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Three-Dimensional Blasius Flow of a Non-Newtonian Fluid

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Abstract

Three-dimensional boundary-layer equations have been modelled for non-Newtonian power-law fluid over a moving flat plate. The governing partial differential equations are transformed into coupled non-linear ordinary differential equations using suitable similarity transformations and then solved numerically using shooting method. The effect of various parameters like shear-to-strain-rate parameter (α) , fluid index (n) $and the velocity ratio parameter (λ) are discussed persuasively in the relevant graphs.

Keywords: Three-dimensional Blasius flow; Boundary layer non-Newtonian fluid; Similarity solution; Three-dimensionality parameter; Numerical solution.

1. INTRODUCTION

There are numerous experimental and computational studies available on non-Newtonian fluid flows in both two-dimension and three-dimension in the literature. The boundary layer flow of non-Newtonian fluids has become an important topic of investigation, ever since its various industrial and engineering applications such as in polymer processing, food processing, biochemical industries etc. The classical theory of fluid dynamics depends upon the hypothesis of a linear relationship between the stress tensor and strain tensor, rate of strain tensor and even rate of the stress tensor.

Newton's viscosity law's states that, the shear stress between adjacent fluid layers is proportional to the velocity gradients between the two layers. The fluids which do not follow this linear relationship are called non-Newtonian fluids and a few examples of non-Newtonian fluids are a slurry paste, printer ink, condensed milk, molten rubber, shampoos, tomato sauce, etc.

Copious literature in the boundary-layer flow problems deals with the traditional Newtonian fluid, but most of the industrially important fluids do not satisfy this classical idea of a linear relationship between viscosity and rate of deformation. Several mathematical models have been proposed to explain the rheological behaviour of non-Newtonian fluids. Some of the models are the power-law or Ostwald-de Waele, Sisko, Ellis, Williamson, Eyring, Powell-Eyring and Reiner-Philippoff fluid model etc. Among these models, the power-law model is the most widely used because it is frequently encountered in allied processing and chemical engineering process applications.

The modelling of non-Newtonian fluids creates interesting challenges to the mathematician. The motivation of present work is to study the behaviour of the motion of Ostwald-de Waele or power-law fluid over a moving surfaces in a three-dimensional boundary-layer, involving various parameters like the power-law index of the fluid (n), the three-dimensionality or shear-to-strain-rate parameter (α).

A sound investigation of non-Newtonian fluid flow in the two-dimensional boundary-layer has been carried out by many researchers. Schowalter (1960) gave applications of power-law fluids in the layer by similar solutions and he also analysed the mechanism of non-Newtonian fluids in Schowalter (1978). Rajagopal *et al.* (1980), (1983) studied boundary-layer theory for non-Newtonian fluids by Falkner-Skan equations. Wu and Thompson (1996) studied non-Newtonian shear-thinning flows past a plate. Liao (2003) gave the analytic solution of MHD non-Newtonian fluids over-stretching sheets. Elgazery and AbdElazem (2009) gave numerical solutions of power-law fluid on unsteady heat and mass transfer through porous medium past semi-infinite vertical plate in presence of magnetic effects and radiation for variable thermal conductivity, surface temperature and viscosity. Hayat *et al.* (2011) investigated the MHD mixed convection non-Newtonian fluid in the stagnation-point flow over a stretching surface.

All the above-mentioned analysis were carried out only on two-dimensional non-Newtonian fluids. Three-dimensional flow is not just a simple extension of two-dimensional flow. Because of its mathematical complexity, this problem is generally avoided. The three-dimensional flow introduces additional effects in the system, i.e, firstly, due to the influence of an additional third dimension effect, it induces a change in boundary-layer thickness.

Secondly, the existence of secondary flow on crossflow. The governing equation for the flow of a fluid in the three-dimensional boundary-layer region was first studied by Howarth (1951a).

Numerical solution of three-dimensional boundary-layer for stagnation point flows is obtained by Howarth (1951b) and shown that nodal points of attachment of the flow. Howarth (1951) obtained the solution for two orthogonal Heimenz flows of non-axisymmetric stagnation point flow depends on a one-parameter $\alpha = \frac{V_\infty}{U_\infty}$. The three-dimensional boundary-layer flow of a Newtonian fluid was considered by Rosenhead (1963). In this book, it shows the solution in the range of $0 < \alpha < 1$. Later Davey

(1961) investigate this problem at saddle point solution of Howarth equation. Then many research has been done on Newtonian three-dimensional boundary-layer i.e, Libby (1967), Scholiefed and Davey (1967), Wang (1984), Duck and Stow (2002), Weidmman (2012) recently Kudenatti and Kirsur (2014) for various other situations.

