:

$$\kappa^{f(\theta)} = e^{\kappa\theta} - 2 \tag{2.21}$$

Step 4:The n in $Eq.$(2.4) is a positive integer determined by balancing the highest order derivatives and the nonlinear terms in $Eq.$(2.3).Substituting $Eq.$(2.4) along with $Eq.$(2.5) into $Eq.$(2.3) and setting coefficients of $\kappa^{f(\theta)}$ to zero,we obtained a system of algebraic equations and calculate parameters $a_r(r = 1, ..., n), b_r(r = 1, ..., n), a_0, k_j(j = 1, 2, 3), c, p, m, q$.

Step 5:Substituting all the parameter values into $Eq.$(2.4) along with $\kappa^{f(\theta)}$ which is a general solution of $Eq.$(2.5),we obtained the required the solutions of $Eq.$(2.1).

3. NEW EXACT SOLUTIONS OF THE TIME-SPACE FRACTIONAL ORDER $(1+3)-$DIMENSIONAL *BURGER* EQUATION

the time-space fractional order $(1+3)-$dimensional *Burger* equation is

$$\begin{aligned} &D_t^{\alpha}u - D_{xx}^{2\alpha}u - D_{yy}^{2\alpha}u - D_{zz}^{2\alpha}u - uD_x^{\alpha}u = 0 \\ &u = u(x,y,z,t), \alpha \in (0,1] \end{aligned} \tag{3.1}$$

Using the following traveling wave transforms:

$$u(x,y,z,t) = U(\theta), \theta = \frac{k_1 x^{\alpha}}{\alpha} + \frac{k_2 y^{\alpha}}{\alpha} + \frac{k_3 z^{\alpha}}{\alpha} + \frac{ct^{\alpha}}{\alpha} \tag{3.2}$$

Substituting (3.2) in (3.1) give:

$$cU' - (k_1^2 + k_2^2 + k_3^2)U'' - k_1 UU' = 0 \tag{3.3}$$

Integrating *Eq.*(3.3) and setting the constant of integration to zero give:

$$2cU - 2(k_1^2 + k_2^2 + k_3^2)U' - k_1 U^2 = 0 \tag{3.4}$$

Applying the homogeneous balance principle on *Eq.*(3.4) we get:

$$n + 1 = 2n \Rightarrow n = 1 \tag{3.5}$$

Substituting $n = 1$ into equation (2.4), we get:

$$U(\theta) = a_1\kappa^{f(\theta)} + a_0 + b_1\kappa^{-f(\theta)}, f'(\theta) = \frac{p\kappa^{-f(\theta)} + m + q\kappa^{f(\theta)}}{ln\kappa} \tag{3.6}$$

Substituting equation (3.6) into equation *Eq.*(3.4), we obtain a set of algebraic equations about $a_0, a_1, b_1, k_j (j = 1,2,3), c, p, m, q$:

$$\begin{cases} 2b_1 p(k_1^2 + k_2^2 + k_3^2) - b_1^2 k_1 = 0 \\ 2cb_1 + 2b_1 m(k_1^2 + k_2^2 + k_3^2) - 2a_0 b_1 k_1 = 0 \\ 2ca_0 - 2(a_1 p - b_1 q)(k_1^2 + k_2^2 + k_3^2) - k_1(2a_1 b_1 + a_0^2) = 0 \\ 2ca_1 - 2a_1 m(k_1^2 + k_2^2 + k_3^2) - 2a_0 a_1 k_1 = 0 \\ -2a_1 q(k_1^2 + k_2^2 + k_3^2) - a_1^2 k_1 = 0 \end{cases} \tag{3.7}$$

By solving the algebraic equations (3.7), we get the values of the corresponding coefficients a_0, a_1, b_1, c. Substituting the general solution of *Eq.*(2.5) and the values of

the corresponding coefficients into $Eq.$(3.6), the solution of $Eq.$(3.1) can be obtained:
For case 1: $a_0 = \frac{(m\pm\sqrt{m^2-4pq})(k_1^2+k_2^2+k_3^2)}{k_1}, a_1 = 0, b_1 = \frac{2p(k_1^2+k_2^2+k_3^2)}{k_1}, c = \pm\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2), i = \sqrt{-1}$.

$$u_1 = \frac{4pq(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{4pq-m^2}tan\{\frac{\sqrt{4pq-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{4pq-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}-k_1m} + \frac{(m\pm i\sqrt{4pq-m^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.8}$$

$$u_2 = \frac{-4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{-4q^2-m^2}tan\{\frac{\sqrt{-4q^2-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{-4q^2-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}-k_1m} + \frac{(m\pm i\sqrt{-4q^2-m^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.9}$$

$$u_3 = \frac{4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{4q^2-m^2}tan\{\frac{\sqrt{4q^2-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{4q^2-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}-k_1m} + \frac{(m\pm i\sqrt{4q^2-m^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.10}$$

$$u_4 = \frac{2pq(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{pq}tan\{\sqrt{pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2i\sqrt{pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}} \pm \frac{2i\sqrt{pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.11}$$

$$u_5 = \frac{4pq(k_1^2+k_2^2+k_3^2)}{-k_1\sqrt{m^2-4pq}coth\{\frac{\sqrt{m^2-4pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}-k_1m} + \frac{(m\pm\sqrt{m^2-4pq})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.12}$$

$$u_6 = \frac{4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{m^2+4q^2}coth\{\frac{\sqrt{m^2+4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}+k_1m} + \frac{(m\pm\sqrt{m^2+4q^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.13}$$

$$u_7 = \frac{-4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{m^2-4q^2}coth\{\frac{\sqrt{m^2-4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}+k_1m} + \frac{(m\pm\sqrt{m^2-4q^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.14}$$