Though there is much work in the three-dimensional boundary layer of non-Newtonian fluid. We lack an efficient model to study a similar solution of non-Newtonian fluid. Na T.Y. and Hansen A. (1967) obtained the numerical solution of three-dimensional boundary-layer of power-law fluids by assuming cross flows, in this he found the two components of mainstream flow is polynomial in variable x. Subba *et al.* (1993) also gave similar solutions of non-Newtonian fluids in three-dimensional stagnation flow. Yurusoy M. and Pakdemirli M. (1997) studied Lie group analysis on unsteady three-dimensional boundary layer of non-Newtonian fluids to obtained solutions by PDE to ODE. Timol M.G. and Patel M. (2013) investigated ten different models of non-Newtonian fluid in both two and three-dimensional in stagnation point flow. Recently by Nadeem *et al.* (2014) analyzed MHD effects in shear-thinning fluid i.e, Casson fluid on three-dimensional boundary-layer on the linearly stretching sheet with convective boundary condition.

The remainder of the present work is presented as follows, in section 2, we model the flow theory of three-dimensional flow over a flat plate of power-law fluid. To study the non-Newtonian effects are reduced the PDE (partial differential equations) using similarity transformation, it converts into coupled non-linear ODE (ordinary differential equations). Due to the coupled and nonlinear nature, the obtained equations are solved by numerically using the shooting technique. The procedure of the method and convergence condition to the problem is given in section 3. In section 4 we discussed the solutions of the problem through graphically. Results of velocity profiles, wall shear stress for various parameters are discussed in detailed.

2. FLOW MODEL

We consider the steady three-dimensional laminar boundary-layer flow of an incompressible Non-Newtonian power-law fluid over a moving flat plate. The plate is semi-infinite, impermeable sheet coinciding with the plane $z = 0$, the flow bring confined to $z > 0$. The governing equations for the flow of incompressible fluid are continuity equation and momentum equations. The non-Newtonian boundary layer equation is derived from Navier-stokes equation. The vectorial form of continuity and momentum equation are:

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\rho\left(\vec{q}\cdot\nabla\right)\vec{q} = -\nabla p + \nabla \tau \tag{2}$$

where μ velocity components of x, y and z axis, ρ is the density, p is the hydrodynamic pressure, and τ is the viscous stress or frictional stress (it is a 2nd order vector or tensor). Usually τ viscous stress are assumed to be proportional to the rate of strain then the proportionality constants is known as viscosity coefficient μ (fluid viscosity).

According to stokes's law of function, the relationship between stress tensor and rate of strain components are,

(1) The stress components are linear functions of the rate of strain components, Newtonian fluid (μ is regarded as constant)

(2) If the stress components i.e. are not linear, then the relation between stress components are invariant to orientation of the co-ordinate axis, i.e. they remains unchanged by a rotation of the system of co-ordinate or by an interchange of axis.

The relationship between viscous stress tensor and rate of strain tensor is given by.

$$\tau = \tau_{i,j} = 2\mu e_{ij} + \lambda e_{kk}\delta_{i,j}. \tag{3}$$

where $\lambda = -2/3\mu, \mu$ non-newtonian viscosity and $g'' = t' = W$ is the rate of deformation tensor. To determine velocity distribution from equation (2) by substitution τ . For Non-Newtonian fluid the equation of motion is governed by,

$$\rho\left(\vec{q}\cdot\nabla\right)\vec{q} = -\nabla p + \nabla\left(2\mu e_{ij} + \lambda e_{kk}\delta_{ij}\right) \tag{4}$$

where rate of deformation tensor defined as ,

$$e_{ij} = 1/2\left(\nabla\vec{q} + \nabla\vec{q}^{T}\right), \tag{5}$$

for incompressible fluid $e_{kk} = 0$ by continuity equation, i.e.

$$\left(\vec{q}.\nabla\right)\vec{q} = -\frac{1}{\rho}\nabla p + \nu\nabla\left(\nabla\vec{q} + \nabla\vec{q}^{T}\right). \tag{6}$$

By Ostwald-de Waele model (power-law model). The non-Newtonian viscosity ν can be expressed in terms of the (Schowalter 1960)

$$\nu = -K\left(e_{ij}^{\frac{n-1}{2}}\right) = -K\left|\left(\nabla\vec{q} + \nabla\vec{q}^{T}\right)^{\frac{n-1}{2}}\right| \tag{7}$$

where e_{ij} in component

$$e_{i,j} = 1/2\sum_{j=1}^{3}\sum_{i=1}^{3}\left[\frac{\partial\vec{q}_i}{\partial x_j}\frac{\partial\vec{q}_j}{\partial x_i} + \left(\frac{\partial\vec{q}_i}{\partial x_i}\right)^2\right] \tag{8}$$

This is the most generalized model for non-Newtonian fluid where K consistency of the fluid and n fluid index. Which can be used to classify the fluids as shear-thinning/ pseudo plastic fluids if $n<1$ and shear-thickening/Dilatant fluids if $n>1$. If $n=1$, ($K = \mu$) then is it known as Newtonian fluid. Further, for large Reynolds number flow, the viscosity are confined to a thin layer near the boundary surface. Within this thin layer, a very large velocity gradient exists, $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y} << \frac{\partial u}{\partial z};\frac{\partial v}{\partial x},\frac{\partial v}{\partial y} << \frac{\partial v}{\partial z}$. Under usual boundary-layer approximations, the continuity and momentum equations are respectively expressed as.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{9}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial x}+K\frac{\partial}{\partial z}\left(\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial z}\right) \tag{10}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial y}+K\frac{\partial}{\partial z}\left(\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial v}{\partial z}\right) \tag{11}$$

$$\frac{\partial p}{\partial z}=0 \tag{12}$$

where u, v, w are velocity components in x, y and z directions and the corresponding relevant boundary conditions as

$$\text{at } z=0: \quad u=U_w(x,y), \quad v=V_w(x,y), \quad w=0, \tag{13}$$

$$\text{as } z\to\infty; \quad u\to U(x,y), v\to V(x,y) \tag{14}$$

where $U_w(x,y)$ and $V_w(x,y)$ are the surface velocities in x and y directions and $U(x,y)$ and $V(x,y)$ are outer stream velocity, to the boundary layer. From Bernoulli's theory the pressure gradient becomes.

$$\frac{-1}{\rho}\frac{\partial p}{\partial x}=U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}, \tag{15}$$

$$\frac{-1}{\rho}\frac{\partial p}{\partial y}=U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}, \tag{16}$$

Therefore the (11-12) is written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+K\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}^{2}+\frac{\partial v}{\partial z}^{2}\right)^{\frac{n-1}{2}}\frac{\partial u}{\partial z}\right] \quad (17)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+K\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}^{2}+\frac{\partial v}{\partial z}^{2}\right)^{\frac{n-1}{2}}\frac{\partial v}{\partial z}\right] \quad (18)$$

Equation (15) shows that the normal pressure gradient is zero i.e. The pressure is assumed constant across the flow region. The velocities of the outside boundary layer is expressed as $U=U_\infty, V=V_\infty$ which is constants then the pressure gradient vanishes in the equations. Then three unknown velocity components in equation (17-18) and can easily be reduced by introducing two unknown stream functions ψ_1 and ψ_2 i.e. $u=\frac{\partial \psi_1}{\partial z}, v=\frac{\partial \psi_2}{\partial z}, w=-\left(\frac{\partial \psi_1}{\partial x}+\frac{\partial \psi_2}{\partial y}\right)$ where ψ_1 and ψ_2 are stream functions, which is defined as

$$\psi_1 = U_\infty f(\eta)\left(\frac{n(n+1)\mu x}{U_\infty^{2-n}}\right)^{\frac{1}{n+1}} \quad (19)$$

$$\psi_2 = V_\infty g(\eta)\left(\frac{n(n+1)\mu x}{U_\infty^{2-n}}\right)^{\frac{1}{n+1}} \quad (20)$$

$$\eta = z\left(\frac{1}{n(n+1)}\frac{U_\infty^{2-n}}{\mu x}\right)^{\frac{1}{n+1}} \quad (21)$$

where $f(\eta)$ and $g(\eta)$, are velocities on *x* and *y* directions with respect to similarity transformations η expressed as $\eta = z\left(\frac{1}{n(n+1)}\frac{U\infty^{2-n}}{\nu x}\right)^{\frac{1}{n+1}}$ (Kanpur and Ramesh 1963).

The number of independent variables (three) can be reduced to one in accordance with similarity transformations. By substituting ψ_1 and ψ_2 in the above equations we get

$$\frac{\partial \psi_1}{\partial z}\frac{\partial^2 \psi_1}{\partial x \partial z}+\frac{\partial \psi_1}{\partial z}\frac{\partial^2 \psi_2}{\partial y \partial z}-\left(\frac{\partial \psi_1}{\partial x}+\frac{\partial \psi_2}{\partial y}\right)\frac{\partial^2 \psi_1}{\partial z^2}=\nu\frac{\partial}{\partial z}\left[\left[\left(\frac{\partial^2 \psi_1}{\partial z^2}\right)^2+\left(\frac{\partial^2 \psi_2}{\partial z^2}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2 \psi_1}{\partial z^2}\right] \quad (22)$$

$$\frac{\partial\psi_1}{\partial z}\frac{\partial^2\psi_2}{\partial x\partial z}+\frac{\partial\psi_2}{\partial z}\frac{\partial^2\psi_2}{\partial y\partial z}-\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}\right)\frac{\partial\psi_2}{\partial z^2}=\nu\frac{\partial}{\partial z}\left[\left[\left(\frac{\partial^2\psi_1}{\partial z^2}\right)^2+\left(\frac{\partial^2\psi_2}{\partial z^2}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2\psi_2}{\partial z^2}\right] \quad (23)$$