$$u_8 = \frac{-2pq(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{-pq}coth\{\sqrt{-pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}} \pm \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.15}$$

$$u_9 = \frac{4pq(k_1^2+k_2^2+k_3^2)}{-k_1\sqrt{m^2-4pq}th\{\frac{\sqrt{m^2-4pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}-k_1m} + \frac{(m\pm\sqrt{m^2-4pq})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.16}$$

$$u_{10} = \frac{4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{m^2+4q^2}th\{\frac{\sqrt{m^2+4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}+k_1m} + \frac{(m\pm\sqrt{m^2+4q^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.17}$$

$$u_{11} = \frac{-4q^2(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{m^2-4q^2}th\{\frac{\sqrt{m^2-4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}\}+k_1m} + \frac{(m\pm\sqrt{m^2-4q^2})(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.18}$$

$$u_{12} = \frac{-2pq(k_1^2+k_2^2+k_3^2)}{k_1\sqrt{-pq}th\{\sqrt{-pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}} \pm \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.19}$$

$$u_{13} = -\frac{2q(k_1^2+k_2^2+k_3^2)th\{-q[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2q(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2q(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.20}$$

$$u_{14} = \frac{2q(k_1^2+k_2^2+k_3^2)cot\{q[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2qi(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2qi(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.21}$$

$$u_{15} = \frac{2pm(k_1^2+k_2^2+k_3^2)}{k_1\{e^{m[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}+\frac{m(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}-p\}} + \frac{2m(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.22}$$

$$u_{16} = \frac{2pm(k_1^2+k_2^2+k_3^2)}{k_1\{e^{m[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}-\frac{m(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}-p\}} \tag{3.23}$$

$$u_{17} = \frac{4\kappa(k_1^2+k_2^2+k_3^2)}{k_1\{e^{\kappa[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}+\frac{\kappa(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}-2\}} + \frac{2\kappa(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.24}$$

$$u_{18} = \frac{4\kappa(k_1^2+k_2^2+k_3^2)}{k_1\{e^{\kappa[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}-\frac{\kappa(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}-2\}} \tag{3.25}$$

For case 2: $a_0 = \frac{(-m\pm\sqrt{m^2-4pq})(k_1^2+k_2^2+k_3^2)}{k_1}, a_1 = \frac{-2q(k_1^2+k_2^2+k_3^2)}{k_1}, b_1 = 0, c = \pm\sqrt{m^2-4pq}(k_1^2+k_2^2+$

$k_3^2), i = \sqrt{-1}$.

$$u_{19} = -\frac{\sqrt{4pq-m^2}(k_1^2+k_2^2+k_3^2)tan\frac{\sqrt{4pq-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{4pq-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{i\sqrt{4pq-m^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.26}$$

$$u_{20} = -\frac{\sqrt{-4q^2-m^2}(k_1^2+k_2^2+k_3^2)tan\frac{\sqrt{-4q^2-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{-4q^2-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{i\sqrt{-4q^2-m^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.27}$$

$$u_{21} = -\frac{\sqrt{4q^2-m^2}(k_1^2+k_2^2+k_3^2)tan\frac{\sqrt{4q^2-m^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{i\sqrt{4q^2-m^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{i\sqrt{4q^2-m^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.28}$$

$$u_{22} = -\frac{2\sqrt{pq}(k_1^2+k_2^2+k_3^2)tan\{\sqrt{pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2i\sqrt{pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2i\sqrt{pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.29}$$

$$u_{23} = \frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)coth\frac{\sqrt{m^2-4pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.30}$$

$$u_{24} = \frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)coth\frac{\sqrt{m^2+4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.31}$$

$$u_{25} = \frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)coth\frac{\sqrt{m^2-4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.32}$$

$$u_{26} = \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)coth\{\sqrt{-pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.33}$$

$$u_{27} = \frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)th\frac{\sqrt{m^2-4pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2-4pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.34}$$

$$u_{28} = \frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)th\frac{\sqrt{m^2+4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2+4q^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.35}$$

$$u_{29} = \frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)th\frac{\sqrt{m^2-4q^2}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]}{2}}{k_1} \pm \frac{\sqrt{m^2-4q^2}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.36}$$

$$u_{30} = \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)th\{\sqrt{-pq}[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2\sqrt{-pq}(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.37}$$

$$u_{31} = \frac{-2q(k_1^2+k_2^2+k_3^2)coth\{-q[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2q(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2q(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.38}$$

$$u_{32} = \frac{-2q(k_1^2+k_2^2+k_3^2)tan\{q[\frac{k_1x^\alpha}{\alpha}+\frac{k_2y^\alpha}{\alpha}+\frac{k_3z^\alpha}{\alpha}\pm\frac{2qi(k_1^2+k_2^2+k_3^2)t^\alpha}{\alpha}]\}}{k_1} \pm \frac{2qi(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.39}$$

$$u_{33} = -\frac{2m(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.40}$$

$$u_{34} = -\frac{2\kappa(k_1^2+k_2^2+k_3^2)}{k_1} \tag{3.41}$$

4. CONCLUSION

In this paper, by applying the modified auxiliary equation method to the space-time fractional $(1+3)-$dimensional *Burger* equation, two sets of new exact solutions are obtained.These general solutions are divided into seven categories, namely trigonometric function solutions and their fractional solutions, hyperbolic function solutions and their fractional solutions, rational number solutions, exponential function solutions, and constant solutions, among which hyperbolic function solutions and their fractional solutions , trigonometric function solution, rational number solution and constant solution are the new exact solutions obtained in this paper, namely (3.12) – (3.21) formula and (3.26) – (3.41) formula.This article expands the application of the modified auxiliary equation method and enriches the solution system of the space-time fractional $(1 + 3)-$dimensional *Burger* equation.

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