Using this transformations, we get a system of the third order coupled non-linear ordinary differential equations.

$$\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-1}{2}} f'''+(n-1)f''\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-3}{2}}\left(f''f'''+\alpha^2 g''g'''\right)+\frac{n}{2}ff''=0 \quad (24)$$

$$\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-1}{2}} g'''+(n-1)g''\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-3}{2}}\left(f''f'''+\alpha^2 g''g'''\right)+\frac{n}{2}fg''=0 \quad (25)$$

with boundary conditions.

$$\begin{aligned} &f(\eta)=g(\eta)=0,\ f'(\eta)=g'(\eta)=\lambda_{1,2} \quad \text{where} \quad \eta=0,\\ &f'(\eta)=g'(\eta)=1 \quad \text{as} \quad \eta\to\infty \end{aligned} \quad (26)$$

where $f(\eta)$ and $g(\eta)$ are non-dimensional stream functions, η is a similarity variable, and a prime differentiation with respect to η. The parameter $\alpha=\frac{V_\infty}{U_\infty}$ is three-dimensionality parameter or shear-to-strain rate ratio parameter, for $\alpha=0$ the system (24-25) correspondence to the classical two dimensional flow. For $\alpha=1$, the above system corresponds to the symmetrically flow *n* is a fluid index, *n<1* shearing thinning and *n>1* shearing thickening $\lambda_1=\frac{U_W}{U_\infty},\lambda_2=\frac{V_w}{V_\infty}$. are the velocity ratios of free stream velocity to boundary velocity, for $\lambda_1\lambda_2>0$ and $\lambda_1\lambda_2<0$ correspond to moving surface in the same and the opposite direction to the free-stream velocity, $\lambda_{1,2}=0$ is the case for flow over fixed surface. The system (24-25) with boundary conditions describe three dimensional boundary-layer flow of a non-Newtonian fluid, These nonlinear coupled differential equations are solved by numerical technique. The solution procedure of numerical method for above couple differential equation discussed in the next section.

3. NUMERICAL METHOD

The system of coupled non-linear differential equations along with the boundary conditions are solved numerically using a Runge-Kutta method with shooting. Since the equations are highly non-linear, a numerical treatment would be more appropriate. Although the problem is a boundary value problem, it is converted in to an initial value problem. In this technique the higher order differential equations are converted in to system of first order differential equations by introducing some additional unknown functions, then the system of IVP solved by Runge-Kutta method, by assign a trial value to the unknown initial condition. Once the result are obtain to IVP its check whether the boundary condition at infinity is satisfied. We repeat the procedure until we find an appropriate convergent initial value with in a tolerance limit of 10^{-8}. The step size $\Delta\eta = 0.001$ is used to obtain the numerical solutions.

To describe this method, the system with boundary conditions is consider in the form of system of First-order ODE which are written as,

$$\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-1}{2}} f''' + (n-1) f''\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-3}{2}}\left(f''f'''+\alpha^2 g''g'''\right) + \frac{n}{2} ff'' = 0 \quad (27)$$

$$\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-1}{2}} g''' + (n-1) g''\left(f''^2+\alpha^2 g''^2\right)^{\frac{n-3}{2}}\left(f''f'''+\alpha^2 g''g'''\right) + \frac{n}{2} fg'' = 0 \quad (28)$$

$$f' = u \quad (29)$$

$$f'' = u' = V \quad (30)$$

$$g' = t \quad (31)$$

$$g'' = t' = W \quad (32)$$

Now equation becomes

$$V' = \frac{-(n-1)WVW'\alpha^2\left(V^2+\alpha^2W^2\right)^{\frac{n-3}{2}} - nfV}{(n-1)V^2\left(V^2+\alpha^2W^2\right)^{\frac{n-3}{2}} + \left(V^2+\alpha^2W^2\right)^{\frac{n-1}{2}}} \quad (33)$$

$$W' = \frac{-(n-1)WVV'\left(V^2+\alpha^2W^2\right)^{\frac{n-3}{2}} - nfW}{(n-1)\alpha^2W^2\left(V^2+\alpha^2W^2\right)^{\frac{n-3}{2}} + \left(V^2+\alpha^2W^2\right)^{\frac{n-1}{2}}} \quad (34)$$

equation (33) and equation (34) are coupled first order equations. We decouple the equations by substituting (34) in (33) and simply for V', once V' is obtain, then substitute V' in equation (34) so, that we get W' in terms of known quality, i.e.

$$V' = \frac{(n-1)W^2V^2\left(V^2+\alpha^2W^2\right)^{\frac{n-3}{2}}\alpha^2 - \left(V^2+\alpha^2W^2\right)^{\frac{n-1}{2}} nfV}{\left\{(n-1)^2\alpha^2V^2W^2\left(V^2+\alpha^2W^2\right)^n - 3 + (n-1)\left(V^2+\alpha^2W^2\right)^n - 1 + \left(V^2+\alpha^2W^2\right)^n - 1\right\}} \quad (35)$$

$$W' = \frac{(n-1)\alpha^2W^2V^2(V^2+\alpha^2W^2)^{(n-3)} - (V^2+\alpha^2W^2)^{\frac{n-1}{2}} nfW}{(n-1)^2\alpha^2V^2W^2(V^2+\alpha^2W^2)^n - 3 + (n-1)(V^2+\alpha^2W^2)^n - 1 + (V^2+\alpha^2W^2)^n - 1} \quad (36)$$

Therefore the solution of non-linear problem (24-25) is the combination of the solution to six, first order initial value problems are equation with initial conditions,

$f(0)=0,\ u(0)=f'(0)=\lambda_1,\ V(0)=P_1, g(0)=0,\ t(0)=g'(0)=\lambda_2,\ W(0)=P_2$ where P_1 and P_2 are unknown initial conditions for equation, which are assumed initially and integrated numerically using Runge-Kutta method. The solution of BVP by using the solution of IVP, with initially assumed inspired initial conditions, each to the equation of IVP involving the different value of P_1, P_2.To obtain the next approximation, we use the secant method with two choices of P_1 and P_2. Once the desired accurate initial conditions are obtained, we checked by comparing with the boundary condition of the problem, to repeat the Runge-Kutta method and secant method. This numerical method used to solve the power-law fluid, the numerical result are obtained for flow indicates n = 0.8,1.0,1,2,1.4 from different value of three-dimensionality parameter α and λ_1, λ_2 are obtained. In each case, the velocity profiles $f'(\eta)$ and $g'(\eta)$ exist and figure are discussed below. This shooting technique would well and can be adjusted by taking a very fine guide in the flow domain to capture and significant flow variations. In all our simulations, the error tolerance was set 10^{-8}.

4. RESULT AND DISCUSSION

The physical behaviour of the problem by the effects of interesting parameters are discussed here. The system (24-25) with boundary condition (26) has been solved numerically using shooting technique which has been described in section 3. The main concern of the analysis is to determine the both velocity profiles $f'(\eta)$ and $g'(\eta)$ and skin-friction $f''(0)$ and $g''(0)$ for various values of the parameter involved in it, such as power index (n), shear-to-strain-rate (α) and velocity ratio parameter ($\lambda_{1,2}$of both the direction. The three-dimensional boundary-layer flow of power-law fluid over a flat surface is considered to be moving either in the same or in the opposite direction to the mainstream flows. The results from numerical methods are compared with the literature value for ($\alpha = 0.0$) i.e, (two-dimensional boundary-layer flow) through tabulated values. The effects of the three-dimensional parameter on the boundary-layer are discussed and solution obtain for range ($-1 < \alpha < \infty$) Davey (1961).

Figure 1 describe the velocity profiles $f'(\eta)$ and $g'(\eta)$ of the boundary-layer in both the directions. For the shearing thinking conditions (i.e, $n=0.8$) and increasing value of shear-to-strain parameters, both the end velocity profiles becoming confined to a region to satisfy end boundary conditions. These effects on the layer decrease the thickness of the boundary.

Figure 2, it shows a similar treatment for both the pseudo-plastic fluid case ($n<1$). Dilatant fluid case ($n>1$), for fixed values of $\alpha = 0.5$ The result for $n=1.0$ is known as Newtonian flow, we see that the velocity profiles do exist for all values of power index and benign. Computation for increasing n indicates that profiles always convected towards the surface. Because of these effect on the surface lead to thinning of the boundary-layer thickness.

Figure 3 and figure 4, illustrates how both the surface and mainstream flows have movements in both directions (i.e, x and y-directions). In fig 3 it shows velocity profiles for velocity ratio λ_1 in x-direction is kept fixed (i.e λ_1=0) and λ_2 in y-direction vary for different speed. Figure 3a, the velocity profiles approach the mainstream flow asymptotically. But in fig 3b shows different profiles, for all λ_2, i.e, λ_2<0 (moving in opposite) and λ_2>0 (moving in same direction to mainstream). Also, boundary-layer is quite large compared to x-direction.

The vice-versa nature is observed in figure 4, here figure 4a is velocity profiles of $f'(\eta)$in x-direction for various value of $\lambda_1$and figure 4b is a profiles of $g'(\eta)$ for λ_2=0.0 fixed surface. Here we saw a variation of profiles satisfy the end condition by making boundary-layer thickness thinner, but slower when compared to fig 4b for λ_2=0.0. In 4b, it shows similar behaviour but reduces boundary-layer thickness pastor compare to 4a. While, by observation fig 3a for λ_1=0.0, and fig 4b for λ_2=0.0 it is slower. In this case, the shooting method code accordingly takes a few extra iterations to produce the profiles.

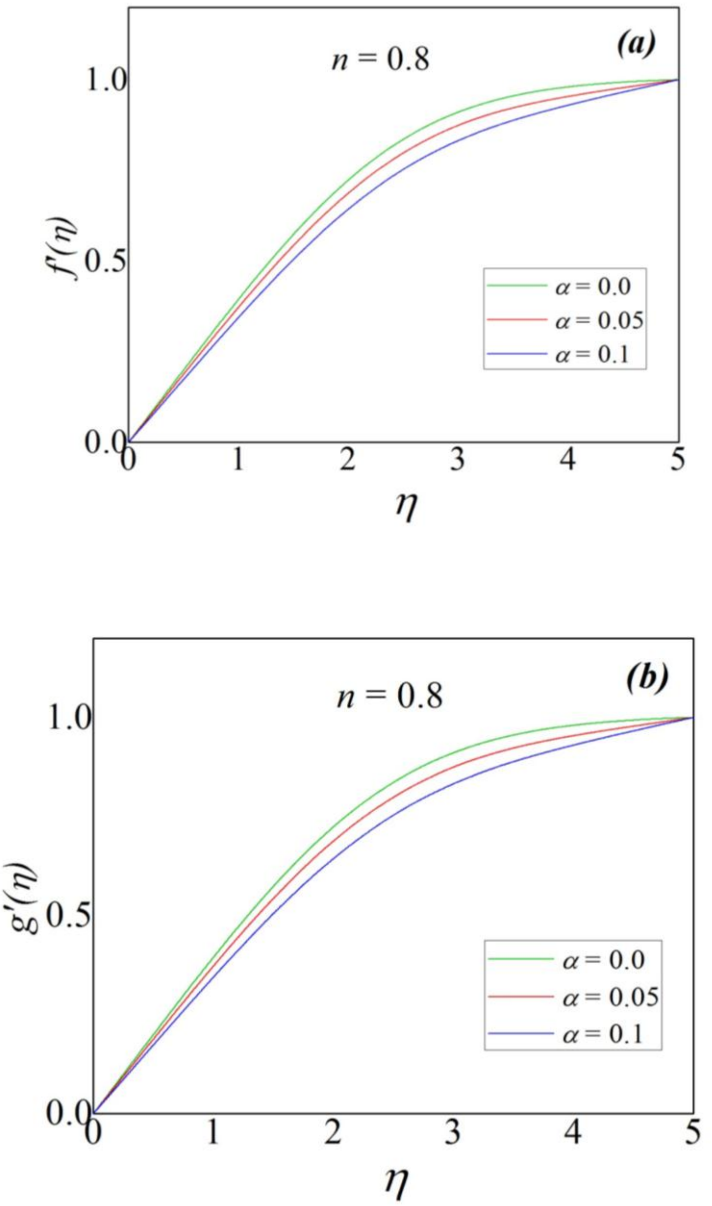


Figure 1: Variation of velocity profile $f'(\eta)$ and $g'(\eta)$ with η for different value of α and fixed n = *0.8*.

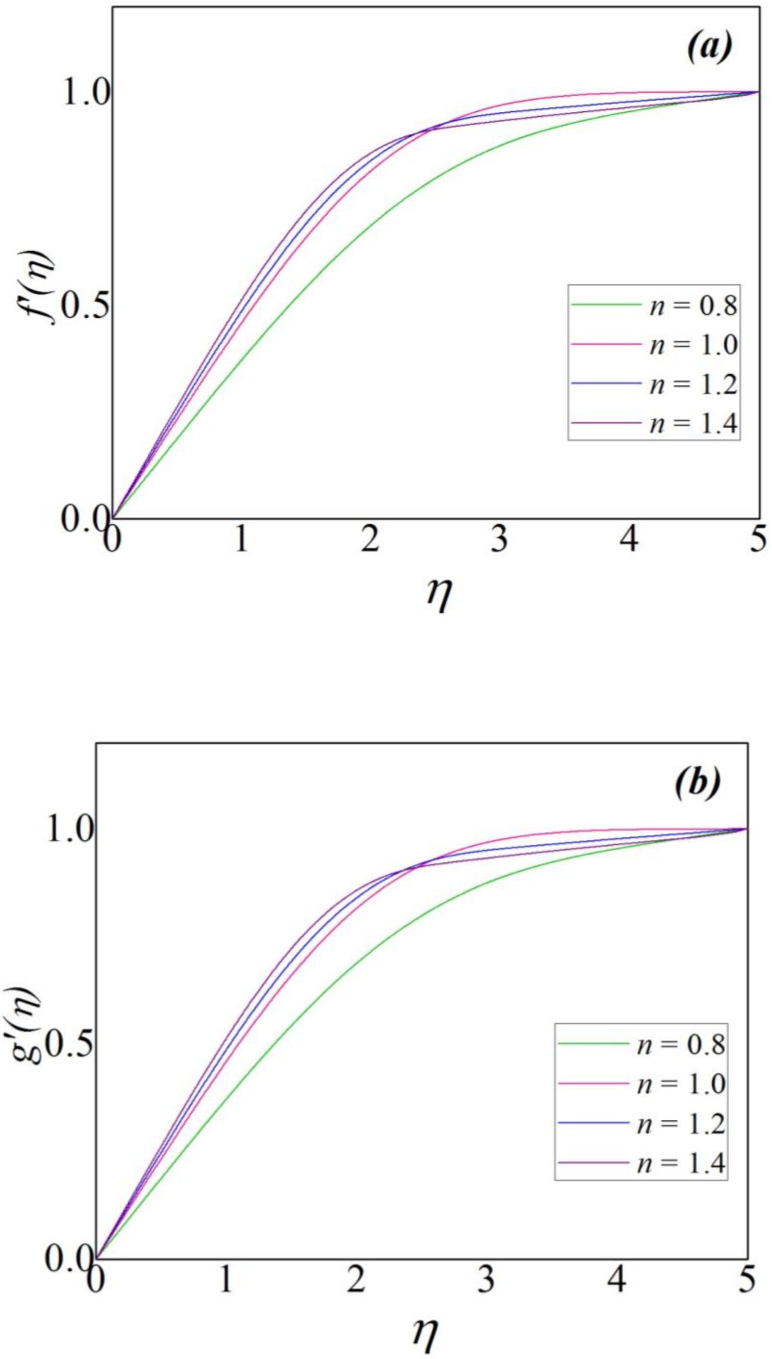


Figure 2: Variation of velocity profile $f'(\eta)$ and $g'(\eta)$ with η for different value of n and fixed α= *0.5*.

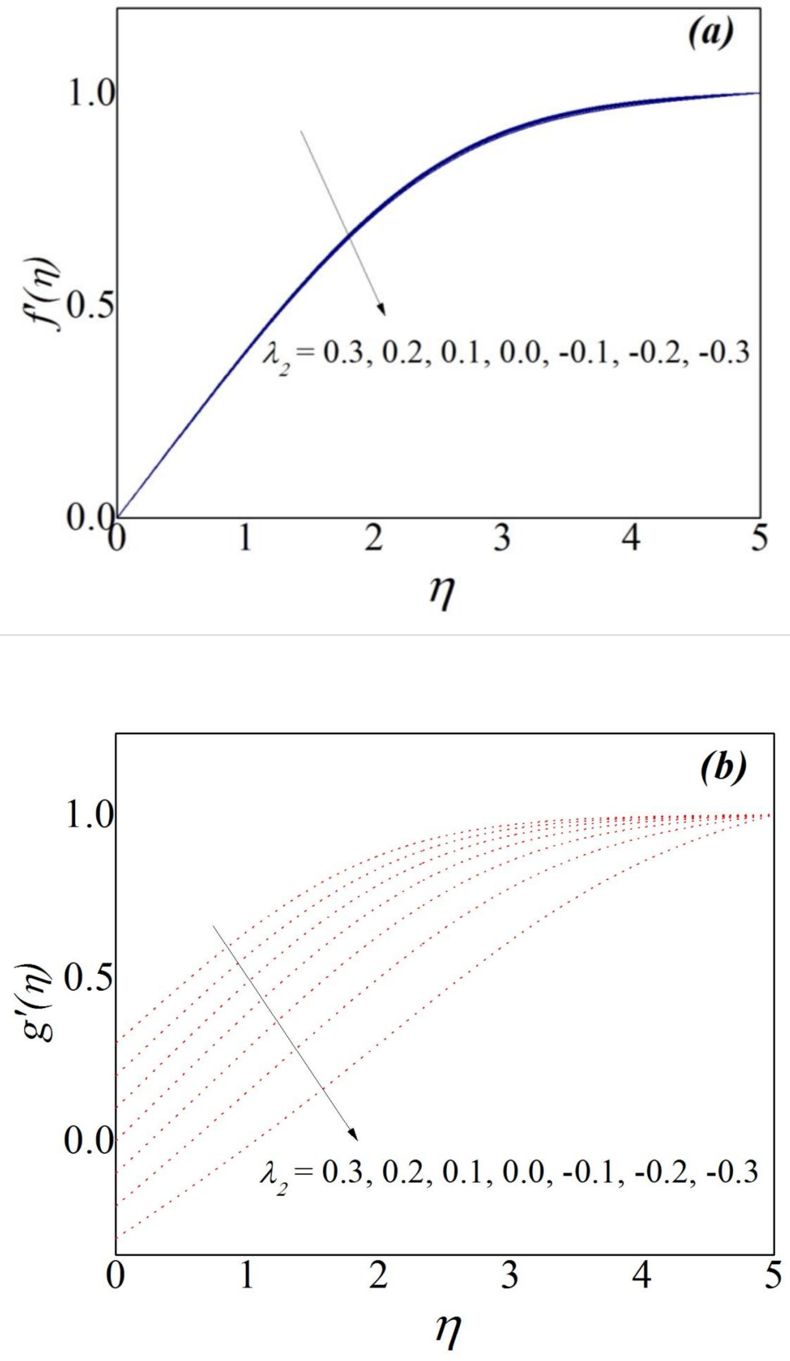
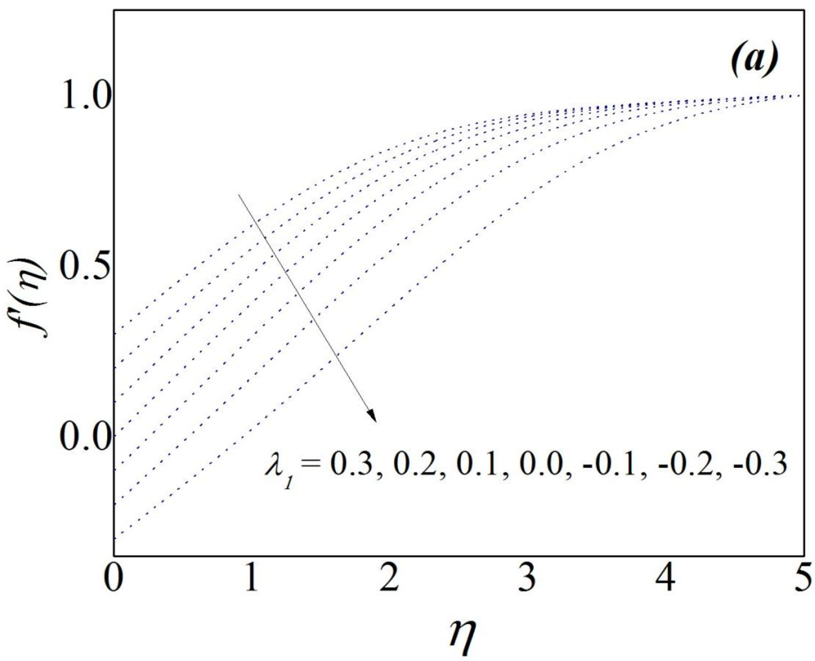


Figure 3: The variation of the velocities in both directions as a function of η for different λ_2 and $\lambda_1 = 0.0$.



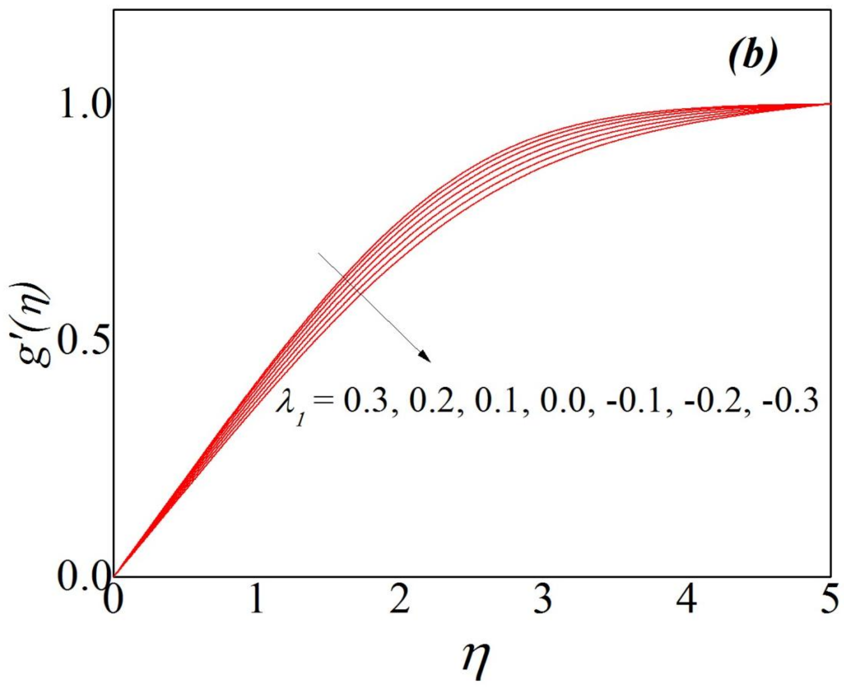


Figure 4: The variation of the velocities in both directions as a function of η for different λ_1 and $\lambda_2 = 0.0$.

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